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On spanning trees with few branch vertices

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A conjecture of Matsuda, Ozeki, and Yamashita posits that, for any positive integer k, a connected claw-free n-vertex graph G must contain either a spanning tree with at most k branch vertices or an independent set of $2k + 3$ vertices whose degrees add up to at most $n - 3$. In other words, G has this spanning tree whenever $\sigma_{2k+3}(G) \geq n - 2$. We prove this conjecture.

1. Introduction

In a tree, vertices of degree one and vertices of degree at least three are called *leaves* and *branch vertices*, respectively. A hamiltonian path can be regarded as a spanning tree with maximum degree at most two, a spanning tree with at most two leaves, or a spanning tree with no branch vertex. A natural extension of the hamiltonian path problem is, therefore, to look for conditions that guarantee the existence of a spanning tree that is "almost" a hamiltonian path in each of these ways. Many researchers have investigated independence number conditions and degree sum conditions for the existence of spanning trees with low maximum degree [3,9,12,15]; few leaves [1,8,14,16]; and few branch vertices [2,4–7,10]. Several of these results are discussed in more detail in a 2011 survey of spanning trees [13].

We denote by $\sigma_m(G)$ the smallest possible sum of degrees of an independent set of m vertices in G. If there is no such independent set, we say $\sigma_m(G) = \infty$. We also denote by $G[V] = G[v_1, v_2, \ldots, v_t]$ the subgraph induced by $V = \{v_1, v_2, \ldots, v_t\}$. A paper of Matsuda, Ozeki, and Yamashita [10] conjectures a condition on connected claw-free graphs which ensures the existence of a spanning tree with at most k branch vertices. As they mention, it is best possible.

Conjecture 1 (*[10]*). *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{2k+3} \geq n - 2$, then G has a spanning tree with at most k branch vertices.*

The $k = 0$ case, as the authors point out, follows from a theorem of Matthews and Sumner [11]. The authors prove the $k = 1$ case in the same paper [10], and the $k = 2$ case has since been shown in [7]. In this paper, we prove the entire conjecture, but must first give some definitions and notation. Throughout this paper, uv denotes the edge between any two vertices u and v.

Definition 1. Let $B = B(T)$ denote the set of branch vertices of a tree T, and let $L(T)$ denote the set of leaves. Let $B_n(T)$ denote the set of branch vertices of T with degree exactly n, and let $B_{\leq n}(T)$ $\left(B_{\geq n}(T)\right)$ denote the set of branch vertices of T with degree at most (at least) n. Any two vertices of T, say u and v, are joined by a unique path, denoted uTv, and we denote $d(u, v) = |E(uTv)|$. Now if $e \in E(T)$, then eTv denotes the unique shortest path containing v and one of the

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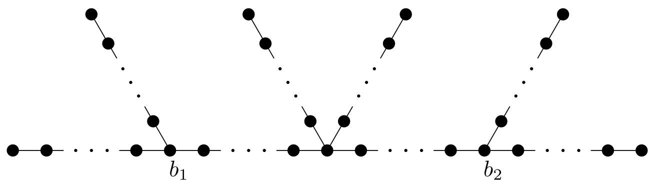

Fig. 1. An example of a tree T. Its internal subtree, in this case, is the path b_1Tb_2.

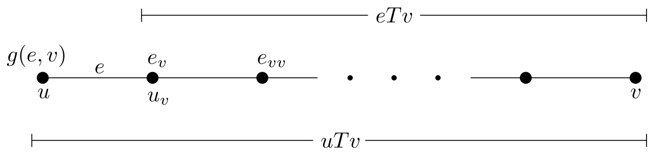

Fig. 2. A path between vertices u and v within some tree T, as described in Definition 1, showing $g(e, v)$ as described in Definition 2. If T is a spanning tree of some graph G, note that v is an oblique neighbor of e with respect to T if and only if $vg(e, v) \in E(G)$.

vertices incident to e, but not the edge e itself. We also denote $\{u_v\} := V(uTv) \cap N_T(u)$ and e_v as the vertex incident to e in the direction toward v. If $e_v \neq v$, then we denote $\{e_{vv}\} := V(e_vTv) \cap N_T(e_v)$, similar to the u_v notation. *(See Fig. 2.)* Lastly, we call the set $S_T = \bigcup_{u,v \in B} uTv$ the **internal subtree** of T. (See Fig. 1.)

Definition 2. Let T be a spanning tree of a graph G and let $v \in V(G)$ and $e \in E(T)$. Denote $g(e, v)$ as the vertex incident to e farthest away from v in T. We say v is an **oblique neighbor** of e **with respect to T** if $vg(e, v) \in E(G)$.

Note that both vertices incident to a given edge of T are among its oblique neighbors.

Definition 3. Let T be a spanning tree of a graph. Two vertices are **pseudoadjacent with respect to T** if there is some $e \in E(T)$ which has them both as oblique neighbors. Similarly, a vertex set is **pseudoindependent** with respect to T if no two vertices in the set are pseudoadjacent with respect to T.

Note that pseudoadjacency (with respect to any tree) is a weaker condition than adjacency, while pseudoindependence is a stronger condition than independence.

Theorem 1. *Let G be a connected, claw-free graph on n vertices, and let k be a non-negative integer. If $\sigma_{2k+3}(G) \geq n - 2$, then G has a spanning tree with at most k branch vertices.*

Proof. Suppose some G, as described in the theorem, has no spanning tree with at most k branch vertices. As the theorem is proved for $k < 3$, we may assume $k \geq 3$. Choose some spanning tree T of G such that:

(T1) $|B(T)|$ is as small as possible.
(T2) We select trees with at least one degree 3 vertex over those with none, subject to (T1).
(T3) If (T2) allows no trees with a degree 3 vertex, $|L(T)|$ is as small as possible.
(T4) If (T2) allows a tree with at least one degree 3 vertex, the vertices with more than 4 neighbors have the smallest possible total number of neighbors *in excess of 4*. That is,

$$\sum_{v \in B_{\geq 5}(T)} (\deg_T(v) - 4)$$

is as small as possible.

Note that T must have at least four branch vertices.

We begin by showing that T must have at least one vertex of degree 3. Suppose T has no vertices of degree 3. The number of leaves in T is therefore:

$$|L(T)| = 2 + \sum_{b \in B(T)} (\deg_T(b) - 2) \geq 2 + \sum_{b \in B(T)} (2) \geq 2 + (k+1)(2) = 2k + 4.$$

We will first establish that $L(T)$ is independent, and then that it is pseudoindependent with respect to T.

Suppose two leaves s and t are adjacent in G. Then s has some nearest branch vertex b, so $T' := T - \{bb_s\} + \{st\}$ has fewer leaves than T, violating either (T2) or (T3) depending on $\deg_T(b)$. Therefore $L(T)$ must be independent in G.

Suppose two leaves s and t are pseudoadjacent with respect to T. Then there is some edge $e \in E(T)$ such that $sg(e,s), tg(e,t) \in E(G)$. Consider two cases.

Case 1: Suppose $g(e,s) = g(e,t)$. Define $a := g(e,s) = g(e,t)$, so $V(sTt) \cap V(sTa) \cap V(tTa) =: \{w\} \not\subseteq \{s,t,a\}$. Since $G[a, e_w, s, t]$ is not a claw, either $se_w \in E(G)$ or $te_w \in E(G)$ (we know $st \notin E(G)$ since $L(T)$ is independent). We may assume the first by symmetry, so $T' := T - \{e, ww_s\} + \{se_w, ta\}$ violates either (T2) or (T3) since two leaves are lost (s and t) while at most one is gained (w_s). This argument works even if $w_s = s$, in which case the leaf t is lost and no leaves are gained.

Case 2: Suppose $g(e,s) \neq g(e,t)$. Then $e_s = g(e,t)$ and $e_t = g(e,s)$, so $se_t, te_s \in E(G)$. This implies that $e_s, e_t \in V(sTt)$. Choose an arbitrary branch vertex $b \in V(sTt)$; assume $b \in V(eTt)$ by symmetry. Then $T' := T - \{e, bb_t\} + \{se_t, te_s\}$ violates either (T2) or (T3) since two leaves are lost (s and t) while at most one is gained (b_t).

Therefore $L(T)$ is pseudoindependent with respect to T, so no edge of T has more than one leaf of T as an oblique neighbor. We next find two edges of T that have no leaves of T as oblique neighbors. Since T has at least four branch vertices, we know $S_T \neq \emptyset$ and $S_T \neq K_1$. Choose a leaf of S_T (note that it is a branch vertex of T) and call it b. As T has no vertices of degree exactly 3, then $\deg_T(b) \geq 4$ and $|N_T(b) \cap S_T| = 1$, so $|N_T(b) \setminus S_T| \geq 3$. Choose three of these vertices and call them u, v, w. Since $G[b, u, v, w]$ is not a claw, $\{u, v, w\}$ cannot be independent in G. By symmetry, assume $uv \in E(G)$. We will show that bu and bv have no leaves as oblique neighbors.

Since $u \notin S_T$, there is some $z \in L(T)$ such that $u = b_z$. If some leaf $l \neq z$ is an oblique neighbor of bu, then $lu \in E(G)$, so $T' := T - \{bu\} + \{lu\}$ violates (T2) if $\deg_T(b) = 4$, or (T3) otherwise. If z is an oblique neighbor of bu, then $bz \in E(G)$, so $T' := T - \{bu, bv\} + \{bz, uv\}$ similarly violates either (T2) or (T3). Note that this works even if $z = u$. Therefore bu has no leaves as oblique neighbors, and by the same argument, neither does bv.

For any $v, x \in V(G)$, we have $vx \in E(G)$ if and only if v is an oblique neighbor of xx_v. Therefore the number of edges with v as an oblique neighbor equals the degree of v. Since no edge has more than one leaf as an oblique neighbor, and two of them have no leaves as oblique neighbors, the degrees of the leaves can add up to at most $|E(T)| - 2 = (n-1) - 2 = n - 3$, contradicting the assumption of the theorem.

Therefore T must have at least one vertex of degree 3, so we can choose a root $r \in B_3(T)$, denoting $N_T(r) =: \{r_1, r_2, r_3\}$. Since no claw can be centered at r, we may assume by symmetry that $r_1 r_2 \in E(G)$. We denote the branch vertex or leaf closest to any $e \in E(T)$ in the direction *away* from the root as $x = x(e)$. For each $i \in [3]$, define $x_i := x(rr_i)$. We will need one more definition.

Definition 4. For any rooted spanning tree T with root $r \in B_3(T)$, denoted (T, r), each branch vertex $x \in B(T) \setminus \{r\}$ has a **distance–degree pair** $(d(x, r), \deg_T(x))$. We define a **pair sequence** on the entire set $B(T) \setminus \{r\}$, which contains the distance–degree pairs of all vertices of $B(T) \setminus \{r\}$ in lexicographically increasing order (shortest distance first, and smallest degree first given equal distance).

Since such an r must exist, choose (T, r) such that:

(T5) The sequence of distance–degree pairs of $B(T) \setminus \{r\}$, as defined above, is lexicographically as small as possible, subject to (T4). That is, given a tree T_A with its root r_A and a tree T_B with its root r_B, we select (T_A, r_A) over (T_B, r_B) if and only if the earliest entry that differs in their pair sequences is "smaller" (lexicographically, as described in Definition 4) for (T_A, r_A) than it is for (T_B, r_B). (See Fig. 3.)

Before completing the proof of Conjecture 1, we introduce several useful lemmas.

Lemma 1. *If a is a child of $b \in B(T) \setminus \{r\}$, then a is adjacent in G to some $c \in N_T(b) \setminus \{a\}$.*

Proof. Suppose there is no such c. To avoid claws centered at b, $N_T(b) \setminus \{a\}$ must be a clique in G, so $T' := T - \{bd : d \in N_T(b) \setminus \{a, b_r\}\} + \{b_r d : d \in N_T(b) \setminus \{a, b_r\}\}$ violates (T1) if $b_r \in B(T)$, or (T5) otherwise since $d(b_r, r) < d(b, r)$. □

Lemma 2. *Let a, x, and y be three distinct vertices of G. If $\deg_T(x) = 3$, $\deg_T(y) \in \{1, 3\}$, $a \in V(rTx)$, and $x \in V(rTy)$, then $ya \notin E(G)$.*

Proof. If $ya \in E(G)$, then $T' := T - \{xx_y\} + \{ya\}$ violates (T1) if $a \in B(T)$ or $x_y \in B_3(T)$, or (T5) otherwise, due to the shorter distance from r to a than there was from r to x. □

Lemma 3. *Let a and x be two distinct vertices of G. If $\deg_T(x) = 3$, $a \in V(rTx)$, and y is a child of x, then $ya \notin E(G)$.*

Proof. Given these conditions, if $ya \in E(G)$, then $T' := T - \{xy\} + \{ya\}$ violates (T1) if $a \in B(T)$, or (T5) otherwise, due to the shorter distance from r to a than there was from r to x. □

Corollary 1. *If $x \in B_3(T) \setminus \{r\}$, then the two children of x are adjacent in G.*

Proof. By Lemma 3, neither child of x is adjacent to x_r. Since no claw can be centered at x, this requires that the two children are adjacent. □

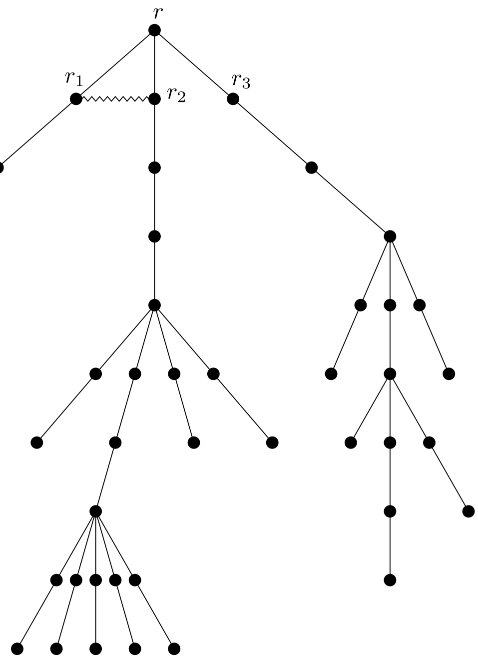

Fig. 3. An example of a rooted spanning tree (T, r) of a connected claw-free graph G with pair sequence $((3, 4), (4, 3), (4, 5), (5, 4), (7, 6))$. Since $G[r, r_1, r_2, r_3]$ cannot be a claw, we assume by symmetry that $r_1r_2 \in E(G)$ (shown as a squiggly line segment). Note that $\sum_{v \in B_{\geq 5}(T)} (\deg_T(v) - 4) = 3$.

Define $X := L(T) \cup B_3(T) \setminus \{r\}$. We will now show that $|X| \geq 2k + 3$. Define $m := |B_3(T)|$, so $|B_{\geq 4}(T)| \geq k + 1 - m$. Therefore:

$$\begin{aligned} |L(T)| &= 2 + \sum_{b \in B(T)} (\deg_T(b) - 2) \geq 2 + m + 2(k + 1 - m) \\ &= 2 + m + 2k + 2 - 2m \\ &= 2k + 4 - m \end{aligned}$$

hence:

$$|X| = |L(T)| + |B_3(T) \setminus \{r\}| \geq (2k + 4 - m) + (m - 1) = 2k + 3.$$

We next show that X is independent. Let $u, v \in X$ and assume $uv \in E(G)$. Now if $r \in V(uTv)$, then $T' := T - \{rr_u\} + \{uv\}$ violates (T1). If $u \in V(rTv)$ (or, symmetrically, $v \in V(rTu)$), then $u \in B_3(T)$, so define $\{u^*\} := N_T(u) \setminus \{u_r, u_v\}$. Now Corollary 1 gives that $u_vu^* \in E(G)$, so $T' := T - \{uu_v, uu^*\} + \{uv, u_vu^*\}$ violates (T1). (This works even if $u_v = v$, in which case the edge uv is deleted and added back, so T' simplifies to $T - \{uu^*\} + \{u_vu^*\}$.) The remaining possibility is that $V(rTu) \cap V(rTv) \cap V(uTv) =: \{w\} \not\subseteq \{r, u, v\}$. Now consider $T' := T - \{ww_u\} + \{uv\}$. If $w \in B_3(T)$ (or $u \neq w_u \in B_3(T)$), then T' violates (T1) since w (or w_u) is no longer a branch vertex. If $w \in B_{\geq 5}(T)$, then T' violates (T4) since w decreases the sum total but neither u nor v increases it (their degrees were originally at most 3 and are now at most 4). The remaining case is that $w \in B_4(T)$, in which case T' violates (T5) since w, which is closer to r than either u or v, has its distance–degree pair decreased.

Having established X as an adequately large independent set, it remains to show that X is pseudoindependent with respect to T, and then find two edges of T with no oblique neighbors in X, as we did for the case $B_3(T) = \emptyset$. To that end, here are several more lemmas.

Lemma 4. *If $y, z \in X$ are both oblique neighbors of $e \in E(T)$, then $e_r = e_y = e_z$.*

Proof. If this is not the case, then either $e_x = e_y = e_z$ (where $x = x(e)$ as described just before Definition 4), or $\{e_y, e_z\} = \{e_r, e_x\}$. Consider both these cases.

Case 1: Suppose $e_x = e_y = e_z$. Lemma 2 ensures that $y \notin V(xTz)$ and $z \notin V(xTy)$. We can therefore define $V(xTy) \cap V(xTz) \cap V(yTz) =: \{w\} \not\subseteq \{y, z\}$ (it is possible that $w = x$), and so $\deg_T(w) \geq 4$ by Lemma 2. If $w \in B_4(T)$, then $T' := T - \{ww_y, ww_z\} + \{ye_r, ze_r\}$ violates (T1) if e_r is a branch vertex, or (T5) if not. Otherwise $w \in B_{\geq 5}(T)$, and then we note that $G[e_r, e_x, y, z]$ is not a claw, so either $ye_x \in E(G)$ or $ze_x \in E(G)$. We may assume the first by symmetry, so $T' := T - \{e, ww_y\} + \{ye_x, ze_r\}$ violates (T4) via w.

Case 2: Suppose $e_y = e_r$ but $e_z = e_x$ (or vice versa, by symmetry). Depending on the location of y, we may have $r \in V(yTx)$, or $y \in V(rTx)$, or neither. If $r \in V(yTx)$, then $T' := T - \{e, rr_x\} + \{ye_x, ze_r\}$ violates (T1) via r. If $y \in V(rTe_r)$, we can define $\{y^*\} := N_T(y) \setminus \{y_r, y_x\}$; we then have from Corollary 1 that $y_xy^* \in E(G)$, implying that $T' := T - \{e, yy_x, yy^*\} + \{ye_x, ze_r, y_xy^*\}$ violates (T1) via y. If neither inclusion is true, we may denote $V(rTx) \cap V(rTy) \cap V(xTy) =: \{w\} \not\subseteq \{r, y\}$. We then have $T' := T - \{e, ww_y\} + \{ye_x, ze_r\}$. This T' violates (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$. □

Lemma 5. *If $y, z \in X$ are both oblique neighbors of some $e \in E(T)$, then neither y nor z is separated from e by r.*

Proof. Suppose at least one of y and z is separated from e by r. If they both are, then to avoid a claw centered at e_x, we must have either $ye_r \in E(G)$ or $ze_r \in E(G)$. We may assume the first by symmetry, and therefore $T' := T - \{e, rr_x\} + \{ye_r, ze_x\}$ violates (T1) via r. Therefore only one of them is separated from x by r (say $r \in V(xTz) \setminus V(xTy)$), and we note that $e_x \neq x$ (otherwise $T' := T - \{rr_x\} + \{zx\}$ violates (T1)), so e_{xx} exists. Now either $y \in V(rTx)$ or $y \notin V(rTx)$, so consider both cases.

Case 1: Suppose $y \in V(rTx)$. We can define $\{y^*\} := N_T(y) \setminus \{y_r, y_x\}$, so Corollary 1 requires that $y_xy^* \in E(G)$. Thus $T' := T - \{yy_x, yy^*, rr_z\} + \{ye_x, ze_x, y_xy^*\}$ violates (T1) since at least two branch vertices are lost (r and y) while only one is gained (e_x).

Case 2: Suppose $y \notin V(rTx)$. We can then define $V(rTx) \cap V(rTy) \cap V(xTy) =: \{w\} \not\subseteq \{r, y\}$. We can assume $G[e_x, e_{xx}, y, z]$ is not a claw, so either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ye_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, rr_z\} + \{ye_{xx}, ze_x\}$ violates (T1) via r. Otherwise $ze_{xx} \in E(G)$, and either $z \in L(T)$ or $z \in B_3(T)$. If $z \in B_3(T)$, then $T' := T - \{e_xe_{xx}, rr_z\} + \{ze_x, ze_{xx}\}$ violates (T1) via r. Otherwise $z \in L(T)$ and then $T' := T - \{e_xe_{xx}, ww_y\} + \{ye_x, ze_{xx}\}$ violates either (T1) if $w \in B_3(T)$ (or $w_y \in B_3(T)$), or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$. □

We now reach the heart of the argument that X is pseudoindependent with respect to T. Suppose some $y, z \in X$ are pseudoadjacent with respect to T, so they are both oblique neighbors of some $e \in E(T)$. As before, we denote $x = x(e)$. Now either both y and z are on the path rTx (and must be closer to the root than e on this path, as we established with Lemma 4), or exactly one of them is, or neither of them is, so consider all three cases.

Case A: Suppose $y, z \in V(rTx)$. Then $y, z \in B_3(T)$. By symmetry, we may assume $y \in V(rTz)$. Define $\{y^*\} := N_T(y) \setminus \{y_r, y_x\}$ and $\{z^*\} := N_T(z) \setminus \{z_r, z_x\}$, so Corollary 1 requires that $y_xy^*, z_xz^* \in E(G)$. Now $T' := T - \{yy_x, yy^*, zz_x, zz^*\} + \{ye_x, ze_x, y_xy^*, z_xz^*\}$ violates (T1) since two branch vertices are lost (y and z) while at most one is gained (e_x). (See Fig. 4.)

Case B: Suppose $y \in V(rTx)$ but $z \notin V(rTx)$. Define $\{y^*\} := N_T(y) \setminus \{y_r, y_x\}$, so Corollary 1 requires that $y_xy^* \in E(G)$. If $e_x = x$, then $T' := T - \{yy_x, yy^*\} + \{xy, y_xy^*\}$ violates (T1), so we may assume e_{xx} exists. In general, while it is possible that $e_{xx} = x$, this will not be a problem since $\deg_T(e_{xx}) = \deg_{T'}(e_{xx})$ in the upcoming subcases. We define $V(rTx) \cap V(rTz) \cap V(xTz) =: \{w\}$. Now Lemma 5 ensures that $w \neq r$, and our starting assumption ensures that $w \neq z$. We know that $w, y \in V(rTx)$, but we do not know which is closer to r, or indeed if they are the same, so consider three cases:

Subcase B1: Suppose $w \in V(rTy_r)$, and consider $T' := T - \{ww_x, yy_x, yy^*\} + \{ye_x, ze_x, y_xy^*\}$. Now T' violates (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$, since at least one branch vertex is lost (y) while exactly one is gained (e_x).

Subcase B2: Suppose $w = y$, and note that $y^* = y_z$. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ze_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_x, yy_z\} + \{ye_x, ze_{xx}, y_xy_z\}$ violates (T1) via y. Otherwise $ye_{xx} \in E(G)$, so since $G[y, y_r, y_x, e_{xx}]$ is not a claw, either $y_re_{xx} \in E(G)$ or $y_xe_{xx} \in E(G)$ (recall that Lemma 3 implies $y_ry_x \notin E(G)$). If $y_xe_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_x\} + \{y_xe_{xx}, ze_x\}$ violates (T1) via y. (Note that this works even if $e = yy_x$.) Otherwise $y_re_{xx} \in E(G)$, and then $T' := T - \{e_xe_{xx}, yy_z\} + \{y_re_{xx}, ze_x\}$ violates (T1) if $y_r \in B(T)$, or (T5) otherwise.

Subcase B3: Suppose $w \in V(y_xTx)$ (note that w is above e on this path by Lemma 4). Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ze_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_x, yy^*\} + \{ye_x, ze_{xx}, y_xy^*\}$ violates (T1) via y. Otherwise $ye_{xx} \in E(G)$, and then we consider $\deg_T(w)$. If $w \in B_{\geq 5}(T)$, then $T' = T - \{e_xe_{xx}, ww_z\} + \{ye_{xx}, ze_x\}$ violates (T1) if $w_z \in B_3(T)$ and otherwise (T4) via w. Otherwise $w \in B_{\leq 4}(T)$, so Lemma 1 requires that w_z must have some neighbor in G among the remaining vertices of $N_T(w)$. If this neighbor is w_r, then $T' := T - \{e_xe_{xx}, ww_r, ww_z\} + \{ye_{xx}, ze_x, w_rw_z\}$ violates (T1) via w. If, instead, this neighbor is w_x, then $T' := T - \{e_xe_{xx}, ww_x, ww_z\} + \{ye_x, ye_{xx}, w_xw_z\}$ violates (T1) via w. (Note that this works even if $e = ww_x$.) If this neighbor is neither w_r nor w_x, then it must be w^*, where $N_T(w) =: \{w_r, w_x, w_z, w^*\}$, and then $T' := T - \{e_xe_{xx}, ww_z, ww^*\} + \{ye_{xx}, ze_x, w_zw^*\}$ violates (T1) via w.

Case C: Suppose $y, z \notin V(rTx)$. Recall that $r \notin V(xTy) \cup V(xTz)$ by Lemma 5. Define $\{w\} := V(rTx) \cap V(rTy) \cap V(xTy)$ and $\{u\} := V(rTx) \cap V(rTz) \cap V(xTz)$, so $\{u, w\} \cap \{r, y, z\} = \emptyset$. Suppose $u = w$. Then the locations of y and z relative to each other have a few possibilities ($y \in V(uTz)$, or $z \in V(uTy)$, or neither), but the following argument will work the same for all of them. Since $G[e_x, e_r, y, z]$ is not a claw, either $ye_r \in E(G)$ or $ze_r \in E(G)$. We may assume the first by symmetry, so as long as $e \neq ww_x$, we have $T' := T - \{e, ww_x\} + \{ye_r, ze_x\}$ violates either (T1), (T4), or (T5), depending on $\deg_T(w)$. If $e = ww_x$, then $T' := T - \{e\} + \{zw_x\}$ similarly violates (T1), (T4), or (T5). Otherwise $u \neq w$, and we may assume $u \in V(rTw)$ by symmetry.

If $e_x = x$, note that $\deg_T(x) \geq 4$ (by our choice of independent set X) and consider $\deg_T(w)$. If $w \in B_{\geq 5}(T)$, then $T' := T - \{ww_y\} + \{xy\}$ violates (T1) if $y \neq w_y \in B_3(T)$, or else (T4) if $w_y \in B_{\geq 5}(T)$, or (T5) otherwise, since the sum total in (T4) would remain the same. Still assuming $e_x = x$, it remains to consider the case $w \in B_{\leq 4}(T)$. Now Lemma 1

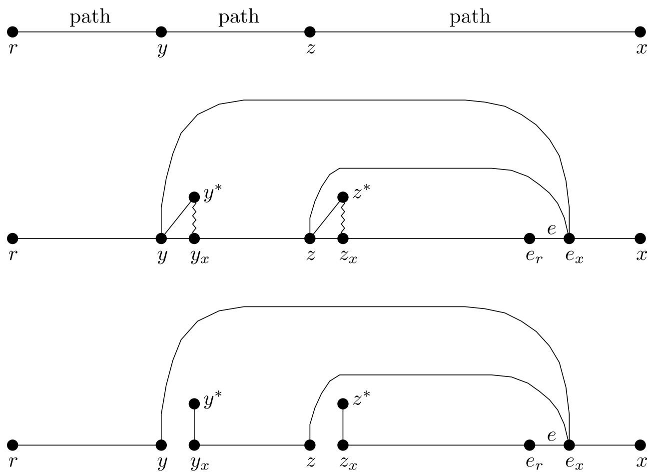

Fig. 4. These pictures show how one might visualize Case A, provided $e_r \neq z$. The first picture shows the relative positions of important vertices, as they are assumed in this case. In the second picture, the straight-line edges are part of the tree, while the curved and jagged edges are known to exist in the graph. The third picture shows the corresponding parts of T', which has one less branch vertex than T.

requires that w_y must have some neighbor in G among the remaining vertices of $N_T(w)$. If this neighbor is w_r, then $T' := T - \{ww_r, ww_y\} + \{xy, w_rw_y\}$ violates (T1) via w. If, instead, this neighbor is w_x, then $T' := T - \{ww_x, ww_y\} + \{xz, w_xw_y\}$ violates (T1) via w. If this neighbor is neither w_r nor w_x, then it must be w^*, where $N_T(w) =: \{w_r, w_x, w_y, w^*\}$, and then $T' := T - \{ww_y, ww^*\} + \{xy, w_yw^*\}$ again violates (T1) via w.

We have thus established that $e_x \neq x$, so e_{xx} exists. In general, as in Case B above, it will not be a problem if $e_{xx} = x$ since $\deg_T(e_{xx}) = \deg_{T'}(e_{xx})$ in the upcoming subcases. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ye_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, uu_x\} + \{ye_{xx}, ze_x\}$ violates either (T1), (T4), or (T5), depending on $\deg_T(u)$ as in the first paragraph of this case. Otherwise $ze_{xx} \in E(G)$, and then we consider $\deg_T(w)$. If $w \in B_{\geq 5}(T)$, then $T' := T - \{e_xe_{xx}, ww_y\} + \{ye_x, ze_{xx}\}$ violates (T4) via w. Otherwise $w \in B_{\leq 4}(T)$, and then Lemma 1 ensures that w_y is adjacent in G to at least one other vertex of $N_T(w)$. If $w_rw_y \in E(G)$, then $T' := T - \{e_xe_{xx}, ww_r, ww_y\} + \{ye_x, ze_{xx}, w_rw_y\}$ violates (T1) via w. If $w_xw_y \in E(G)$, we consider $\deg_T(z)$. If $z \in L(T)$, then $T' := T - \{e_xe_{xx}, ww_y\} + \{ye_x, ze_{xx}\}$ violates either (T1) or (T5). Otherwise $z \in B_3(T)$, and then $T' := T - \{e_xe_{xx}, ww_x, ww_y\} + \{ze_x, ze_{xx}, w_xw_y\}$ violates (T1) via w. The remaining possibility is that w_yw^*, where $N_T(w) =: \{w_r, w_x, w_y, w^*\}$. Then $T' := T - \{e_xe_{xx}, ww_y, ww^*\} + \{ye_x, ze_{xx}, w_yw^*\}$ violates (T1) via w. This concludes case C and thus the pseudoindependence argument.

Therefore X is a pseudoindependent set with respect to T. We will now show that rr_1 (and rr_2, by symmetry) has no oblique neighbors in X. Suppose some $x \in X$ is an oblique neighbor of rr_1. Now either $r \in V(r_1Tx)$ or $r_1 \in V(rTx)$. If $r \in V(r_1Tx)$, then $xr_1 \in E(G)$, so $T' := T - \{rr_1\} + \{xr_1\}$ violates (T1) via r. Otherwise $r_1 \in V(rTx)$, and then $xr \in E(G)$, so $T' := T - \{rr_1, rr_2\} + \{xr, r_1r_2\}$ violates (T1) via r.

Therefore rr_1 and rr_2 have no oblique neighbors in X. As before, the number of edges with any $v \in X$ as an oblique neighbor equals the degree of v, so the degrees of X add up to at most $|E(T)| - 2 = (n-1) - 2 = n - 3$, contradicting the assumption of the theorem. Therefore the theorem is proven. □

2. Concluding remarks

Having shown than any connected claw-free graph must contain either a spanning tree with at most k branch vertices or an independent $(2k+3)$-set with at most $n-3$ outgoing edges, perhaps an algorithm can be found to search a given graph for one feature or the other. One can undoubtedly be constructed directly out of this proof, but perhaps its run time can be shortened.

Declaration of competing interest

No author associated with this paper has disclosed any potential or pertinent conflicts which may be perceived to have impending conflict with this work. For full disclosure statements refer to https://doi.org/10.1016/j.disc.2019.06.037.

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References

[1] H. Broersma, H. Tuinstra, Independence trees and Hamilton cycles, J. Graph Theory 29 (1998) 227–237.
[2] E. Flandrin, T. Kaiser, R. Kužel, H. Li, Z. Ryjáček, Neighborhood unions and extremal spanning trees, Discrete Math. 308 (2008) 2343–2350.
[3] J. Fujisawa, H. Matsumura, T. Yamashita, Degree bounded spanning trees, Graphs Combin. 26 (2010) 695–720.
[4] L. Gargano, M. Hammar, There are spanning spiders in dense graphs (and we know how to find them), Lect. Notes Comput. Sci. 2719 (2003) 802–816.
[5] L. Gargano, M. Hammar, P. Hell, L. Stacho, U. Vaccaro, Spanning spiders and light-splitting switches, Discrete Math. 285 (2004) 83–95.
[6] L. Gargano, P. Hell, L. Stacho, U. Vaccaro, Spanning trees with bounded number of branch vertices, Lect. Notes Comput. Sci. 2380 (2002) 355–365.
[7] R. Gould, W. Shull, On a conjecture on spanning trees with few branch vertices, J. Combin. Math. Combin. Comput. (2017) in press.
[8] M. Kano, A. Kyaw, H. Matsuda, K. Ozeki, A. Saito, T. Yamashita, Spanning trees with small number of leaves in a claw-free graph, Ars. Combin. 103 (2012) 137–154.
[9] H. Matsuda, H. Matsumura, On a k-tree containing specified leaves in a graph, Graphs Combin. 22 (2006) 371–381.
[10] H. Matsuda, K. Ozeki, T. Yamashita, Spanning trees with a bounded number of branch vertices in a claw-free graph, Graphs Combin. 30 (2014) 429–437.
[11] M.M. Matthews, D.P. Sumner, Longest paths and cycles in $K_{1,3}$-free graphs, J. Graph Theory 9 (1985) 269–277.
[12] V. Neumann-Lara, E. Rivera-Campo, Spanning trees with bounded degrees, Combinatorica 11 (1991) 55–61.
[13] K. Ozeki, T. Yamashita, Spanning trees: A survey, Graphs Combin. 27 (2011) 1–26.
[14] M. Tsugaki, T. Yamashita, Spanning trees with few leaves, Graphs Combin. 23 (2007) 585–598.
[15] S. Win, Existenz von Gerüsten mit vorgeschriebenem Maximalgrad in Graphen, Abh. Math. Semin. Univ. Hambg. 43 (1975) 263–267 (in German).
[16] S. Win, On a conjecture of Las Vergnas concerning certain spanning trees in graphs, Result. Math. 2 (1979) 215–224.