

CHAPTER 5

Recurrence Relations

5.1. Recurrence Relations

Here we look at recursive definitions under a different point of view. Rather than definitions they will be considered as equations that we must solve. The point is that a recursive definition is actually a definition when there is one and only one object satisfying it, i.e., when the equations involved in that definition have a unique solution. Also, the solution to those equations may provide a *closed-form* (explicit) formula for the object defined.

The recursive step in a recursive definition is also called a *recurrence relation*. We will focus on *kth-order linear recurrence relations*, which are of the form

$$C_0 x_n + C_1 x_{n-1} + C_2 x_{n-2} + \cdots + C_k x_{n-k} = b_n,$$

where $C_0 \neq 0$. If $b_n = 0$ the recurrence relation is called *homogeneous*. Otherwise it is called *non-homogeneous*.

The basis of the recursive definition is also called *initial conditions* of the recurrence. So, for instance, in the recursive definition of the Fibonacci sequence, the recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

or

$$F_n - F_{n-1} - F_{n-2} = 0,$$

and the initial conditions are

$$F_0 = 0, F_1 = 1.$$

One way to solve some recurrence relations is by *iteration*, i.e., by using the recurrence repeatedly until obtaining a explicit close-form formula. For instance consider the following recurrence relation:

$$x_n = r x_{n-1} \quad (n > 0); \quad x_0 = A.$$

By using the recurrence repeatedly we get:

$$x_n = r x_{n-1} = r^2 x_{n-2} = r^3 x_{n-3} = \cdots = r^n x_0 = A r^n,$$

hence the solution is $x_n = A r^n$.

In the following we assume that the coefficients C_0, C_1, \dots, C_k are constant.

5.1.1. First Order Recurrence Relations. The homogeneous case can be written in the following way:

$$x_n = r x_{n-1} \quad (n > 0); \quad x_0 = A.$$

Its general solution is

$$x_n = A r^n,$$

which is a *geometric sequence* with *ratio* r .

The non-homogeneous case can be written in the following way:

$$x_n = r x_{n-1} + c_n \quad (n > 0); \quad x_0 = A.$$

Using the summation notation, its solution can be expressed like this:

$$x_n = A r^n + \sum_{k=1}^n c_k r^{n-k}.$$

We examine two particular cases. The first one is

$$x_n = r x_{n-1} + c \quad (n > 0); \quad x_0 = A.$$

where c is a constant. The solution is

$$x_n = A r^n + c \sum_{k=1}^n r^{n-k} = A r^n + c \frac{r^n - 1}{r - 1} \quad \text{if } r \neq 1,$$

and

$$x_n = A + c n \quad \text{if } r = 1.$$

Example: Assume that a country with currently 100 million people has a population growth rate (birth rate minus death rate) of 1% per year, and it also receives 100 thousand immigrants per year (which are quickly assimilated and reproduce at the same rate as the native population). Find its population in 10 years from now. (Assume that all the immigrants arrive in a single batch at the end of the year.)

Answer: If we call x_n = population in year n from now, we have:

$$x_n = 1.01 x_{n-1} + 100,000 \quad (n > 0); \quad x_0 = 100,000,000.$$

This is the equation above with $r = 1.01$, $c = 100,000$ and $A = 100,000,000$, hence:

$$\begin{aligned} x_n &= 100,000,000 \cdot 1.01^n + 100,000 \frac{1.01^n - 1}{1.01 - 1} \\ &= 100,000,000 \cdot 1.01^n + 1000(1.01^n - 1). \end{aligned}$$

So:

$$x_{10} = 110,462,317.$$

The second particular case is for $r = 1$ and $c_n = c + dn$, where c and d are constant (so c_n is an arithmetic sequence):

$$x_n = x_{n-1} + c + dn \quad (n > 0); \quad x_0 = A.$$

The solution is now

$$x_n = A + \sum_{k=1}^n (c + dk) = A + cn + \frac{dn(n+1)}{2}.$$

5.1.2. Second Order Recurrence Relations. Now we look at the recurrence relation

$$C_0 x_n + C_1 x_{n-1} + C_2 x_{n-2} = 0.$$

First we will look for solutions of the form $x_n = cr^n$. By plugging in the equation we get:

$$C_0 cr^n + C_1 cr^{n-1} + C_2 cr^{n-2} = 0,$$

hence r must be a solution of the following equation, called the *characteristic equation* of the recurrence:

$$C_0 r^2 + C_1 r + C_2 = 0.$$

Let r_1, r_2 be the two (in general complex) roots of the above equation. They are called *characteristic roots*. We distinguish three cases:

1. *Distinct Real Roots.* In this case the general solution of the recurrence relation is

$$x_n = c_1 r_1^n + c_2 r_2^n,$$

where c_1, c_2 are arbitrary constants.

2. *Double Real Root.* If $r_1 = r_2 = r$, the general solution of the recurrence relation is

$$x_n = c_1 r^n + c_2 n r^n,$$

where c_1, c_2 are arbitrary constants.

3. *Complex Roots.* In this case the solution could be expressed in the same way as in the case of distinct real roots, but in order to avoid the use of complex numbers we write $r_1 = r e^{\alpha i}$, $r_2 = r e^{-\alpha i}$, $k_1 = c_1 + c_2$, $k_2 = (c_1 - c_2) i$, which yields:¹

$$x_n = k_1 r^n \cos n\alpha + k_2 r^n \sin n\alpha.$$

Example: Find a closed-form formula for the Fibonacci sequence defined by:

$$F_{n+1} = F_n + F_{n-1} \quad (n > 0); \quad F_0 = 0, \quad F_1 = 1.$$

Answer: The recurrence relation can be written

$$F_n - F_{n-1} - F_{n-2} = 0.$$

The characteristic equation is

$$r^2 - r - 1 = 0.$$

Its roots are:²

$$r_1 = \phi = \frac{1 + \sqrt{5}}{2}; \quad r_2 = -\phi^{-1} = \frac{1 - \sqrt{5}}{2}.$$

They are distinct real roots, so the general solution for the recurrence is:

$$F_n = c_1 \phi^n + c_2 (-\phi^{-1})^n.$$

Using the initial conditions we get the value of the constants:

$$\begin{cases} (n=0) & c_1 + c_2 & = & 0 \\ (n=1) & c_1 \phi + c_2 (-\phi^{-1}) & = & 1 \end{cases} \Rightarrow \begin{cases} c_1 & = & 1/\sqrt{5} \\ c_2 & = & -1/\sqrt{5} \end{cases}$$

Hence:

$$F_n = \frac{1}{\sqrt{5}} \{ \phi^n - (-\phi)^{-n} \}.$$

¹Reminder: $e^{\alpha i} = \cos \alpha + i \sin \alpha$.

² $\phi = \frac{1+\sqrt{5}}{2}$ is the *Golden Ratio*.

Solving Recurrence Relations

Part 1. Homogeneous linear 2nd degree relations with constant coefficients.

Consider the recurrence relation

$$(*) \quad T(n) + aT(n-1) + bT(n-2) = 0$$

This is called a *homogeneous linear 2nd degree recurrence relation with constant coefficients*:

- *2nd degree* because it gives $T(n)$ in terms of $T(n-1)$ and $T(n-2)$,
- *linear and constant coefficients* because of the form of the left side, and
- *homogeneous* because of the zero on the right hand side.

The idea for solving this relation is to “guess” a solution of the form $T(n) = x^n$ for some number x , and then to simply substitute this expression into the equation to determine the value(s) of x that work. Since $T(n-1) = x^{n-1}$ and $T(n-2) = x^{n-2}$ we get the equation

$$x^n + ax^{n-1} + bx^{n-2} = 0$$

Since x is clearly not zero, we can divide by x^{n-2} to get

$$x^2 + ax + b = 0$$

which is called the *characteristic equation* for the recurrence relation (*).

Case 1: If this equation factors as $(x-r_1)(x-r_2) = 0$ with $r_1 \neq r_2$ (so that the characteristic equation has two distinct roots), then the general solution to (*) is

$$T(n) = c_1(r_1)^n + c_2(r_2)^n$$

where c_1 and c_2 are constants. This is called a *general solution* because *every* function $T(n)$ that is a solution to the relation (*) has this form. If we also have initial conditions $T(0) = t_0$ and $T(1) = t_1$ then we can determine the values of c_1 and c_2 and thus get the exact formula for $T(n)$. We'll illustrate the process with a typical example.

Example 1. Solve the recurrence relation

$$T(n) - 4T(n-1) + 3T(n-2) = 0, \quad T(0) = 0, \quad T(1) = 2$$

Note: This could also be written as

$$T(n) = \begin{cases} 0 & n = 0 \\ 2 & n = 1 \\ 4T(n-1) - 3T(n-2) & n > 1 \end{cases}$$

Solution: The characteristic equation is $x^2 - 4x + 3 = 0$, or $(x-3)(x-1) = 0$, so the general solution is $T(n) = c_1 3^n + c_2 1^n = c_1 3^n + c_2$. To find c_1 and c_2 we plug in the initial conditions to get two equations in those two variables:

$$\begin{aligned} 0 &= T(0) = c_1 3^0 + c_2 = c_1 + c_2 \\ 2 &= T(1) = c_1 3^1 + c_2 = 3c_1 + c_2 \end{aligned}$$

It's easy to solve these equations for the solution $c_1 = 1$, $c_2 = -1$, so the final answer is

$$\boxed{T(n) = 3^n - 1}$$

Check: We can check our answer quickly and easily. The recurrence formula gives us

$$\begin{aligned} T(2) &= 4T(1) - 3T(0) = 4(2) - 0 = 8 \\ T(3) &= 4T(2) - 3T(1) = 4(8) - 3(2) = 26 \\ T(4) &= 4T(3) - 3T(2) = 4(26) - 3(8) = 80 \end{aligned}$$

It appears that the sequence is indeed giving us numbers that are one less than the powers of 3 (1, 3, 9, 27, 81, ...), so the formula $T(n) = 3^n - 1$ seems to be correct.

Case 2: If the characteristic equation factors as $(x - r)^2 = 0$ (with a single root), then we use as our general solution the formula

$$T(n) = c_1 r^n + c_2 n r^n$$

As before, initial conditions can be used to solve for c_1 and c_2 . Here is a typical example.

Example 2. Solve the recurrence relation

$$T(n) = \begin{cases} 3 & n = 0 \\ 17 & n = 1 \\ 10T(n-1) - 25T(n-2) & n > 1 \end{cases}$$

Solution: The characteristic equation is $x^2 - 10x + 25 = 0$, or $(x - 5)^2 = 0$, so the general solution is $T(n) = c_1 5^n + c_2 n 5^n$. As before, we find c_1 and c_2 by plugging in the initial conditions:

$$\begin{aligned} 3 &= T(0) = c_1 5^0 + c_2(0) = c_1 \\ 17 &= T(1) = c_1 5^1 + c_2(1)5^1 = 5c_1 + 5c_2 \end{aligned}$$

The solution here is $c_1 = 3$ and $c_2 = 2/5$, so the exact solution is

$$\boxed{T(n) = (3)5^n + (2/5)n5^n = (15 + 2n)5^{n-1}}$$

Check: The recurrence formula gives us values

$$\begin{aligned} T(2) &= 10T(1) - 25T(0) = 10(17) - 25(3) = 95 \\ T(3) &= 10T(2) - 25T(1) = 525 \end{aligned}$$

These are indeed the values given by our formula as well: $T(2) = (15 + 4)5^1 = 95$, and $T(3) = (15 + 6)5^2 = 525$.

Case 3: It can happen in general recurrence relations that the characteristic equation $x^2 + ax + b = 0$ has no real roots, but instead has two complex number roots. There is a method of solution for such recurrences, but we will not concern ourselves with this case since it does not typically arise in recurrences that come from studying recursive algorithms.

Part 2. Non-homogeneous linear 2nd degree relations with constant coefficients.

Now consider what happens when the right side of equation (*) (from page 1) is *not* zero. We get an equation of the form

$$(**) \quad T(n) + aT(n-1) + bT(n-2) = f(n)$$

We'll learn how to solve (**) in the special case that $f(n) = (\text{polynomial in } n)r^n$.

Actually, this requires only a slight modification of the method for the homogeneous case. We simply multiply the characteristic equation by $(x - r)^{k+1}$ where k is the degree of the polynomial part of $f(n)$. The solution method then proceeds as before. This is best illustrated by examples.

Example 3. Solve the recurrence relation

$$T(n) = \begin{cases} 1 & n = 0 \\ 3T(n-1) + 2^n & n > 0 \end{cases}$$

Note: This is actually a 1st degree relation (since $T(n-2)$ does not appear), but the same method applies.

Solution: If the nonhomogeneous term 2^n were not present, the characteristic equation would be simply $x - 3 = 0$. In the presence of the nonhomogeneous term, however, we must multiply this by $(x - 2)^{0+1}$ (In this case $f(n) = 2^n$ is a 0-degree polynomial [the constant 1] times a 2^n term.) So, the characteristic equation is actually $(x - 3)(x - 2) = 0$, so the general solution is $T(n) = c_1 3^n + c_2 2^n$. Note that since the relation is only 1st degree, we only have one initial condition. Yet we'll need two equations to find the constants c_1 and c_2 . We can get a second value to use by simply applying the recurrence formula for $n = 1$: $T(1) = 3T(0) + 2^1 = 3(1) + 2 = 5$. We now proceed as usual:

$$\begin{aligned} 1 &= T(0) = c_1 3^0 + c_2 2^0 = c_1 + c_2 \\ 5 &= T(1) = c_1 3^1 + c_2 2^1 = 3c_1 + 2c_2 \end{aligned}$$

These equations have solution $c_1 = 3$ and $c_2 = -2$, so the exact solution is

$$T(n) = (3)3^n - (2)2^n = 3^{n+1} - 2^{n+1}$$

Check: The recurrence formula gives us values

$$\begin{aligned} T(2) &= 3T(1) + 2^2 = 3(5) + 4 = 19 \\ T(3) &= 3T(2) + 2^3 = 3(19) + 8 = 65 \end{aligned}$$

These are borne out by our solution: $T(2) = 3^3 - 2^3 = 27 - 8 = 19$, and $T(3) = 3^4 - 2^4 = 81 - 16 = 65$.

Example 4. Solve the recurrence relation

$$T(n) = \begin{cases} n + 1 & n = 0, 1 \\ 5T(n-1) - 6T(n-2) + 3 \cdot 2^n & n > 1 \end{cases}$$

Solution: Here again, the nonhomogeneous term involves a zero-degree polynomial, so the modified characteristic equation will be $(x^2 - 5x + 6)(x - 2)^{0+1} = 0$, or $(x - 3)(x - 2)^2 = 0$. The general solution must now involve *three* terms: a 3^n term, a 2^n term, and (because the $(x - 2)$ factor appears twice in the characteristic equation) an $n2^n$ term. Thus: $T(n) = c_1 3^n + c_2 2^n + c_3 n 2^n$.

We'll need three equations to solve for the three constants, yet we have only the two initial conditions $T(0) = 1$ and $T(1) = 2$ to use. As in the last example, we can generate one more value by using the recurrence formula: $T(2) = 5T(1) - 6T(0) + 3 \cdot 2^2 = 5(2) - 6(1) + 12 = 16$. This gives us the three equations

$$\begin{aligned} 1 &= T(0) = c_1 3^0 + c_2 2^0 + c_3(0) = c_1 + c_2 \\ 2 &= T(1) = c_1 3^1 + c_2 2^1 + c_3(1)2^1 = 3c_1 + 2c_2 + 2c_3 \\ 16 &= T(2) = c_1 3^2 + c_2 2^2 + c_3(2)2^2 = 9c_1 + 4c_2 + 8c_3 \end{aligned}$$

A little effort gives the solution $c_1 = 12$, $c_2 = -11$, and $c_3 = -6$. The exact solution is then

$$T(n) = 12 \cdot 3^n - 11 \cdot 2^n - 6n2^n$$

Note that we could have safely concluded that $T(n)$ is in $\Theta(3^n)$ simply from its general solution (without even solving for c_1 , c_2 , and c_3).

Check: In addition to $T(2) = 16$, the recurrence formula gives us

$$T(3) = 5T(2) - 6T(1) + 3 \cdot 2^3 = 5(16) - 6(2) + 24 = 92$$

Checking these against our solution, we find $T(2) = 12 \cdot 3^2 - 11 \cdot 2^2 - 6(2)2^2 = 108 - 44 - 48 = 16$ and $T(3) = 12 \cdot 3^3 - 11 \cdot 2^3 - 6(3)2^3 = 324 - 88 - 144 = 92$.

Example 5. Predict the big-Theta behavior of a function $T(n)$ satisfying the recurrence relation $T(n) = 7T(n-1) - 10T(n-2) + (2n+5)3^n$.

Solution: The modified characteristic equation is $(x^2 - 7x + 10)(x - 3)^{1+1} = 0$, or factoring, $(x - 2)(x - 5)(x - 3)^2 = 0$. The general solution is thus $T(n) = c_1 2^n + c_2 5^n + c_3 3^n + c_4 n 3^n$. This will clearly be in $\Theta(5^n)$.

Part 3. Change of variable technique for decrease-by-constant-factor recurrence relations.

Many recursive algorithms (such as “binary search” or “merge sort”) work by dividing the input in half and calling itself on the now-half-sized input. Analyzing the efficiency of such algorithms leads to recurrence relations that give $T(n)$ in terms of $T(\lfloor \frac{n}{2} \rfloor)$ instead of in terms of $T(n-1)$. These “*decrease by a constant-factor*” recurrence relations can be converted into standard linear recurrence relations by applying a simple change of variable.

Let's say we have a recurrence relation of the form

$$T(n) = aT(\lfloor \frac{n}{b} \rfloor) + f(n)$$

where b is some positive integer (usually $b = 2$). We will define a new function $S(k)$ by the rule $S(k) = T(b^k)$. Our recurrence relation then gives

$$\begin{aligned} S(k) = T(b^k) &= aT\left(\left\lfloor \frac{b^k}{b} \right\rfloor\right) + f(b^k) \\ &= aT(b^{k-1}) + f(b^k) \\ &= aS(k-1) + f(b^k) \end{aligned}$$

which is a first-order linear recurrence relation for S . This can then be solved by the methods we've already covered.

Example 6. Solve the recurrence relation

$$T(n) = \begin{cases} 2 & n = 1 \\ 3T(\lfloor \frac{n}{2} \rfloor) + n \log_2(n) & n > 1 \end{cases}$$

Solution: Using the substitution $S(k) = T(2^k)$ this recurrence formula becomes

$$\begin{aligned} S(k) = T(2^k) &= 3T\left(\left\lfloor \frac{2^k}{2} \right\rfloor\right) + (2^k) \log_2(2^k) \\ &= 3T(2^{k-1}) + (2^k)k \\ &= 3S(k-1) + k2^k \end{aligned}$$

The characteristic equation of this relation is $(x-3)(x-2)^{1+1} = 0$, so its general solution is $S(k) = c_1 3^k + c_2 2^k + c_3 k 2^k$.

What about initial conditions for S ? We can use the recurrence formula for T to get $T(1) = 2$, $T(2) = 3T(1) + 2 = 8$, and $T(4) = 3T(2) + 8 = 32$. So, $S(0) = T(2^0) = T(1) = 2$, $S(1) = T(2^1) = T(2) = 8$, and $S(2) = T(2^2) = T(4) = 32$. We can now use these values to get equations for c_1 , c_2 , and c_3 :

$$\begin{aligned} 2 &= S(0) = c_1 3^0 + c_2 2^0 + 0 \\ 8 &= S(1) = c_1 3^1 + c_2 2^1 + c_3(1)2^1 = 3c_1 + 2c_2 + 2c_3 \\ 32 &= S(2) = c_1 3^2 + c_2 2^2 + c_3(2)2^2 = 9c_1 + 4c_2 + 8c_3 \end{aligned}$$

Solving this three-by-three system of equations (by whatever method you prefer) leads to the solution $c_1 = 8$, $c_2 = -6$, and $c_3 = -2$. This gives the solution to the recurrence relation for S as

$$S(k) = 8 \cdot 3^k - 6 \cdot 2^k - 2k2^k$$

But we want a formula for T , not S . Remembering that $S(k) = T(2^k)$ and assuming n is a power of 2, we can say

$$\begin{aligned} T(n) = S(\log_2 n) &= 8 \cdot 3^{\log_2 n} - 6 \cdot 2^{\log_2 n} - 2(\log_2 n)2^{\log_2 n} \\ &= 8 \cdot 3^{\log_2 n} - 6n - 2n \log_2 n \end{aligned}$$

valid for all values of n that are powers of 2. Here we have used the familiar logarithm law that says $2^{\log_2 n}$ is equal to n . What can we do to simplify the term $3^{\log_2 n}$? This just requires a bit of logarithmic trickery:

$$3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n} = (2^{\log_2 n})^{\log_2 3} = n^{\log_2 3}$$

So, we can rewrite our answer as

$$T(n) = 8n^{\log_2 3} - 6n - 2n \log_2 n \text{ for } n = 1, 2, 4, 8, \dots$$

Two notes about this solution are in order:

- First, since $\log_2 3$ is clearly between 1 and 2 (since 3 is between 2^1 and 2^2) we can say that our solution is in $O(n^2)$ (at least for the values $n = 1, 2, 4, 8, \dots$).
- Second, we know that our formula for $T(n)$ is valid for powers of 2 – that is, it correctly computes $T(1), T(2), T(4), T(8)$, and so on. What about the other values of n ? Thanks to something called the *Smoothness Rule* we can at least say that our formula for $T(n)$ has the correct big-Theta behavior. See your text for the exact statement, but the basic idea is this:

Smoothness Rule: Suppose $T(n)$ is in $\Theta(f(n))$ for values of n that are powers of a constant $b \geq 2$. Then if $f(n)$ is a “nice” function (here “nice” includes the polynomials n, n^2, n^3 , etc. but not exponential or factorial functions) we can say that $T(n)$ really *is* in $\Theta(f(n))$.

Homework Problems

Solve each of the following recurrence relations.

$$(A) \quad T(n) = \begin{cases} 1 & n = 0 \\ 4 & n = 1 \\ 8T(n-1) - 15T(n-2) & n > 1 \end{cases}$$

$$(B) \quad T(n) = \begin{cases} 5 & n = 0 \\ 9 & n = 1 \\ 6T(n-1) - 9T(n-2) & n > 1 \end{cases}$$

$$(C) \quad T(n) = \begin{cases} 1 & n = 0 \\ 2T(n-1) + 3^n & n > 0 \end{cases}$$

$$(D) \quad T(n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 4T(n-1) - 3T(n-2) + 1 & n > 1 \end{cases}$$

Hint: The nonhomogeneous term is 1^n .

$$(E) \quad T(n) = \begin{cases} 2 & n = 1 \\ T(\lfloor \frac{n}{3} \rfloor) + 7n & n > 1 \end{cases}$$

$$(F) \quad T(n) = \begin{cases} 2 & n = 1 \\ 5T(\lfloor \frac{n}{2} \rfloor) + 3n & n > 1 \end{cases}$$