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Intersections of Longest Cycles in k-Connected Graphs

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Let G be a connected graph, where $k \geqslant 2$. S. Smith conjectured that every two longest cycles of G have at least k vertices in common. In this note, we show that every two longest cycles meet in at least $ck^{3/5}$ vertices, where $c \approx 0.2615$.

In this note, we provide a lower bound on the number of vertices in the intersection of any two longest cycles in a k-connected graph ($k \geqslant 2$). This work is inspired by the following conjecture due to Scott Smith; see [2, 6].

Conjecture 1. In a k-connected graph, two longest cycles meet in at least k vertices.

According to Grötchel [6], the conjecture has been verified up to $k = 10$. Theorem 1.2(a) of [6] showed the conjecture is true up to $k = 6$. Further, Grötchel and Nemhauser [7] studied the properties of two longest cycles meeting in exactly 2 vertices in 2-connected graphs and Grötchel [6] studied the properties of two longest cycles meeting in k vertices for $k = 3, 4, 5$. For

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general k, S. Burr and T. Zamfirescu (private communication) mentioned the following result.

THEOREM 1 (Burr and Zamfirescu). *If G is a k-connected graph with $k \geqslant 2$, then every pair of different longest cycles meet in at least $\sqrt{k}-1$ vertices.*

The purpose of this paper is to improve the above result for large k as follows.

THEOREM 2. *If G is a k-connected graph, then any two different longest cycles meet in at least $ck^{3/5}$ vertices, where $c = 1/(\sqrt[3]{256}+3)^{3/5} \approx 0.2615$.*

The proof of Theorem 2 will be deferred until we present several lemmas. We will generally follow the notation of Bondy and Murty [3]. All graphs considered in this paper are simple graphs. For convenience we will use the notation $G \subseteq K_{m,n}$ to signify a bipartite graph in which one part has m vertices and the other part has n vertices. Let X be a path or a cycle on a graph. We will usually give an orientation to X. In this case, for any pair of vertices u and $v \in V(X)$, we will use the notation $X[u, v]$ to signify the segment of X from u to v along the orientation (if such a segment exists). The same segment with reversed orientation will be denoted by $\bar{X}[v, u]$.

LEMMA 1 (Hylton–Cavallius [8]). *Let $G \subseteq K_{n,n}$ be a bipartite graph. Then G contains $K_{s,t}$ as a subgraph if*

$$e(G) \geqslant (s-1)^{1/t}\,(n-t+1)\,n^{1-1/t} + (t-1)\,n.$$

For the purpose of this paper, we will only need the following special case of Lemma 1:

COROLLARY 1. *Let $G \subseteq K_{n,n}$. Then G contains $K_{3,257}$ if*

$$e(G) \geqslant \sqrt[3]{256}\,(n-2)\,n^{2/3} + 2n.$$

The following classical Ramsey-type result is due to Erdős and Szekeres.

LEMMA 2 (Erdős and Szekeres [5]). *Every sequence of n^2+1 real numbers contains a monotone subsequence of length $n+1$.*

The following result slightly generalizes Lemma 2.

LEMMA 3. *Let Σ be a set of n permutations of a sequence of S of $2^{2^n}+1$ elements. Then there is a subsequence (a, b, c) of S on which each* *permutation* $\sigma \in \Sigma$ *is monotonic* (*that is*, *either* $\sigma(a) < \sigma(b) < \sigma(c)$ *or* $\sigma(a) > \sigma(b) > \sigma(c)$).

Proof. We proceed by induction on n. The lemma is true for $n = 1$ by Lemma 2. Suppose it is true for $n - 1$, where $n \geqslant 2$. Let $\sigma \in \Sigma$. By Lemma 2, there is a subsequence S' of S of $2^{2^{n-1}} + 1$ elements on which σ is monotonic. By induction, there is a subsequence (a, b, c) of S' on which each permutation in $\Sigma - \{\sigma\}$ is monotonic. ▌

The following classical result due to Dirac will be also be used in the proof.

LEMMA 4 (Dirac [4]). *Let G be a 2-connected graph of minimum degree δ on n vertices, where $n \geqslant 3$. Then G contains either a cycle of length at least 2δ or a Hamiltonian cycle.*

The Proof of Theorem 2. We will prove Theorem 2 by contradiction. Let G be a k-connected graph which contains two longest cycles C and D such that $|V(C) \cap V(D)| < ck^{3/5}$, where $c = 1/(\sqrt[3]{256} + 3)^{3/5}$. Let $V(C) \cap V(D) = A = \{a_1, a_2, ..., a_m\}$ where $m < ck^{3/5}$. Let $X_1, X_2, ..., X_m$ be the segments of $C - A$ (some of them may be empty) and let $Y_1, Y_2, ..., Y_m$ be the segments of $D - A$ (some of them may be empty).

Since $|A| < ck^{3/5}$, G is not Hamiltonian. By Lemma 4, we have $|V(G)| > 2k$ and $|V(C)| = |V(D)| \geqslant 2k$. Thus,

$$|X_1 \cup X_2 \cup \cdots \cup X_m| = |Y_1 \cup Y_2 \cup \cdots \cup X_m| \geqslant k.$$

Since G is k-connected, $G - A$ is $(k - m)$-connected and thus contains $k - m$ pairwise vertex-disjoint paths $P_1, P_2, ..., P_{k-m}$ from $X_1 \cup X_2 \cup \cdots \cup X_m$ to $Y_1 \cup Y_2 \cup \cdots \cup Y_m$. Let $\mathscr{P} = \{P_1, P_2, ..., P_{k-m}\}$.

CLAIM 1. *There do not exist two paths P_i and P_j both of whose initial vertices are in the same segment of $C - A$ and both of whose end vertices are in the same segment of $D - A$.*

Proof. To the contrary, without loss of generality, we may assume that $P_1 = P_1[u_1, v_1]$ and $P_2 = P_2[u_2, v_2]$ are two paths with u_1 and u_2 in X_1 and v_1 and v_2 in Y_1. Furthermore, we assume that $C[u_1, u_2] \subseteq X_1$ and $D[v_1, v_2] \subseteq Y_1$. Note that G contains two cycles

$$C^* = C[u_2, u_1]\, P_1[u_1, v_1]\, D[v_1, v_2]\, \overleftarrow{P}_2[v_2, u_2]$$

and

$$D^* = C[v_2, v_1]\, \overleftarrow{P}_1[v_1, u_1]\, C[u_1, u_2]\, P_2[u_2, v_2]$$

Then,

$$|V(C^*)| + |V(D^*)| = |V(C)| + |V(D)| + 2(|V(P_1)| - 1) + 2(|V(P_2)| - 1)$$
$$> |V(C)| + |V(D)|,$$

a contradiction. ▮

We now construct an auxiliary graph H with vertex set $\{X_1, X_2, ..., X_m, Y_1, Y_2, ..., Y_m\}$ and for each path $P_\ell \in \mathscr{P}$ from X_i to Y_j we insert an edge e_ℓ joining X_i and Y_j in H. By Claim 1, H contains no multiple edges. Thus, H is a simple bipartite graph where each partite set has m vertices.

Note that H has at most m^2 edges. Thus, $k - m \leqslant m^2$, which implies that $m \geqslant \sqrt{k} - 1$. Theorem 1 is proved at this stage. The strategy for the remainder of the proof of Theorem 2 is a refinement if this idea.

Since $H \subseteq K_{m,m}$ and H contains at least $k - m \geqslant (\sqrt[3]{256} + 2)\, m^{5/3}$ edges, by Corollary 1, H contains $K_{3,257}$ as a subgraph. Relabelling the segments if necessary, we assume the vertices $X_1, X_2, X_3, Y_1, Y_2, ..., Y_{257}$ induce a $K_{3,257}$ and let $P_{i,j}[u_{i,j}, v_{i,y}]$ denote the path from X_i to Y_j for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 257$ in $\mathscr{P}$.

We orient both cycles C and D. Beginning with vertex a_1, the orientation of C gives a linear order of $V(C)$, that is, for any pair of vertices x_1 and $x_2 \in V(C)$, we define $x_1 \prec x_2$ if $x_1 \in C[a_1, x_2]$. Furthermore, if S and T are two disjoint segments of $C - \{a_1\}$, we define $S \prec T$ if S lies between a_1 and T along the orientation of C. We define analogous notation with respect to the cycle D.

With the above definition, we lose no generality by assuming that $X_1 \prec X_2 \prec X_3$ and $Y_1 \prec Y_2 \prec \cdots \prec Y_{257}$. Along the orientation of C, the 257 vertices $u_{i,1}, u_{i,2}, ..., u_{i,257}$ give a permutation σ_i of $1, 2, ..., 257 = 2^{2^3} + 1$ for each $i = 1, 2, 3$. By Lemma 4, there are three integers a, b, c from $\{1, 2, ..., 257\}$ on which each permutation σ_i is monotonic, that is, either $\sigma_i(a) > \sigma_i(b) > \sigma_i(c)$ or $\sigma_i(a) < \sigma_i(b) < \sigma_i(c)$ for each $i = 1, 2, 3$. Without loss of generality, we may assume that $a = 1, b = 2, c = 3$. By the Pigeonhole Principle, we may assume that $\sigma_i(1) > \sigma_i(2) > \sigma_i(3)$ for $i = 1, 2$. Thus, we have

$$u_{1,1} \prec u_{1,2} \prec u_{1,3} \tag{1}$$

and

$$u_{2,1} \prec u_{2,2} \prec u_{2,3} \tag{2}$$

and either

$$u_{3,1} \prec u_{3,2} \prec u_{3,3} \tag{3}$$

or

$$u_{3,3} \prec u_{3,2} \prec u_{3,1}. \tag{4}$$

Recall that the end vertex of the path $P_{i,j}$ is $v_{i,j}$ for each path $P_{i,j}$. We may assume that $v_{1,1} \prec v_{2,1}$, otherwise we reverse the roles of X_1 and X_2.

CLAIM 2. *There are i and j with* $1 \leqslant i \leqslant j \leqslant 3$ *such that either*

$$v_{1,i} \prec v_{2,i} \qquad \textit{and} \qquad v_{1,j} \prec v_{2,j} \tag{5}$$

or

$$v_{2,i} \prec v_{1,i} \qquad \textit{and} \qquad v_{2,j} \prec v_{1,j}. \tag{6}$$

Note that the existence of a large $K_{2,m}$ in the auxiliary graph H would not be enough to provide statements of (5) and (6). This is the reason we need the fact that H contains a $K_{3,257}$ in our proof.

Proof. Since $v_{1,1} \prec v_{2,1}$, Claim 2 follows if either $v_{1,2} \prec v_{2,2}$ or $v_{1,3} \prec v_{2,3}$. We assume that $v_{2,2} \prec v_{2,1}$ and $v_{2,3} \prec v_{1,3}$. Clearly, Claim 2 follows in this case also. ∎

By reversing the orientation of D if necessary, without loss of generality, we assume that $v_{1,1} \prec v_{2,1}$ and $v_{1,2} \prec v_{2,2}$. Then, G contains two cycles C^* and D^* listed below:

$$\begin{aligned} C^* = {} & P[u_{1,1}, v_{1,1}]\, D[v_{1,1}, v_{2,1}] \leftarrow \overleftarrow{P}_{2,1}[v_{2,1}, u_{2,1}] \leftarrow \overleftarrow{C}[u_{2,1}, u_{1,2}] \\ & \times P[u_{1,2}, v_{1,2}]\, D[v_{1,2}, v_{2,2}] \leftarrow \overleftarrow{P}_{2,2}[v_{2,2}, u_{2,2}]\, C[v_{2,2}, u_{1,1}] \end{aligned}$$

and

$$\begin{aligned} D^* = {} & \leftarrow \overleftarrow{P}[v_{1,1}, u_{1,1}]\, C[u_{1,1}, u_{2,1}]\, P_{2,1}[u_{2,1}, v_{2,1}] \leftarrow \overleftarrow{D}[v_{2,1}, v_{1,2}] \\ & \leftarrow \overleftarrow{P}[v_{1,2}, u_{1,2}]\, C[u_{1,2}, u_{2,2}]\, P_{2,2}[u_{2,2}, v_{2,2}]\, D[v_{2,2}, v_{1,1}]. \end{aligned}$$

However,

$$\begin{aligned} |V(C^*)| + |V(D^*)| &= |V(C)| + |V(D)| + 2 \sum_{1 \leqslant i,\, j \leqslant 2} (|V(P_{i,j})| - 1) \\ &> |V(C)| + |V(D)| \end{aligned}$$

which contradicts the assumption that both C and D are longest cycles of G. ∎

Applications

We shall now state an application of Theorem 2 to vertex-transitive graphs. Babai [1] proved that every connected vertex-transitive graph with at least four vertices contains a cycle of length greater than $(3n)^{1/2}$. Following the (nice and simple) proof of Babai [1], we can conclude:

PROPOSITION 1. *If G is a vertex-transitive graph of order n such that every two different longest cycles meet in at least $f(k)$ vertices, then G contains a cycle of length greater than $(f(k)n)^{1/2}$.*

Combining the above result and Theorem 2, we obtain the following:

THEOREM 3. *If G is a k-connected vertex-transitive graph of order n, then G contains a cycle of length greater than $ck^{3/10}n^{1/2}$, where $c \approx 0.2615$.*

Mader [9] and Watkins [10] proved that the connectivity of a connected vertex-transitive d-regular graph is at least $\frac{2}{3}(d+1)$. Note that vertex transitive graphs are regular, hence we can conclude:

THEOREM 4. *If G is a connected vertex-transitive graph of degree d, then G contains a cycle of length at least $cd^{3/10}n^{1/2}$, where $c \approx 0.2615$.*

Although the folklore belief of many is that all but a finite number of connected vertex-transitive graphs are Hamiltonian, it seems that the lower bounds on the circumference given above are the best known at present for large d.

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