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Extremal Theory for Cliques in Graphs

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Abstract

In this paper we consider the foundations and development of one of the most beautiful subareas of graph theory, classical extremal theory, sometimes called Turán type extremal theory. We concentrate on results dealing with complete graphs (cliques). Particular results for extremal numbers and examples of extremal graphs are presented, along with supersaturation results on the number of copies of a complete graph contained in a given graph. These results are sharpest for triangles. We also briefly consider ramsey perturbation for cliques. Finally, we present a brief overview of the general theory of the structure of extremal graphs.

1 Introduction

Extremal graph theory, in its most general form, concerns any problem which attempts to determine the relation between graph invariants (such as order, size or minimum degree) and a graph property (like being hamiltonian, containing a perfect matching or containing a particular subgraph G_1). Typically, given a graph property P, an invariant i and a class of graphs $\mathcal{H}$, one tries to determine the least value m such that every graph G in $\mathcal{H}$ with $i(G) > m$ has property P.

We shall consider the question generally credited with starting extremal theory and to the beginnings of the research that sprang from this question. This study is rich in counting techniques and estimations. We shall use elementary results about convex functions to obtain some bounds. Facts about this can be found in [31]. We also assume a fundamental knowledge of graph theory and unless otherwise stated, follow the notation and terminology of [31]. We include proofs, where possible, for completeness and to help the reader get a better understanding for the types of techniques commonly used.

We shall limit our investigation to the particular type of extremal problem whose initiation is generally credited to Turán [46]. As is often the case, Turán was not the first to ask a question of this type, but his work did provide the real impetus in this area. In this problem, we ask the following: Given a graph G, determine the

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maximum number of edges $ex(n;G)$ in a graph of order n that does not contain G as a subgraph. A graph E of order n with $ex(n;G)$ edges and not containing G as a subgraph is called an *extremal graph* for this problem. The complete solution of any such extremal problem ordinarily requires two things. First, we must produce an extremal graph on n vertices and $ex(n;G)$ edges that does not contain G as a subgraph. Second, we must show that any graph on n vertices and with at least $ex(n;G)+1$ edges must contain a copy of G.

The investigation of this extremal problem naturally leads us to the study of the structure of extremal graphs. A rather beautiful theory has been developed that essentially tells us that the exact structure of the forbidden graphs themselves is not really as important as their chromatic number.

2 Complete Subgraphs

Historically, the first real result in extremal graph theory seems to be due to Mantel in 1907. Mantel [37] posed as an exercise, determining the maximum number of edges in a graph without triangles, then along with several others [38], obtained a solution to the exercise. We present this result here, as it has a remarkably simple proof.

Theorem 1 *If G has order n and contains no triangles, then* $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$.

Proof: Suppose G is as described and number the vertices of G from 1 to n. Assign vertex i a weight of $w_i \geq 0$ such that $\sum_{i=1}^{n} w_i = 1$. Our goal is to maximize

$$S = \sum_{ij \in E(G)} w_i w_j$$

(where the sum is taken over all edges in G). Suppose vertices u and v are not adjacent in G. Let the neighbors of u have total weight x and let the neighbors of v have total weight y, where we assume without loss of generality that $x \geq y$.

Since $(w_u+\epsilon)x + (w_v - \epsilon)y \geq w_u x + w_v y$ we do not decrease the value of S if we shift some weight from the vertex v to the vertex u. It follows that S is maximal if all the weight is concentrated on some complete subgraph of G, in fact, on one edge. But then $S \leq \frac{1}{4}$ (applying standard convexity). On the other hand, taking all $w_i = n^{-1}$, we see that $S \geq n^{-2}|E|$. But then these two inequalities imply that $|E| \leq \frac{n^2}{4}$. □

We now turn to Turán's problem: What is the maximum number of edges q in a graph of order n that does not contain the complete graph K_p, $p \geq 3$? We begin by producing the extremal graph for the Turán problem, as this graph is easy to describe. For the forbidden graph K_{p+1} (with chromatic number $p+1$), we begin with the complete p-partite graph $K_{n_1,n_2,\ldots,n_p}$ where $n = \sum n_i$. It is easy to show that among all such graphs, the one with the maximum number of edges is that graph with partite sets as nearly equal as possible (convexity again). In fact, among all graphs on n vertices with chromatic number p, it has the maximum number of edges. Thus, if $n = kp + r$, $0 \leq r < p$, then $p - r$ of the partite sets contain k vertices and the remaining r of the partite sets contain $k+1$ vertices. We denote this graph as $T_{n,p}$, and call it the *Turán graph.* We further note that

$$|E(T_{n,p})| = \binom{n}{2} - \frac{k(n-p+r)}{2}.$$

We are now ready to state Turán's theorem [46].

Theorem 2 *Among the graphs of order n which do not contain K_p, there exists exactly one with the maximum number of edges, namely $T_{n,p-1}$.*

We will sketch two proofs to Turán's theorem, showing two common and useful techniques in extremal theory. The first technique is called *chopping* and resembles Turán's original proof. The strategy is to chop off a useful subgraph and work around this structure to complete the proof, carefully avoiding the chopped graph. For convenience and to maintain a notation common in extremal theory, we denote by G^n a graph of order n.

Turán's Chopping Proof: We proceed by induction on n, the order of the extremal graph under construction. The anchor is trivial so assume the result holds for orders less than n and suppose the extremal graph G^n is K_p—free. Since G is extremal, it follows that $H = K_{p-1}$ is a subgraph of G^n and define q_1, q_2, q_3 as follows:

$$q_1 = |E(H)| = \binom{p-1}{2},$$

$$q_2 = \text{no. of edges between } H \text{ and } V - H \leq (n-p+1)(p-2)$$

$$q_3 = |E(V-H)| \leq |E(T_{n-p+1,p-1})|$$

(the bound in the third expression follows from the inductive assumption).

It is clear that $|E(G^n)| = q_1 + q_2 + q_3$, and by summing the bounds given on q_1, q_2 and q_3 we see that

$$|E(G^n)| \leq |E(T_{n,p-1})|.$$

It remains to show that if equality holds, then $G^n = T_{n,p-1}$. Clearly, $q_2 = (n-p+1)(p-2)$. This determines a partition of $V(G^n)$ into $p-1$ classes, defined according to their $p-2$ adjacencies in H. These classes are clearly independent, so G^n is a complete $(p-1)$ partite graph defined by these classes, that is, $G = T_{n,p-1}$. □

The second proof technique, known as *symmetrization*, has become a powerful tool in extremal theory. The process of symmetrization proceeds as follows: Given nonadjacent vertices v and u, we delete all the edges incident to the vertex u and make u adjacent to all vertices in $N(v)$. The vertex u is then said to be *symmetric* to v. First, it is clear that no K_p is formed during this process, since only the vertices of $N(v)$ have new adjacencies and u and v are not adjacent. Thus, if a K_p now exists, it must have existed prior to symmetrization. Second, if $deg\ u < deg\ v$, then we have increased the number of edges in the graph without producing the forbidden K_p. Clearly, under certain conditions, symmetrization can be useful in extremal problems.

We now sketch a second proof of Turán's theorem, using symmetrization, due to Zykov [48].

Sketch of Zykov's Symmetrization Proof: We assume the anchor and inductive steps have been performed and consider the extremal graph G^n which is K_p-free. Let v have maximum degree in G^n and symmetrize all of $V - N(v)$ to v. Let S_1 denote these vertices along with v. Clearly, S_1 is an independent set of vertices. Further, since v had maximum degree, our new graph has at least as many edges as G^n. Now, repeat this process on $G^n - S_1$, forming the set S_2. Continue the procedure, forming the sets $S_3, \ldots, S_d$. As we noted earlier, since G^n was K_p-free, this new graph formed by symmetrization is also K_p-free. (That is, $d \leq p-1$.) Thus, any K_p-free graph can be transformed into a d-partite ($d \leq p-1$) graph. Further, to maximize the number of edges in such a graph, standard convexity arguments (as noted before) imply that the graph is actually $T_{n,p-1}$. □

We now state an extension of Turán's theorem due to Erdös [12] which can be used to provide yet another proof of Turán's theorem.

Theorem 3 *Let G^n be a K_p-free graph with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n$. Then there exists a $(p-1)$ chromatic graph H^n which is K_p-free with degrees $s_1 \geq s_2 \geq \ldots \geq s_n$ and such that $s_i \geq d_i$ for every i.*

We continue our investigation of complete subgraphs with a theorem from Dirac [4] that shows that we actually get more than a K_p once we have more than the extremal number of edges. To this end, we say that a graph H is *saturated* (in particular, we say a graph H is *G-saturated*) if H does not contain G and if the addition of any edge to H results in a graph that does contain G. If H contains more than $ex(n, G)$ we say that H is *supersaturated.* By an $(n, q)-$graph G we mean that G has order n and size q.

Theorem 4 *If $n \geq r+1$, then every $(n, ex(n, K_r)+1)-$graph G contains a $K_{r+1} - e$.*

Proof: We proceed by induction on n, the order of G. For $n = r+1$, it is clear that having one more than the extremal number of edges forces $G = K_{r+1} - e$, and so we have the anchor step.

Now, we assume the result holds on all such graphs of order less than n and consider an $(n, ex(n; K_r) + 1)$ graph G. Let x have minimum degree $\delta(G)$. Then it is easily seen that $\delta(G) \leq \delta(T_{n,r})$, and so $|E(G-x)| \geq ex(n-1; K_r) + 1$. Hence, by induction we see that $G - x$ contains $K_{r+1} - e$, and the result holds. □

For completeness, we now state the following corollary to Turán's theorem due to Zarankiewicz [47].

Corollary 5 *If G^n is K_r-free, then*

$$\delta(G^n) \leq \left(1 - \frac{1}{r-1}\right) n = \frac{r-2}{r-1} n.$$

With effort, one can improve upon the above corollary (see [2]); however, we shall simply state this improvement. Here, $\chi(G^n)$ is the chromatic number of G^n.

Theorem 6 *If $\chi(G^n) \geq r$ and G^n is K_r-free, then*

$$\delta(G^n) \leq \frac{(3r-7)}{(3r-4)}n.$$

3 Counting Triangles

We continue our investigation of complete subgraphs by concentrating on triangles. Turán's theorem tells us when we can be sure one triangle exists, but our goal is to establish bounds on the number of triangles that exist in general. In first attacking this problem, we will find it useful to change our setting and sum the number of triangles that must be contained in a graph and its complement. Let $k_r(G)$ equal the number of copies of K_r contained in the graph G. Independent work of several people, including Goodman [30], Moon and Moser [39] and Lovász [35] all lead to the following result.

Theorem 7 *Given an $(n,q)-$graph G with $(n,\overline{q})$ complement $\overline{G}$,*

$$\begin{aligned} k_3(G) + k_3(\overline{G}) &= \binom{n}{3} - (n-2)q + \sum_{i=1}^{n} \binom{deg\ v_i}{2} \\ &= \binom{n}{3} - (n-2)\overline{q} + \sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2}. \end{aligned}$$

Proof: Consider the degree sequence of G. There are $\sum_{i=1}^{n} \binom{deg\ v_i}{2}$ pairs of adjacent edges of G and $\sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2}$ pairs of adjacent edges in $\overline{G}$. The sum of these two numbers can be counted in another way as well. Each of the triangles in G and $\overline{G}$ contains three pairs of adjacent edges, and each of the remaining

$$L = \binom{n}{3} - k_3(G) - k_3(\overline{G})$$

triples of vertices contains exactly one such pair. Hence,

$$\sum_{i=1}^{n} \binom{deg\ v_i}{2} + \sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2} = 3k_3(G) + 3k_3(\overline{G}) + L.$$

Solving for our desired sum yields,

$$k_3(G) + k_3(\overline{G}) = \frac{1}{2}\left(\sum_{i=1}^{n} \binom{deg\ v_i}{2} + \sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2} - \binom{n}{3} \right).$$

But note that

$$\sum_{i=1}^{n} \binom{n-1-deg\, v_i}{2} = \sum_{i=1}^{n} \frac{(n-1-deg\, v_i)(n-2-deg\, v_i)}{2}$$
$$= \sum_{i=1}^{n} \left(\binom{n-1}{2} - (n-2)deg\, v_i + \binom{deg\, v_i}{2} \right).$$

Now, substituting and rearranging terms completes the result. □

Corollary 8 *The graphs G^n and $\overline{G}^n$ contain a total of at least $\frac{n(n-1)(n-5)}{24}$ triangles.*

We note that this bound is attained if, and only if, $\overline{G}$ is $\frac{2\overline{q}}{n}$ regular. In particular, we must have $n = 2\delta + 1$ and δ must be even.

Yet another refinement of Corollary 8 due to Goodman [30] (see also Sauvé [42]) allows us to count the minimal number of triangles in a graph of order n and its complement. In this case we obtain:

$$2\binom{l}{3} \quad \text{if n} = 2\text{l}$$

$$\frac{2}{3}k(k-1)(4k+1) \text{ if n} = 4\text{k} + 1$$

$$\frac{2}{3}k(k+1)(4k-1) \text{ if n} = 4\text{k} + 3.$$

Lorden [34] was able to show the following:

Theorem 9 *Let $G = G^n$ and suppose that $\overline{G}$ does not contain a triangle, then*

$$k_3(G) \geq \binom{\lfloor \frac{n}{2} \rfloor}{3} + \binom{\lfloor \frac{n+1}{2} \rfloor}{3}.$$

Clapham [8] further specialized the problem of triangles in a graph and its complement.

Theorem 10 *Suppose G is self complementary (that is, isomorphic to its complement), then $k_3(G)$ is at least*

$$\frac{2k}{3}(k-1)(2k-1) \quad \text{if } n = 4k$$

or

$$\frac{k}{3}(k-1)(4k+1) \quad \text{if } n = 4k+1,$$

and this inequality is best possible.

Recently, Erdös [17] conjectured that if the edges of a K_n are 2-colored, then the number of edge disjoint monochromatic triangles in a graph of sufficiently large order n is at least $\frac{n^2}{12}$. Jacobson [33] conjectured that if the edges of a K_n are 2-colored, then there are at least $n^2/20$ edge disjoint monochromatic triangles in one of the colors. Finally, Erdös and Gyárfás [20] conjecture that if $\overline{G}$ contains no triangles, then G will contain at least $\frac{n^2}{12}$ edge disjoint triangles.

Returning to the general problem of finding bounds on the number of triangles present in a graph, Moon and Moser [39] obtained the following bound.

Theorem 11 *An (n,q)–graph contains at least $\frac{q}{3n}(4q-n^2)$ triangles.*

Proof: Suppose that $uv \in E$. Then there are at least $deg\, u + deg\, v - n$ vertices adjacent to both u and v. Thus, we see that

$$k_3(G) \geq \frac{1}{3}\sum_{uv\in E}(deg\, u + deg\, v - n).$$

But since each $deg\, u$ term appears $deg\, u$ times in this sum, we have that

$$k_3(G) \geq \frac{1}{3}\sum_{u\in V} deg^2\, u - nq.$$

So by the Cauchy inequality,

$$k_3(G) \geq \frac{1}{3}\left(\frac{(2q)^2}{n} - nq\right) = \frac{q}{3n}(4q-n^2).$$

□

The next result is due to Rademacher (see [24]) and extends Mantel's Theorem.

Theorem 12 *For every even n, a graph on n vertices with $\frac{n^2}{4}+1$ edges contains at least $\frac{n}{2}$ triangles. Furthermore, this result s best possible.*

The graph $K_{n/2,n/2} + e$ shows that Rademacher's Theorem is best possible, as it contains exactly $\frac{n}{2}$ triangles, each containing the edge e.

Our next result was originally conjectured by Nordhaus and Stewart [40]. The result is due to Bollobás [3].

Theorem 13 *If G is a graph on n vertices and $\frac{n^2}{4} \leq |E(G)| \leq \frac{n^2}{3}$ edges then G contains at least $\frac{n}{9}(4|E(G)| - n^2)$ triangles.*

The best lower bound for the number of triangles was proved by Fisher [29] who gave an asymptotically sharp bound on the number of triangles t. That bound is off from the optimal one only by a lower order term (in most cases).

Theorem 14 *For a graph on n vertices and q edges with $\frac{n^2}{4} \leq q \leq \frac{n^2}{3}$ the number of triangles t in G is*

$$t \geq \frac{9qn - 2n^3 - 2(n^2-3q)^{3/2}}{27}.$$

Comparing these theorems over the range of possible values for q we see that Rademacher's Theorem is most accurate for $q = \frac{n^2}{4} + 1$ edges; the bounds of Moon and Moser and of Bollobás are equal when $q = \frac{n^2}{3}$; finally, Moon and Moser's Theorem yields the exact number of triangles when $q = \binom{n}{2}$.

We complete this part of the discussion with a result due to Erdös [16]. We begin with a sequence of lemmas.

Lemma 1 *Every $(n, ex(n-1, K_3)+2)-$graph G which contains an odd cycle, contains a triangle.*

Proof. Let G be as described and let $C : u_1, u_2, \ldots, u_{2k+1}$ be the vertices of a shortest odd cycle in G. We can assume that $3 \leq 2k+1 \leq n$. Now $< u_1, u_2, \ldots, u_{2k+1} >$ can have no other edges, for otherwise a shorter odd cycle would be formed. Let $v_1, v_2, \ldots, v_{n-2k-1}$ be the other vertices of G. Any v_i $(1 \leq i \leq n-2k-1)$ can be adjacent to at most two u_j $(1 \leq j \leq 2k+1)$, for otherwise an odd cycle shorter than C would be formed. Finally, Turán's Theorem implies $< v_1, \ldots, v_{n-2k-1} >$ can have at most $ex(n-2k-1, K_3)$ edges. Thus, the number of edges in G is at most

$$2k + 1 + 2(n-2k-1) + ex(n-2k-1, K_3) \leq ex(n-1, K_3) + 1,$$

contradicting our assumptions. □

Lemma 2 *There exists a constant $c_2 > 0$ such that every $(n, ex(n, K_3) + 1)$ graph G contains at least $\lfloor c_2 n \rfloor$ triangles having a common edge (u, v).*

Proof. Let $T = \{(u_i, v_i, w_i) | 1 \leq i \leq r\}$ be a maximal system of disjoint triangles in G. Thus, in $G - T$ the remaining $n - 3r$ vertices contain no triangles and therefore have at most $ex(n-3r, K_3)$ edges.

Denote by $G(i)$ the graph obtained from G by deleting the first $i-1$ triangles of T. Further, let $deg_i\, u_i$, $deg_i\, v_i$ and $deg_i\, w_i$ be the degrees of u_i, v_i and w_i in $G(i)$.

We now show that for some i $(1 \leq i \leq r)$ we must have

$$deg_i\, u_i + deg_i\, v_i + deg_i\, w_i > n(1 + 9c_2) - 3i, \tag{1}$$

for if this failed to hold for any i, then the number of edges in G would be at most

$$\sum_{i=1}^{r} \{n(1+9c_2) - 3i\} + ex(n-3r, K_3) < ex(n, K_3)$$

by a simple calculation for sufficiently small c_2. But this contradicts the fact G contains at least $ex(n, K_3) + 1$ edges. Thus, (1) holds for say $i = i_0$. Then a simple calculation shows that there are at least $3\lfloor c_2 n \rfloor$ vertices of $G(i_0)$ which are adjacent to more than one of the vertices u_{i_0}, v_{i_0}, w_{i_0}. Therefore, there are at least $\lfloor c_2 n \rfloor$ vertices adjacent to the same pair, which completes our proof. □

When k triangles share a common edge we call such a graph a *k-book* (or a book with k pages) and denote it B_k. Note that Edwards ([10] unpublished but often cited) showed that a graph of order n with $ex(n, K_3) + 1$ edges contains a $B_{n/6}$

Lemma 3 *Let $\delta > 0$ be a fixed number. Consider any (n, q)-graph G with $q > ex(n, K_3) - \frac{n}{2}(1-\delta)$, $n > n_0(\delta)$, which contains a triangle. Then G contains an edge (u, v) and $r = \lfloor c_3 n \rfloor + 1$ ($c_3 = c_3(\delta)$) vertices w_i ($i = 1, 2, \ldots, r$) so that all the triangles (u, v, w_i) ($i = 1, 2, \ldots, r$) are in G.*

Proof. By assumption, G contains a triangle (u, v, w). Assume first that

$$deg\ u + deg\ v + deg\ w > n(1 + 9c_3) + 9. \tag{2}$$

Then the result follows from Lemma 2.

If (2) fails to hold, then $G-u-v-w$ has $n-3$ vertices and at least $q-n(1+9c_3)-9$ edges. But if $c_3 < \frac{\delta}{18}$, then for $n > n_0$,

$$q - n(1+9c_3) - 9 > ex(n, K_3) - \frac{n}{2}(1-\delta) - n(1+9c_3) - 9 > ex(n-3, K_3).$$

But then by Lemma 2, $G - u - v - w$ contains the desired configuration of triangles, which completes the proof. □

We are finally ready to present our goal, a theorem due to Erdös [16].

Theorem 15 *There exists a constant $c_1 > 0$ such that for n sufficiently large and $t < c_1 n/2$, if a graph G on n vertices contains at least $\lfloor \frac{n^2}{4} \rfloor + t$ edges, then G contains at least $t \lfloor \frac{n}{2} \rfloor$ triangles.*

Proof Suppose G is as above and $t < c_1 \frac{n}{2}$. We first assume that after the omission of any $r = \lfloor c_1 \frac{n}{2c_3} \rfloor$ edges, the graph still contains a triangle. (Note: $c_3 = c_3(\delta)$, for $\delta = \frac{1}{4}$ in the last lemma.) For sufficiently small c_1, $\frac{c_1}{2c_3} < \frac{1}{4}$; thus it will be permissible to apply Lemma 3 during the omission of these edges.

By Lemma 3 (or Lemma 2) there exists an edge e_1 contained in $\lfloor c_3 n \rfloor + 1$ triangles of G. Again by Lemma 3 in $H_1 = G - e_1$, there exists an edge e_2 contained in at least $\lfloor c_3 n \rfloor + 1$ triangles of H_1. Suppose we have already chosen the edges $e_1, \ldots, e_r$ each of which is contained in at least $\lfloor c_3 n \rfloor + 1$ triangles. By our earlier assumption $H_r = G - e_1 - \ldots - e_r$ contains at least one triangle. But then by Lemma 3 there is an edge e_{r+1} in H_r which is contained in at least $\lfloor c_3 n \rfloor + 1$ triangles of H_r. These triangles incident on the edges $e_1, \ldots, e_{r+1}$ are clearly distinct, thus G contains at least

$$(r+1)(\lfloor c_3 n \rfloor + 1) > c_1 \frac{n^2}{2} > t\frac{n}{2}$$

triangles, which completes the proof in this case.

Therefore, we may assume that there are $s \le r < \frac{n}{4}$ edges $e_1, e_2, \ldots, e_s$ so that the graph $H = G - e_1 - e_2 - \ldots - e_s$ contains no triangles and we may assume s is the smallest integer with this property. By the fact that $s \le r < \frac{n}{4}$, H has

$$ex(n, K_3) + t - s > ex(n, K_3) - \frac{n}{4} > ex(n-1, K_3) + 1$$

edges. Thus, by Lemma 1, H must contain only even cycles.

By Theorem 2 , $s \ge t$. Suppose $s = t$. Then H has $ex(n, K_3)$ edges and by Turán's Theorem, $H = T_{n,2}$. Clearly, the addition of any edge creates at least $\lfloor \frac{n}{2} \rfloor$ distinct triangles. A simple argument shows that the addition of every further edge introduces at least $\lfloor \frac{n}{2} \rfloor$ triangles and that these triangles are distinct. Thus, G contains at least $t\lfloor \frac{n}{2} \rfloor$ triangles and our result is shown in this case as well.

Finally assume $s = t + w$, $0 < w < \frac{n}{4}$ (since $s < n/4$). We also assume n is even, say $n = 2m$. Now since H contains only even cycles, it is a subgraph of a bipartite graph B whose vertices are say $\alpha_1, \dots, \alpha_{m-u}$ and $\beta_1, \dots, \beta_{m+u}$ (since H has more than $ex(2m, K_3) - \frac{m}{2}$ edges, we have $0 \leq u < (m/2)^{1/2}$).

Clearly, every one of the edges $e_1, \dots, e_s$ join two of the α's or two of the β's, for otherwise for some e_i, the graph $G - e_1 - \dots - e_{i-1} - e_{i+1} - \dots - e_s$ would still have only even cycles and hence no triangles, which contradicts the minimum property of s.

By our assumption, H is a subgraph of B. Assume H is obtained from B by the omission of x edges. Then we clearly have

$$s = x + u^2 + t \quad (\text{or } w = x + u^2),$$

and G is obtained from H by adding s edges $e_1, \dots, e_s$ which are all of the form $(\alpha_{i_1}, \alpha_{i_2})$ or $(\beta_{i_1}, \beta_{i_2})$. Let $e_i = (\beta_{i_1}, \beta_{i_2})$ and let us estimate the number of triangles $(\beta_{i_1}, \beta_{i_2}, \alpha_j)$ in B. Clearly, at most x of the edges (β_{i_1}, α_j), (β_{i_2}, α_j) are not in B; thus $B + e_i$ contains at least $m - u - x$ triangles (if e_i connects two α's, then $B + e_i$ contains at least $m + u - x$ triangles). For different e_i's these triangles are clearly different; thus $G = H + e_1 + \dots + e_s$ contains at least

$$(m - u - x)s = (m - u - x)(x + u^2 +) \geq tm = t(n/2)$$

triangles. The above follows by simple computation from $s = u^2 + x + t < m/2$. The above equation completes the proof in the $n = 2m$ case. For $n = 2m + 1$ the proof is almost identical and hence we omit it here. This completes our proof. □

This result has been improved by Lovász and Simonovits (see [4]) who showed that the theorem holds for $c_1 = 1$.

Returning to the notion of books mentioned earlier, define

$$f(n, q) = \min \max \{\sum_{i=1}^{3} deg\ v_i : v_1, v_2, v_3 \text{ induces a triangle in } G\}.$$

where the maximum is taken over all triangles in G and the minimum is taken over all graphs of order n and size q. Turan's Theorem tells us that $f(n, q) = 0$ if $q \leq n^2/4$. Clearly, $f(n, q) > 0$ when $q > n^2/4$. Bollobás and Erdös (see [11]) raised the problem of determining $f(n, q)$ for $q > n^2/4$.

As mentioned earlier, Edwards [10] and [11] proved that $f(n, q) = \frac{6q}{n}$ for $q > \frac{n^2}{3}$. For the situation that $\frac{n^2}{4} < q < \frac{n^2}{3}$, a construction in [23] shows that $f(n, q) < 4\sqrt{3q} - 2n + 5$. G. Fan [27] proved that $f(n, q) \geq \frac{21q}{4n}$. In particular he showed that $f(n, ex(n, K_3) + 1)) > \frac{21n}{16}$, which improves upon a result of Erdös and Laskar [23] . When $q > .26n^2$, Fan obtained a slightly stronger result as well.

For generalized books, Faudree [28] showed that if G is a graph on n vertices with $n > \frac{k^2(k-1)}{4}$ and $q \geq |E(T_{n,k})|$, then G contains a complete graph K_k such that the sum of the degrees of the vertices is at least $\frac{2kq}{n}$. This result is sharp in an asymptotic sense in that the sum of the degrees of the vertices of a K_k is not larger in general, and if the number of edges in G is at most

$$|E(T_{n,k})| - \epsilon n$$

(for an appropriate ϵ), then the conclusion fails to hold.

We conclude this section with a new result concerning triangles that share a vertex. A graph on $2k+1$ vertices consisting of k triangles which intersect in exactly one common vertex is called a *k-fan* and is denoted by F_k. We wish to determine $\mathrm{ex}(n, F_k)$ for every fixed k. This result was recently provided in [18].

Theorem 16 *For every $k \geq 1$, and for every $n \geq 50k^2$, if a graph G on n vertices has more than*

$$\lfloor \frac{n^2}{4} \rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even} \end{cases} \tag{3}$$

edges, then G contains a copy of a k-fan. Furthermore, the number of edges is best possible.

To prove the lower bound for $\mathrm{ex}(n, F_k)$, consider the following graph $G_{n,k}$. For odd k (where $n \geq 4k-1$) $G_{n,k}$ is constructed by taking $T_{n,2}$, the complete equi-bipartite graph and embedding two vertex disjoint copies of K_k in one side.

For even k (where now $n \geq 4k-3$) $G_{n,k}$ is constructed by taking $T_{n,2}$ and embedding a graph with $2k-1$ vertices, $k^2 - (3/2)k$ edges with maximum degree $k-1$ in one side.

Obviously, $\mathrm{ex}(n, F_k) = \binom{n}{k}$ for $1 \leq n \leq 2k$, and it is easy to check that $\mathrm{ex}(2k+1, F_k) = 2k^2 - 1$ (if $k \geq 2$), which is smaller than (3) for odd k and larger than 3 for even k ($k \geq 4$).

However, in [18] it is conjectured that (3) gives $\mathrm{ex}(n, F_k)$ for all $n \geq 4k$ (rather than $n \geq 50k^2$).

One final note about Theorem 16. If $\nu(G)$ is the edge independence number of G and $\Delta(G)$ is the maximum degree in G, define $f(\nu, \Delta) = \max\{|E(G)| : \nu(G) \leq \nu, \Delta(G) \leq \Delta\}$. Chvátal and Hanson [7] proved that for every $\nu \geq 1$ and $\Delta \geq 1$,

$$f(\nu, \Delta) = \nu\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor \leq \nu\Delta + \nu. \tag{4}$$

Theorem 16 uses the following special case proved by Abbott, Hanson and Sauer [1]:

$$f(k-1, k-1) = \begin{cases} k^2 - \frac{3}{2}k & \text{if } k \text{ is even,} \\ k^2 - k & \text{if } k \text{ is odd.} \end{cases} \tag{5}$$

The extremal graphs are exactly those we embedded into $T_{n,2}$ to obtain the extremal F_k-free graph $G_{n,k}$.

4 Arbitrary Cliques

We now turn to the problem of larger cliques. It is a simple observation that for $s \leq p$

$$k_s(T_{n,p}) \approx \binom{p}{s}\left(\frac{n}{p}\right)^s.$$

What can we say in general about the number of cliques of order s contained in an arbitrary G^n?

We begin with a result of Erdös and Hanani [13] that estimates the maximal number of complete subgraphs of order r contained in a graph of order n and size q. This problem is quite different in nature than those we have been considering, especially in view of the fact that the maximum does not depend on n.

Theorem 17 *Let r and q be natural numbers, $r \geq 3$. Let $q = \binom{s}{2} + t$, $0 < t \leq s$. Then,*

$$\max\{k_r(G) \mid |E(G)| = q\} = \binom{s}{r} + \binom{t}{r-1}.$$

Proof: Let G be obtained from K_s by joining a new vertex to t of the vertices. Then G has size q and $k_r(G) = \binom{s}{r} + \binom{t}{r-1}$. Thus, the maximum is at least as large as claimed.

To see the reverse inequality, we proceed by induction q. Note that for $q \leq \binom{r}{2}$ the result is trivial. Assume now that $q > \binom{r}{2}$ (so $s \geq r$) and the result holds for all smaller values of q. Let G be a graph of size q. Without loss of generality we may suppose that G has no isolated vertices. Now suppose that G has a vertex x with $deg\ x = d < s$. Then there are at most $\binom{d}{r-1}$ K_r's containing x. If $d < t$ then by applying the induction hypothesis to $G - x$ we see that

$$k_r(G) \leq \binom{d}{r-1} + k_r(G-x) \leq \binom{d}{r-1} + \binom{s}{r} + \binom{t-d}{r-1} < \binom{s}{r} + \binom{t}{r-1}.$$

Similarly, if $t \leq d < s$ we have

$$k_r(G) \leq \binom{d}{r-1} + \binom{s-1}{r} + \binom{s+t-d}{r-1} < \binom{s}{r} + \binom{t}{r-1}.$$

Thus, we may suppose that every graph G has minimum degree at least s. Therefore the order n and size q of G satisfy

$$n \geq s+1$$

and

$$q \geq \frac{1}{2}s(s+1).$$

But then from $q \leq \binom{s+1}{2}$ we see that $G = K_{s+1}$, $t = s$ and so

$$k_r(G) = \binom{s+1}{r} = \binom{s}{r} + \binom{t}{r-1}. \ \square$$

Returning to our question on supersaturated graphs, the next result is from [36].

Theorem 18 *If t is defined by $|E(G)| = \frac{1}{2}(1-\frac{1}{t})n^2$, then G^n contains at least*

$$\binom{t}{p}\left(\frac{n}{t}\right)^p$$

copies of K_p.

Note that the complete t-partite graph $K_{(t)(m)}$ (t sets of m vertices) shows that this result is sharp.

Our next result, again due to Lovász and Simonovits [36], deals with the stability of the extremal graph $T_{n,p}$. Roughly speaking this result says that if the size of G^n exceeds the extremal number by t, then either the structure of G^n is very regular and similar to $T_{n,p-1}$ or else the structure is unusual and so many copies of K_p are present.

Theorem 19 *For every $c > 0$, there exists $\delta > 0$ and $c' > 0$ such that, if t is defined by $q = |E(G^n)| = \frac{1}{2}(1-\frac{1}{t})n^2$, if $d = \lfloor t \rfloor$, and if $q = |E(T_{n,p})| + k$ for some $k \in [0, \delta n^2]$, then either G^n contains at least*

$$\binom{t}{r}\left(\frac{n}{t}\right)^r + ckn^{r-2}$$

copies of K_r, or it can be obtained from $T_{n,d}$ by changing at most $c'k$ edges.

Finally we turn to a general bound provided by Bollobás [3, 4]. For $2 \le p < r \le n$, let $k_r(k_p^n \ge x) = \min\{k_r(G) \mid k_p(G) \ge x\}$. We wish to estimate $k_r(k_p^n \ge x)$. First note that since $k_2(G^n) = |E(G^n)|$, then $k_r(k_2^n \ge x)$ is our standard extremal problem. We state here a slightly weaker form of the result, but one more appropriate to our investigation.

Theorem 20 *Let $\psi(x) = \psi(p,r,x,n)$ be the maximal convex function defined in $0 \le x \le \binom{n}{p}$ such that*

$$\psi\left(\left(\frac{n}{s}\right)^p \binom{s}{p}\right) \ge \left(\frac{n}{s}\right)^r \binom{s}{r} \quad s = 1,2,\ldots,n.$$

Then

$$k_r(k_p^n \ge x) \ge \psi(x).$$

Turning to the investigation of the set of complete graphs contained in a graph, we ask the following: At least how many edge disjoint K_r's are contained in every G^n of size q? The bound we give here is rough, but does extend Turan's Theorem. This result was shown by Erdős, Goodman and Pósa [19] for $r = 3$ and by Bollobás [3] for $r \ge 4$.

Theorem 21 *Let $r \ge 3$. Every G^n can be covered with at most $|E(T_{n,r-1})|$ edge disjoint K_r's and edges. If $r > 3$, then $T_{n,r-1}$ is the only graph that cannot be covered with fewer edge disjoint K_r's and edges. For $r = 3$ the extremal graphs are K_4, K_5 and $T_{n,2}$, $(n = 1,2,\ldots)$.*

Recall that G is K_r-saturated if G does not contain K_r, but the addition of any edges to G results in a graph that does contain K_r. We can extend this definition by saying G is *strongly K_r saturated* if $k_r(G) < k_r(G^*)$ whenever G^* is obtained from G by the addition of an edge.

Turán's Theorem tells us the maximal size of a K_r-saturated graph. Erdös, Hajnal and Moon [21] determined the minimal size of a strongly K_r-saturated graph of order n.

Theorem 22 *The minimal size of a strongly K_r-saturated ($r \geq 3$) graph of order n is*

$$(r-2)(n-r+2) + \binom{r-2}{2}.$$

The K_r-saturated graph $G^n = K_{r-2} + (n-r+2)K_1$ is the only strongly K_r-saturated graph of order n and minimal size.

Finally, a complete generalization of Theorem 16 on k-fans is not known. However, we can say a little. The following result was conjectured by Busolini [6] (see [4]).

Theorem 23 *Let $r \geq 3$. If n is sufficiently large, then every G^n of size at least $|E(T_{n,r-1})| + 2$ contains a $K_1 + 2K_{r-1}$ (that is, two complete graphs of order r with exactly one vertex in common).*

The following extension takes on a different form yet again.

Theorem 24 *Let $r \geq 3$ and $k \geq 1$ be natural numbers. If n is sufficiently large then every G^n of size at least*

$$|E(T_{n,r-1})| + k$$

contains k complete graphs of order r, say $H_1, \ldots, H_k$ such that $\cup_{i=1}^k H_i$ is connected and any two H_i's have at most one vertex in common.

We conclude this section with a powerful result due to Hajnal and Szemerédi [32]. This result is usually thought of in terms of graph coloring. Essentially it states that if the maximum degree Δ of the graph is not too large, then we can $\Delta + 1$ color the graph and obtain color classes as balanced as possible. Another way of stating this is:

Theorem 25 *If G is a graph of order n with $\Delta(G) \leq t$, then $G \subseteq T_{n,t+1}$.*

This result was originally conjectured by Erdös. Clearly, for our purposes, this says that $\overline{G}$ must contain sK_{t+1}, where $n = s(t+1) + r$.

5 Ties to Ramsey Theory

When one considers the structure of the Turán graph and its overall importance to our standard extremal question, a natural variation comes to mind. This variation, called *Ramsey perturbation*, brings Ramsey Theory into play. For example, Erdös [13] showed the following:

Theorem 26 *If G^n ($n = 3k$) contains no 4 independent vertices, then G contains at least $3\binom{k}{3}$ triangles.*

One critical feature of the Turán graph $T_{n,p}$ is that it contains large independent sets. Conditions like those imposed in Theorem 26 take this property away. Here we consider what happens under this type exclusion.

We denote by $ex(n, H, f)$ the maximum number of edges in a graph G^n containing no H and at most $f(n)$ independent vertices. Our interest again centers on forbidding cliques of order $p+1$ and when $T_{n,p}$ is the only extremal graph. Clearly, if $f(n) = \lceil \frac{n}{p} \rceil$, we are back to Turán's Theorem. On the other hand, if $f(n)$ is a constant or $f(n)$ tends to infinity very slowly, then Ramsey's Theorem implies the set of possible forbidden subgraphs will be empty, that is, in avoiding large independent sets, we will force whatever graph we wish to exclude to actually be present. This raises several interesting questions.

Can we determine whether $ex(n, K_p, f)$ is significantly smaller than $ex(n, K_p)$ when $f(n) = \lfloor \frac{n}{p-1} \rfloor - cn$?

This question was answered affirmatively by Erdös and Sös [25].

Theorem 27 *For every $c > 0$, there exists $c' > 0$ such that*

$$ex(n, K_p, f) \leq ex(n, K_p) - c'n^2.$$

If we vary the problem by allowing $f(n) = o(n)$ the question becomes much harder. In fact, there is a difference between the odd and even cases. Again Erdös and Sös [] handled the general odd case.

Theorem 28 *There is a constant $c > 0$ such that if $g(n) = c\sqrt{n}\log n$ and $g(n) \leq f(n) = o(n)$, then*

$$\begin{aligned} ex(n, K_{r+1}) &\leq ex(n, K_{2r+1}, g) \\ &\leq ex(n, K_{2r+1}, f) \\ &\leq ex(n, K_{r+1}) + o(n^2) \\ &= (1 - \frac{1}{r} + o(1))\binom{n}{2}. \end{aligned}$$

The even case is harder. It was originally shown for K_4 by Szemeredi [45] and the lower bound was established by Bollobás and Erdös [5]. The result is from [22]

Theorem 29

$$ex(n, K_{2r}, o(n)) = \frac{1}{2}\left(\frac{3k-5}{3k-2}\right)n^2 + o(n^2).$$

6 On the Structure of Extremal Graphs

In this section we begin an investigation of the structure of extremal graphs. After determining the extremal values of various forbidden graphs, it is natural to try to gain further information about the extremal graphs themselves. It is not surprising that a great deal can be said and that this information opens still other avenues of investigation. It should be noted that there is a fundamental difference between extremal problems in which one of the forbidden graphs is bipartite (called a *degenerate extremal problem*), and one where none of the graphs is bipartite. The reasons for this will become more apparent as we progress. For now, simply note that in the degenerate case $ex(n, H) = o(n^2)$, while in the nondegenerate case,

$$ex(n, H) \geq \lfloor \frac{n^2}{4} \rfloor.$$

The foundation for this section is primarily the work of Erdös and Stone [26]. Our goal is to show that for a class of graphs H, the extremal number $ex(n; H)$ depends only loosely on the graphs in H. That is, the exact structure of the forbidden subgraphs is not the critical issue, but rather the dominant feature is the minimum chromatic number of a graph in the class H. In what follows we use the notation $K_{(s)(t)}$ to mean the complete s-partite graph with t vertices in each partite set. We begin with two beautiful results from Erdös and Stone [26].

Theorem 30 *Let $\epsilon > 0$ and $k, t \geq 1$ be given. Then, for n sufficiently large, every graph of order n and with $\delta \geq (1 - \frac{1}{k} + \epsilon)n$ contains $K_{(k+1)(t)}$.*

Proof. (By induction on k). For $k = 1$, the statement claims that $\delta \geq \epsilon n$ and, hence, G has at least $\frac{\epsilon}{2}n^2$ edges. That G contains a $K_{t,t}$ follows from Corollary 5 and Theorem 10.2.5.

Now let $k \geq 2$ and $s = \lfloor \frac{1}{\epsilon} t \rfloor$. If n is sufficiently large, then by our induction assumption, we can find a $K_{(k)(s)}$ in G. Let $Y = V(G) - K_{(k)(s)}$ and let X be those vertices of Y that are adjacent to at least t vertices in each of the partite sets of the $K_{(k)(s)}$. Then the number of missing edges between $Y - X$ and $K_{(k)(s)}$ is at least

$$(|Y| - |X|)(s - t) \geq (|Y| - |X|)(1 - \epsilon)s = (n - ks - |X|)(1 - \epsilon)s.$$

Also, the number of edges missing from any vertex in $K_{(k)(s)}$ is at most $(\frac{1}{k} - \epsilon)n$. Thus, the number of edges missing from the vertices in $K_{(k)(s)}$ is at most

$$ks(\frac{1}{k} - \epsilon)n = (1 - k\epsilon)sn.$$

Thus, the preceding two inequalities imply that

$$(n - ks - |X|)(1 - \epsilon)s \geq (1 - k\epsilon)sn,$$

and solving we see that

$$|X| \geq \frac{\epsilon(k-1)}{(1-\epsilon)}n - ks.$$

Since $k \geq 2$ and $\epsilon > 0$, we see that $|X|$ grows large as n grows large. Then, if

$$|X| > \binom{s}{t}^k (t-1)$$

we can select t vertices that will form the final partite set we desire. □

Theorem 31 *Let G be a graph of order n with at least $(1 - \frac{1}{k} + \epsilon)\frac{n^2}{2}$ edges. Then for n sufficiently large, G contains a $K_{(k+1)(t)}$.*

Proof. Remove a vertex of degree less than $(1 - 1/k + \frac{\epsilon}{2})|V(G)|$ if any exist. Now, in the graph that remains, repeat this process and continue to repeat this process as often as possible. Suppose that at some point in this process we are unable to continue; that is, suppose we are left with a graph H in which all vertices have degree at least

$$(1 - \frac{1}{k} + \frac{\epsilon}{2})|V(H)|.$$

Let $|V(H)| = N$; then if N is sufficiently large, the result will follow from our last theorem. Then, all that remains is for us to show that N cannot be "too small." That is, we wish to show that N is bounded below by a function that grows as n grows.

In the construction of H, the number of edges we removed is at most

$$\begin{aligned}\sum_{j=N+1}^{n} j(1 - \frac{1}{k} + \frac{\epsilon}{2}) &= \left(\binom{n+1}{2} - \binom{N+1}{2}\right)(1 - \frac{1}{k} + \frac{\epsilon}{2}) \\ &\leq \left(\binom{n}{2} - \binom{N}{2}\right)(1 - \frac{1}{k} + \frac{\epsilon}{2}) + (n - N).\end{aligned}$$

The graph H has at most $\binom{N}{2}$ edges, and, thus,

$$(1 - \frac{1}{k} + \epsilon)\binom{n}{2} \leq |E(G)| \leq (1 - \frac{1}{k} + \frac{\epsilon}{2})\left(\binom{n}{2} - \binom{N}{2}\right) + (n - N) + \binom{N}{2}.$$

Thus,

$$\frac{\epsilon}{2}\binom{n}{2} \leq \left(\frac{1}{k} - \frac{\epsilon}{2}\right)\binom{N}{2} + (n - N).$$

Hence, we see that N grows large if n grows large.

Finally, to see that the process of removing vertices of small degree must stop, suppose that it does not stop and examine the sum on the number of edges removed (as we did above). In this case we would have at most $(1 - \frac{1}{k} + \frac{\epsilon}{2})\frac{n^2}{2}$ edges in G, a contradiction. Hence, the process must stop and the result is proved . □

The next result has been the goal of our work in this section. It tells us that the forbidden subgraph's structure is only somewhat responsible for the extremal number. That is, the exact structure of the graph is not as important as the chromatic number.

The significance of the next result has lead to the following definition: Given a family of graphs F, the *subchromatic number* is defined to be

$$\Psi(F) = min\{\chi(G) : G \in F\} - 1.$$

The following result of Erdös and Simonovits [24] is an easy consequence of the Erdös - Stone Theorems.

Theorem 32 *If F is a family of graphs with $\Psi(F) = p$, then*

$$ex(n, F) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

Proof. Since each $G \in F$ is not p-colorable, we see that G is not a subgraph of $T_{n,p}$. Hence,

$$ex(n, F) \geq |E(T_{n,p})| = \left(1 - \frac{1}{p}\right)\frac{n^2}{2} + O(n).$$

On the other hand, there is some $G_0 \in F$ with $\chi(G_0) = p + 1$ and say $|V(G_0)| = m$. Now the Erdös - Stone Theorem asserts that

$$ex(n, K_{(p+1)(m)}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

Since G_0 is a subgraph of $K_{(p+1)(m)}$, we have that

$$ex(n, F) \leq ex(n, K_{(p+1)(m)}) \leq \left(1 - \frac{1}{p} + o(1)\right)\binom{n}{2}.$$

□

The following is an immediate corollary.

Corollary 33

$$lim_{n\to\infty}\frac{ex(n; G)}{n^2} = \frac{1}{2}\left(1 - \frac{1}{\chi(G) - 1}\right).$$

The structure of extremal graphs is fairly stable, in the sense that graphs that are nearly extremal (that is, do not contain the forbidden graph or graphs but have nearly as many edges as the extremal graphs) have a structure that is close to that of extremal graphs. That is, we need not make a great many changes in the edge set of a nearly extremal graph to obtain an extremal graph. This idea is expressed in our next result, the combined efforts of Erdös [14] [15] and Simonovits [44].

Theorem 34 *(The First Stability Theorem) Let F be a family of forbidden graphs with subchromatic number p. For every $\epsilon > 0$, there exists a $\delta > 0$ and an n_ϵ such that if G^n is F-free and if, for $n > n_\epsilon$,*

$$|E(G^n)| > ex(n; F) - \delta n^2,$$

then G^n can be obtained from $T_{n,p}$ by changing at most ϵn^2 edges.

The name "first stability theorem" clearly implies that there are others. These results are beyond the scope of this paper, but the interested reader is advised to see [43] and [4].

Our next theorem can be proven using the first stability theorem and is due to the combined work of Erdös and Simonovits [14] [15] and [44].

For our next result we need the following idea. Consider a partition of the vertex set of G^n as say $S_1, \ldots, S_p$ and the p-partite graph $K_{s_1,\ldots,s_p}$ corresponding to this partition of $V(G^n)$, where $s_i = |S_i|$. An edge vw is called an *extra edge* if it is not in $K_{s_1,\ldots,s_p}$ but is in G^n (similarly, an edge is missing if it is in $K_{s_1,\ldots,s_p}$ but not in G^n). For a given p, the partition $S_1, \ldots, S_p$ is *optimal* if the number of missing edges is minimum. Finally, for a given vertex v, let $b(v)$ denote the number of extra edges at v.

Theorem 35 *(The Asymptotic Structure Theorem) Let F be a family of forbidden subgraphs with $\Psi(F) = p$. If S^n is any extremal graph for F, then it can be obtained from $T_{n,p}$ by deleting and adding at most $o(n^2)$ edges. Furthermore, if F is a finite family, then*

$$\frac{\delta(S^n)}{n} = 1 - \frac{1}{p} + o(1).$$

Sketch of Proof. The first part of the theorem follows from Theorem 32 and the First Stability Theorem.

For the second part, consider an optimal partition $R_1, R_2, \ldots, R_p$ of $V(S^n)$ and assume R_1 has minimum order. Then, $|R_1| \leq \frac{n}{p}$ and by the First Stability Theorem

$$\sum_{v \in R_1} b(v) = o(n^2).$$

If r denotes the maximum order of a graph in F, take r vertices $v_1, \ldots, v_r$ with $\sum_{i=1}^{r} b(v_i)$ minimum. Clearly for some $c > 0$, $|R_1| > cn$. Thus,

$$\sum_{i=1}^{r} b(v_i) \leq \frac{r}{|R_1|} \sum_{v \in R_1} b(v) = o(n).$$

Now apply symmetrization in a slightly modified form. For an arbitrary vertex v in S^n, delete all incident edges and join v to all vertices adjacent to each of $v_1, \ldots, v_r$. The resulting graph S^* contains no member of F. Further, $|E(S^n)| \leq |E(S^*)|$. Hence,

$$deg\, v \geq |\cap_{i=1}^{r} N(v_i)| \geq |\cup_{j=2}^{r} R_j| - \sum_{i=1}^{r} b(v_i) \geq n - \frac{n}{p} - o(n)$$

and the result follows. □

We next present an easy but useful result on the behavior of $ex(n; F)$.

Theorem 36 *For every family F, $\frac{ex(n;F)}{\binom{n}{2}}$ is decreasing as $n \to \infty$.*

Proof. For a fixed extremal graph H^m, take all $\binom{m}{n}$ subgraphs of order n, say $G_1, \ldots, G_t$. Each edge of H^m is in $\binom{m-2}{n-2}$ of the G_i's and, thus,

$$\binom{m-2}{n-2}|E(H^m)| \leq \sum_{i<t} |E(G_i)| \leq \binom{m}{n}|E(H^n)|.$$

But this implies that

$$\frac{|E(H^m)|}{\binom{m}{2}} \leq \frac{|E(H^n)|}{\binom{n}{2}}.$$

□

We finish our study of the structure of extremal graphs by trying to determine when the Turán graph is the extremal graph for a family of graphs F. We will see that $T_{n,p}$ is fundamental to extremal graphs. Once again this result is due to Simonovits [44].

Theorem 37 *A family F has $T_{n,p}$ as an extremal graph (for n sufficiently large) if, and only if, some $G \in F$ has an edge e such that $p = \chi(G-e) = \Psi(F)$. Furthermore, if $T_{n,p}$ is extremal for F for infinitely many values of n, then it is the only extremal graph (again, provided n is sufficiently large).*

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