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ASCENDING SUBGRAPH DECOMPOSITION FOR FORESTS

R. J. FAUDREE*, MEMPHIS STATE UNIVERSITY
R. J. GOULD*, EMORY UNIVERSITY

Abstract. It has been conjectured that if a graph G has $\binom{n+1}{2}$ edges, then the edge set of G can be partitioned into n graphs $G_1, G_2, \ldots, G_n$ such that G_i has i edges $(1 \leq i \leq n)$, and G_i is isomorphic to a subgraph of G_{i+1} $(1 \leq i < n)$. Such a graph G is said to have an *ascending subgraph decomposition (ASD)*. It will be shown that any forest with $\binom{n+1}{2}$ edges has an ASD such that all the graphs in the decomposition are star forests.

1. INTRODUCTION

A graph G is decomposed into subgraphs $G_1, G_2, \ldots, G_n$ if the edge set $E(G)$ of G is partitioned into the n sets $E(G_i)$, $(1 \leq i \leq n)$. A graph G with $\binom{n+1}{2}$ edges has an *ascending subgraph decomposition* (abbreviated ASD) if G can be decomposed into subgraphs G_i $(1 \leq i \leq n)$ such that G_i has size i and G_i is isomorphic to a subgraph of G_{i+1} for $i < n$. The most important problem concerning an ASD is given in the following conjecture which was stated in [1].

CONJECTURE: [1]. *If G is a graph with $\binom{n+1}{2}$ edges, then G has as ASD.*

The conjecture has been verified for several special classes of graphs. In [3], star forests and dense graphs were considered and the following two theorems were proved.

THEOREM A [3]. *If F is a star forest with $\binom{n+1}{2}$ edges, then F has an ASD.*

THEOREM B [3]. *If G is a graph with $\binom{n+1}{2}$ edges, and at most $n+2$ vertices, then G has an ASD.*

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In each of the two previous theorems, as well as the following theorem, all of the graphs in the decomposition are star forests. Using matchings as the graphs in the decomposition, the following theorem and some related results were verified by Fu in [4].

THEOREM C [4]. *If a graph G has $\binom{n+1}{2}$ edges and maximum degree $\Delta(G) \leq \lfloor (n-1)/2 \rfloor$, then G has an ASD.*

This last result was generalized in [2] by increasing the upper bound on the maximum degree condition that implies an ASD and by allowing the graphs in the decomposition to be the vertex disjoint union of short paths, and not just matchings. In particular, the following theorem was proved.

THEOREM D [2]. *If G is a graph with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (2-\sqrt{2})n \rfloor$, then G has an ASD. Also, each of the graphs in the decomposition of G is the disjoint union of paths of length at most 3.*

If the graph G is a forest, a weaker restriction on the maximum degree can be shown to imply an ASD.

THEOREM E [2]. *If G is a forest with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (3-\sqrt{3})n/2 \rfloor$, then G has an ASD. Also, each of the graphs in the decomposition of G is the disjoint union of paths of length at most 2.*

We will generalize this last result with the following theorem, which verifies that all forests have an ASD.

THEOREM 1. *If F is a forest with $\binom{n+1}{2}$ edges, then F has an ASD. Also, each of the graphs in the decomposition of F is the disjoint union of stars.*

2. MAIN RESULTS

Notation and definitions not explicitly mentioned will follow [5]. Frequently used notation will be described in the first part of this section, and any other special notation or definitions will be presented as they are needed. In preparation for the proof of Theorem 1, we introduce some terminology.

An ASD of a forest F with $\binom{n+1}{2}$ edges, which we will denote by $\mathcal{D}$, is a partition of the edge set $E(F)$ of F into n sets $E(F_i)$, $(1 \leq i \leq n)$ such that F_i has i edges and F_i is isomorphic to a subgraph of F_{i+1} for $i < n$. It is sometimes convenient to consider the decomposition $\mathcal{D}$ as a coloring of the edges of F with n colors $\{1, 2, \cdots, n\}$ such that the F_i is the subgraph of edges colored i. We will routinely use the coloring intrepretation in the proof of the main theorem.

If a graph G is the union of two vertex disjoint subgraphs H_1 and H_2, then this will be denoted by $G = H_1 \cup H_2$. Thus if e is any edge of a forest F, then $F - e = H_1 \cup H_2$ for appropriate (not necessarily unique) subforests H_1 and H_2. Of course, if F is a tree, then the subtrees H_1 and H_2 are unique.

In the proof of the main theorem an induction proof will be used, and subforests will be obtained by deleting vertices and edges from the initial forest. Thus, it will be convenient to develop rather specialized notation to easily describe such graphs. Associated with a vertex v of degree $t+1$ in a tree T are $t+1$ edges $e_1, e_2, \cdots, e_{t+1}$ and corresponding components $C_1, C_2, \cdots, C_{t+1}$ such that

$$T - v = C_1 \cup C_2 \cup \cdots \cup C_{t+1}.$$

The edge e_i joins v and the component C_i. These edges and components depend on the vertex v and the tree T, but no reference will be made to v or T, when this leads to no confusion.

Another decomposition of T is obtained by deleting the edge $e = e_{t+1}$, (or any other edge e_i) which gives the following:

$$T - e = C_e \cup H_e,$$

where $C_e = C_{t+1}$ and H_e is the tree spanned by a star of degree t with center at the vertex v and the t subtrees $C_1 \cup C_2 \cup \cdots \cup C_t$. Some of the components C_j could consist of just a single vertex, so we will assume for each i that $c_i = |C_i|$, that $c_1 \geq c_2 \geq \cdots \geq c_t$, and that for some s (possibly t) $c_s > 1$, but $c_{s+1} = 1$. Thus, in this case, the tree H_e has $t - s$ endedges adjacent to the vertex v, and there are s nontrivial components in $H_e - v$. Note that in the remainder of the paper, anytime a vertex v and incident edge e is selected in a forest F, we will assume that we have the associated e_i's, C_i's, c_i's and H_e without having to explicitly describe them.

Since any edge e of a forest F is in some tree T that is a component of the forest, there is a natural decomposition of $F - e$ given by

$$F - e = H_e \cup C_e \cup F',$$

where F' is the forest of components of F distinct from T, and H_e and C_e are the trees of the previous paragraph. If e and f are distinct edges such that H_e and H_f are vertex disjoint, then there is a subforest F' of F such that

$$F - e - f = H_e \cup H_f \cup F'.$$

We are now prepared to give the proof of the following theorem.

THEOREM 1. *If F is a forest with $\binom{n+1}{2}$ edges, then F has an ASD. Also, each of the graphs in the decomposition of F is the disjoint union of stars.*

PROOF: The proof will be by induction on n. It is trivial to verify the result for $n = 1$, 2, and 3. We proceed by induction on n, and assume that all forests with $\binom{m+1}{2}$ edges for $m < n$ have an ASD with star forests for each of the terms. Several cases, which we will show are exhaustive, will be considered.

Case 1: *For some edge e of F, the forest $F - e = F' \cup H$, where H is a subgraph with $n-1$ edges.*

The forest F' has $\binom{n+1}{2} - n = \binom{n}{2}$ edges, so by induction, F' has an ASD of star forests, which we will denote by $\mathcal{D}'$. The ASD for F' can be extended to $\mathcal{D}$, an ASD, for F by replacing for each i the i^{th} color in $\mathcal{D}'$ by the $(i+1)^{th}$ color, coloring the edge e with 1, and arbitrarly coloring the $n-1$ edges of H with the colors $\{2, 3, \cdots, n\}$. Note that a vertex disjoint edge has been added to each of the color classes, so $\mathcal{D}$ consists of star forests, which completes the proof of case 1.

Case 2: *For some edges $\{f_1, f_2, \cdots, f_r\}$ and a union of components H of F, the forest $F - \{f_1, f_2, \cdots, f_r\} = H_{f_1} \cup H_{f_2} \cup \cdots \cup H_{f_r} \cup H \cup F'$ for some subforest F' with $n-2 \geq |E(H_{f_1})| \geq |E(H_{f_2})| \geq \cdots \geq |E(H_{f_r})| \geq 1$, $n-2 \geq |E(H)|$, and $|E(H)| + \sum_{i=1}^{r} |E(H_{f_i})| \geq n-1$.*

We can assume, with no loss of generality, that a set A of n edges can be selected that contain e_1, all of the edges H and of the H_{f_i} for $1 \leq i < r$, and an appropriate number of edges from H_{f_r}. The forest $F'' = F - A$ has $\binom{n}{2}$ edges, and therefore by induction has a star forest ASD, which we denote by $\mathcal{D}'$.

Change the color of each edge in F'' from i to $i+1$. Next, we describe an assignment of a different color to each edge in A such that an independent edge is added to each color class of the translated color classes of $\mathcal{D}'$, which will result in a star forest ASD for F. Color edge e_1 with 1. For each i $(1 \leq i < r)$, assign the color of edge e_{i+1} in F'', if it has not already been assigned, to an edge in H_{e_i}. For each of the colors of an edge in $H_{e_r} \cap F''$, if it has not already been assigned, assign this color to an edge of H_{e_1}. This can be done since $|E(H_{f_1})| \geq |E(H_{f_r})|$. Assign the remaining colors arbitrarly to the remaining edges of A. This gives a coloring that implies that F has a star forest ASD, and completes the proof of case 2.

Case 3: *There is a vertex v of degree $t+1$ with associated edges $\{e_1, e_2, \cdots, e_{t+1} = e\}$ and components $\{C_1, C_2, \cdots, C_{t+1}\}$ such that $t \leq n-1$, $n-2 \geq c_1 \geq c_2 \geq \cdots \geq c_t$, and $|E(H_e)| \geq n-1$.*

Subcase i: *There is an integer $r \leq s$ such that $n-1-t+s \leq c_1+c_2+\cdots+c_r+r < n-1$.*

Select a set A of n edges that contains $\{e, e_1, e_2, \cdots, e_r\}$, the edges of the components $\{C_1, C_2, \cdots, C_r\}$, and $n-r-1-\sum_{i=1}^{r} c_i$ edges e_j for $j > s$ (these will be endedges). The forest $F' = F - A$ has $\binom{n}{2}$ edges, and therefore by induction has a star forest ASD, which we denote by $\mathcal{D}'$.

We will change the color of each edge in F' from i to $i+1$, and show that an assignment of a distinct color can be made to each edge in A such that an independent edge is added to each color class of the translated color classes of $\mathcal{D}'$. This will give a star forest ASD for F. In this assignment, color edge e with 1. There are $t-n+1+\sum_{i=1}^{r} c_i < \sum_{i=1}^{r} c_i$ edges in F' that are adjacent to v. Assign the colors that appear on these edges of F' to edges in $\cup_{i=1}^{r} C_i$, and assign the remaining colors arbitrarly to the remaining edges of A. This gives the required result, and completes the proof of subcase i.

Subcase ii: *There is an integer $r < s$ such that $c_1 + c_2 + \cdots + c_r + r < n-1 < c_1 + c_2 + \cdots + c_{r+1} + r + 1$.*

We will again select a set A of n edges that contain $\{e_1, e_2, \cdots, e_{r+1}\}$, the edges of the components $\{C_1, C_2, \cdots, C_r\}$, and $n-r-1-\sum_{i=1}^{r} c_i$ arbitrary edges from C_{r+1}. The forest $F' = F - A$ has $\binom{n}{2}$ edges, and therefore by induction has a star forest ASD, which we denote by $\mathcal{D}'$.

As in the previous subcase, we will change the color of each edge in F' from i to $i+1$, and show that an assignment of colors can be made to the edges in A such that an independent edge is added to each color class of the translated color classes of $\mathcal{D}'$. Start by coloring th edge e_{r+1} with 1. There are $t - r \leq n - r - 1$ edges in F' that are adjacent to v. Assign the colors that appear on these edges of F' to edges in $\cup_{i=1}^{r+1}(C_i \cap A)$. In making this assignment, make sure that any color assigned to an an edge in $C_{r+1} \cap A$ is distinct from the colors of the edges in $C_{r+1} \cap F'$. This can be done, since $c_1 \geq c_{r+1}$, so any color appearing in $C_{r+1} \cap F'$ can be assigned to an edge of C_1. Assign the remaining colors arbitrarly to the remaining edges of A. This is a coloring that implies that F has a star forest ASD, and completes the proof of subcase *ii*.

Since the conditions of either subcase *i* or subcase *ii* must be satisfied, this completes the proof of case 3.

Case 4: *There are vertices v and v' such that v is of degree $t+1 > n$ with associated edges $\{e_1, e_2, \cdots, e_{t+1} = e\}$ and components $\{C_1, C_2, \cdots, C_{t+1}\}$ with $c_1 \geq c_2 \geq \cdots \geq c_t$ and $\sum_{i=1}^{t} c_i \leq n-2$, and such that v' has corresponding parameters that are marked by a $'$.*

Let $e = e_{t+1}$, $e' = e'_{t'+1}$ (possibly $e = e'$), $q = \sum_{i=1}^{t} c_i$, and $q' = \sum_{i=1}^{t'} c'_i$. With no loss of generality we can assume that $t \geq t'$ and the subtrees H_e and $H_{e'}$, which have $t+q$ and $t'+q'$ edges respectively, are vertex disjoint. Let $m = t + t' + q + q' + 1 + \delta$ (where $\delta = 0$ if $e = e'$ and $\delta = 1$ otherwise), which is the number of edges in the graph H spanned by $H_e \cup H_{e'} \cup \{e, e'\}$.

Select p and r such that

$$m = n + (n-1) + \cdots + (n-p+1) + r,$$

for $0 \leq r < n-p$. We will select an appropriate subgraph L of H with $m - r$ edges, and consider the subforest $F' = F - L$, which has $\binom{n+p+1}{2}$ edges. By induction F' has a star forest ASD, and we will show that this can be extented to be a star forest ASD of F. Two subcases will be considered.

Subcase i: $r \geq q + q'$

Let X be a set of r edges that contain all of the edges in $C_1 \cup C_2 \cup \cdots \cup C_t \cup C'_1 \cup C'_2 \cup \cdots \cup C'_{t'}$ and an additional $r - q - q'$ edges of H that are not adjacent to any of the previously chosen edges. Let $L = H - X$, and consider the forest $F' = F - L$ that has $\binom{n-p+1}{2}$ edges. By the induction assumption F' has a star forest ASD, which we will denote by $\mathcal{D}'$. Change the coloring of each edge in F' from i to $i + p + 1$. We will extend this coloring to a star forest ASD of F.

The graph L has $m - r = n + (n-1) + \cdots + (n-p+1)$ edges that are in two stars. We want to color these edges with n colors such that for $1 \leq i \leq p$, the edges of color i form a star with i edges, and that the edges in each of the remaining colors form a star with p edges. This can clearly be done. In addition, we want to make this coloring such that any edge assigned color j is not adjacent to an edge of F' with color j, and such that edges e and e' have colors from $\{1, 2, \cdots, p\}$. This can be done using the same coloring techniques employed in cases 2 and 3, since there are at most $r < n - p$ colors on the edges in X, but it may require that the edges of X adjacent to v or v' be appropriately chosen. This completes the proof of subcase ii.

Subcase ii: $r < q + q'$

Order the edges in $C_1 \cup C_2 \cup \cdots \cup C_t \cup C'_1 \cup C'_2 \cup \cdots \cup C'_{t'}$ by starting with the edges in C_1, following with the edges in C_2 and continuing until reaching the edges of $C'_{t'}$. Select the first r edges in this order, and delete these edges from H to obtain a graph L that has $n + (n-1) + \cdots + (n-p+1) = pn + \binom{p}{2}$ edges. Therefore, the forest $F' = F - L$ has $\binom{n-p+1}{2}$ edges, so by the induction assumption F' has a star forest ASD, which we will denote by $\mathcal{D}'$. Change the color of each edge in F' from i to $i + p$.

As in the previous subcase, the graph L has $m - r = n + (n-1) + \cdots + (n-p+1)$ edges. We want to color these edges with n colors such that for $1 \leq i \leq p$, the edges of color i form a star with $i - 1$ edges, and an independent edge, and that the edges in each of the remaining colors form a star with $p - 1$ edges and an independent edge. In addition, we want to make this coloring such that any edge assigned color j is not adjacent to an edge of F' with color j, and such that edges e and e' have colors from $\{1, 2, \cdots, p\}$. Just as in the previous case, this can be done since $r < n - p$. This completes the proof of subcase ii, and of case 4.

Case 5: *For some vertex v of F, the forest $F - v$ has at most $n - 2$ edges.*

If v has degree t, then associated with v are edges $e_1, e_2, \cdots, e_t$ along with corresponding components $C_1, C_2, \cdots, C_t$, and therefore

$$T - v = C_0 \cup C_2 \cup \cdots \cup C_t,$$

where C_0 is the subforest of components (possibly empty) of F distinct from the component containing v. By assumption, there are at $r \leq n - 2$ edges in all of the C_i's.

Arbitrarly assign colors $\{n, n-1, \cdots, n-r+1\}$ to the r edges that are not incident to v. Since $r \leq n - 2$, an assignment of colors $\{1, 2, \cdots, n\}$ can be made to the n edges $e_1, e_2, \cdots, e_n$, such that the color assigned to e_i is different from all of the colors assigned to edges in C_i. Note that at most the first r components $C_1, C_2, \cdots, C_r$ have an edge, so the e_j's for $n < j \leq t$ are not incident to any edges of the components of $F - v$. Assign colors to these remaining edges such that there precisely k edges of color k for each k. This coloring implies that F has a star forest ASD, and completes the proof of case 5.

To complete the proof, it is sufficient to show that any forest F will fall into one of the five cases considered. To do this we make the following observation. If v is a vertex of degree $t+1$ in a forest F with associated edges $e_1, e_2, \cdots, e_{t+1}$ and corresponding components $C_1, C_2, \cdots, C_{t+1}$, then

$$F - v = C_0 \cup C_2 \cup \cdots \cup C_{t+1},$$

where C_0 is the forest of components of F not containing v. Then observe that if $|E(H_{e_{t+1}})| \geq n$, then either $|C_i| \leq n - 2$ for $1 \leq i \leq t$, or one can consider the vertex v_1 of C_1 that is adjacent to v and look at the decomposition of F obtained from this vertex. A repetition of this argument will eventually yield a vertex, say v, such that $|E(H_{e_{t+1}})| \geq n$, and $|C_i| \leq n - 2$ for $1 \leq i \leq t$.

If the forest F does not satisfy the conditions of case 1, or case 3, then by the observation of the previous paragraph there must be a vertex v of degree exceeding n such that the components associated with v, with one possible exception, all have at most $n - 2$ edges. If there are two such vertices v and v' with this property, then the conditions of case 4 are satisfied. Therefore, if none of the conditions of the first four cases are satisfied, case 5 must apply. Thus the five cases exhaust all the possibilites for F, and so F must have a star forest ASD. This completes the proof of Theorem 1. ∎

3. OPEN QUESTIONS

There are numerous open questions concerning the existence and nature of ASD's for graphs. The major and most difficult question is, of course, to determine if every graph with $\binom{n+1}{2}$ edges has an ASD. There is an interesting class of graphs related to forests for which the same question can be considered; for example, do all "sparse" graphs (or unicyclic graphs) have an ASD? One can also restrict the nature of the graphs that are allowed in the decompositions of the ASD. There is some evidence to indicate that any graph has an ASD with star forests as the decomposition graphs.

Since it seems to be difficult to prove that every graph G has an ASD, one approach is to determine how closely one can "approximate" an ASD for G. If one only requires that the subgraphs G_i $(1 \leq i \leq n)$ in the decomposition of G satisfy the subgraph property and that G_i has size at most i, then the decomposition has at least n terms. How small can you make the number of terms with this definition? On the other hand, if you require that each G_i has size at least i, then the number of terms in the decomposition is at most n. How large can you make the number of terms in this case? Thus, there are natural upper and lower bound questions on the length of an ASD for a general graph.

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