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NEIGHBORHOOD CONDITIONS

AND

EDGE DISJOINT HAMILTONIAN CYCLES

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Abstract

A graph G satisfies the neighborhood condition $NC(G) \geq m$, if for each pair of nonadjacent vertices of G, the union of their neighborhoods has at least m vertices. For k a fixed positive integer, let G be a graph of order n which satisfies the following conditions: $\delta(G) \geq 4k + 1$, $\kappa_1(G) \geq 2k$, $\kappa_1(G-v) \geq k$ for any vertex v in G, and $NC(G) \geq 2(n + C)/3$ for some constant $C = C(k)$. It is shown that if n is sufficiently large, then G contains k edge disjoint Hamiltonian cycles. Similar conditions are shown to give disjoint perfect matchings.

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Introduction

A graph G of order n is *Hamiltonian* if it has a cycle C_n containing all of the vertices of G. Many conditions, especially degree conditions, have been shown to be sufficient for a graph to be Hamiltonian. One of the earliest conditions involved the sum of degrees of nonadjacent vertices and was due to O. Ore. A graph G satisfies the *degree condition* $DC(G) \geq m$, if for each pair of nonadjacent vertices u and v of G, $d_G(u) + d_G(v) \geq m$.

__Theorem A__: (Ore [9]) Let G be a graph of order $n \geq 3$. If $DC(G) \geq n$, then G is Hamiltonian.

The graph H of order n obtained from a complete graph K_{n-1} by adjoining a new vertex that is adjacent to a single vertex of the complete graph is not Hamiltonian. This example implies the degree condition in Theorem A is necessary, since $DC(H) = n-1$.

There have been numerous generalizations of this degree condition that have been shown to be sufficient for a graph to be Hamiltonian. Another condition that is patterned after the one of Ore, is the following condition that involves the neighborhoods of vertices. A graph G satisfies the *neighborhood condition* $NC(G) \geq m$, if for each pair of nonadjacent vertices u and v of G,

$$|N_G(u) \cup N_G(v)| \geq m,$$

where $N_G(w)$ is the set of vertices adjacent to w and is called the *neighborhood* of w in G.

__Theorem B__: [4] If G is a 2-connected graph of order $n \geq 3$, and $NC(G) \geq (2n - 1)/3$, then G is Hamiltonian.

The result in Theorem B is also sharp in that neither the connectivity condition nor the neighborhood condition can be weakened without losing the Hamiltonian property. These conditions will be discussed in more detail later.

One way to generalize the result of Ore is to determine a similar condition which gives, not just one, but multiple edge disjoint Hamiltonian cycles. In this direction, the following result was proved for large order graphs.

Theorem C: [5] Let k be a fixed positive integer, and G a graph of order n. If $DC(G) \geq n + 2k - 2$, and n is sufficiently large, then G contains k edge disjoint Hamiltonian cycles.

The degree condition in Theorem C is necessary. The graph H obtained from a complete graph K_{n-1} by adjoining a new vertex that is adjacent to a $2k - 1$ vertices of the complete graph does not contain k edge disjoint Hamiltonian cycles, but $DC(H) = n + 2k - 3$. However, the reason the graph H does not contain k edge disjoint Hamiltonian cycles is because of the vertex of degree $2k - 1$. It is natural to ask if the degree condition DC can be weakened, even to the condition of Ore, if some restriction is placed on the minimum degree in the graph. The following result supports a conjecture of this type.

Theorem D: [7] If G is a graph of order $n \geq 20$ with minimal degree $\delta(G) \geq 5$ and $DC(G) \geq n$, then G contains two edge disjoint Hamiltonian cycles.

A natural extension of Theorem D is that any graph G of sufficiently large order n with minimal degree $\delta(G) \geq 2k + 1$ and $DC(G) \geq n$ contains k edge disjoint cycles. The corresponding extension of Theorem B would involve determining appropriate connectivity and minimum degree conditions that, along with the neighborhood condition $NC(G) \geq (2n-1)/3$, insure that the graph G has k edge disjoint Hamiltonian cycles. In what follows $\kappa(G)$ is the *connectivity*, $\kappa_1(G)$ is the *edge connectivity*, and

$\delta(G)$ is the *minimum degree* of a graph G. We will prove the following extension of Theorem B.

Theorem 1: Let k be a fixed positive integer, and G a graph of order n which satisfies the following conditons:

1) $NC(G) \geq 2(n + C)/3$ for some $C = C(k)$,

2) $\delta(G) \geq 4k + 1$,

3) $\kappa_2(G) \geq 2k$, and

4) $\kappa_2(G - v) \geq k$ for all vertices v of G.

Then, if n is sufficiently large, G contains k edge disjoint Hamiltonian cycles.

A immediate corollary of Theorem 1 is that if n is even, then the graph G contains $2k$ edge disjoint perfect matchings. In the case of matchings some of the restrictions of Theorem 1 can be weakened or removed. These weaker conditions are stated in the following analogous result.

Theorem 2: Let m be a fixed positive integer, and G a graph of even order n that satisfies the following conditons:

1) $NC(G) \geq (2n + C)/3$ for some $C = C(k)$,

2) $\delta(G) \geq 2m$, and

3) $\kappa_2(G) \geq m$.

Then, if n is sufficiently large, G contains m edge disjoint perfect matchings.

The necessity and sharpness of the conditions on minimal degree, connectivity, and edge connectivity in both Theorem 1 and Theorem 2 will be discussed in the next section.

Examples

Any theorem that gives a sufficient condition for a graph G to have k edge disjoint Hamiltonian cycles, and is based on a neighborhood condition NC, must have all of the types of restrictions listed in Theorem 1. However the restrictions on some of the parameters are not sharp. We follow with four examples of graphs that do not contain k edge disjoint Hamiltonian cycles, satisfy all but one of the four conditions of Theorem 1, and give a necessary lower bound on the parameter considered in that condition. In each case, n is considered to be sufficiently large in order to avoid exceptions for a few small order cases.

(1) Let H be the disjoint union of complete graphs $K_a \cup K_b \cup K_c$ with $\lfloor (n-2)/3 \rfloor \leq a \leq b \leq c \leq \lceil (n-2)/3 \rceil$ and $a + b + c = n - 2$, and let $G_1 = H + K_2$. The graph G_1 satisfies (2), (3) and (4), is not Hamiltonian, and $NC(G_1) \geq 2\lfloor (n-2)/3 \rfloor$.

(2) The graph $G_2 = (K_{2k-1} \cup K_{n-2k-1}) + K_2$ has $\delta(G) = 2k$, but it does not contain k edge disjoint Hamiltonian cycles, because there are not enough edges in the subgraph K_{2k-1} to construct the necessary k edge disjoint Hamiltonian paths. Conditions (1), (3), and (4) are satisfied by G_2.

(3) A Hamiltonian graph is 2-connected, but the neighborbhood condition NC does not imply any connectivity in the graph. For example, the disconnected graph $H = K_{\lfloor n/2 \rfloor} \cup K_{\lceil n/2 \rceil}$ has $NC(H) = n - 2$. The graph G_3 obtained from H by adding a matching with $2k - 1$ edges between the components of H does not have k edge disjoint Hamiltonian cycles, satisfies conditions (1), (2) and (4), and $\kappa_1(G) = 2k - 1$.

(4) Let G_4 be the graph obtained from the graph H, which was just described in (3), by adding all of the edges from a vertex v in the first component to all of the vertices in the second component, and an additional $k - 1$ edges betweem a second vertex in the first component and $k - 1$ vertices of the second component. This graph does

not contain k edge disjoint Hamiltonian cycles, satisfies conditions (1), (2), and (3), and $\kappa_1(G-v) = k - 1$.

The examples just described indicate that conditions (3) and (4) of Theorem 1 cannot be weakened. Also, probably conditions (1) and (2) can be weakened to agree with the examples, but the proof techniques that will be used require the additional strength.

Each of the types of conditions given in (1), (2) and (3) of Theorem 2 are also necessary for a graph G to have m perfect matchings, and the examples which verify this are either identical or similar to those described for Theorem 1. If $n = 3p + 1$ and p is odd, then $H = K_1 + (K_p \cup K_p \cup K_p)$ contains no perfect matching and $NC(H) = (2n - 5)/3$. Clearly, any graph with m edge disjoint perfect matchings must have minimum degree m. Also, any graph which is made up of two vertex disjoint odd order graphs with only $m - 1$ edges between them, cannot have m edge disjoint perfect matchings, hence $\kappa_1(G) \geq m$. Thus, condition (3) of Theorem 2 is sharp. However, conditions (1) and (2) are probabley not sharp, with examples similar to those previously given suggesting appropriate values for the parameters.

Preliminary Results and Notation

In the proof of Theorem 1, disjoint Hamiltonian cycles in a graph G will, in some cases, be constructed by patching together paths that have been built in dense subgraphs of G. The following result is very useful in proving the existence of such paths. Recall, a path in a graph G that contains all of the vertices of G is called a *Hamiltonian path*, and G is *Hamiltonian Connected* if there is a Hamiltonian path between each pair of vertices of the graph.

Theorem E: [10] *Let G be a graph of order $n \geq 3$. If $DC(G) \geq n + 1$, then G is Hamiltonian connected.*

Repeated application of Theorem A yields the following useful corollary.

Corollary F: Let k be a fixed positive integer and K_m a complete graph of order $m \geq 4k - 1$. For any collection (not necessarily distinct) of k pairs of vertices of K_m, there are k edge disjoint Hamiltonian paths whose endvertices are the k pairs of vertices.

The previous result deals with the case when the paths must terminate in predetermined pairs of vertices. However, in some cases such stringent conditions need not be placed on the endvertices of the paths. Well known 2-factorizations of complete graphs give the following result.

Theorem G: [8] *Let k be a fixed positive integer and K_m a complete graph of order $m \geq 2k + 1$. Then K_m contains k edge disjoint Hamiltonian paths whose endvertices are disjoint pairs of vertices.*

Before giving the proof of Theorem 1, we will introduce some notation that will be used. Most will be standard and can be found in [2], but some will be specialized notation that is convenient for this proof.

If H is a subgraph of G and e is an edge of G, then $G - H$ will denote the graph with the same vertex set as G and with edges that are in G but not H, and $G + e$ is the graph with the edge e added to the edge set of G. In general, we will not distinquish between G, the vertex set $V(G)$, and the edge set $E(G)$, unless doing so will cause confusion. For $x \in G$, $N_H(x)$ will denote the vertices of H which are adjacent to x, and will be called the *neighborhood* of x in H. Also, the *degree*

$d_H(x)$ is the number of elements in $N_H(x)$. When $H = G$ and it is clear which graph is being considered, N_G and d_G will be shortened to just N and d.

An edge with endvertices x and y will be written xy. Likewise, a path P_t with t vertices $\{x_1, x_2, \ldots, x_t\}$ will be expressed as $x_1x_2\ldots x_t$. In some cases the nature of the intermediate vertices is obvious or not crucial, but the endvertices of the path are important. In situations like this, P_t will be expressed as just $P(x_1, x_t)$.

Proofs

Proof of Theorem 1: We suppose that there exists an edge maximal counterexample graph G to the Theorem, and show that this leads to a contradiction. The proof of the Theorem will consist of a series of facts about G, ending with the nonexistence of G.

The maximality of G implies that for any pair of nonadjacent vertices x and y of G, $G + xy$ contains k edge disjoint Hamiltonian cycles. Associated with each edge $xy \notin G$, there are $k - 1$ edge disjoint Hamiltonian cycles $H_1, H_2, \ldots, H_{k-1}$ in G. Let H denote the subgraph of G generated by the edges of these cycles, and let $L = G - H$. The graph L contains no Hamiltonian cycle, but it has a Hamiltonian path $P = x_1x_2\ldots x_n = P(x_1, x_n)$ with $x = x_1$ and $y = x_n$.

In the remainder of the proof we will associate with each pair of nonadjacent vertices x and y, the subgraphs H and L, and the path P. Since it will be clear which graphs G and L we are dealing with, d_G and N_G will be shortened to just d and N respectively, and d_L and N_L by d' and N' respectively. Thus, $d'(v) = d(v) - 2k + 2$ for all $v \in G$.

<u>Fact 1</u>: If $xy \notin G$, then $d(x) + d(y) \leq n + 4k - 5$.

If this is not true, then $d'(x) + d'(y) \geq n$. Since L contains no Hamiltonian cycles, $xx_j \in L$ with $j \leq n$ implies $yx_{j-1} \notin L$. Therefore, $d'(x) + d'(y) < n$, a contradiction.

<u>Fact 2</u>: No two complete subgraphs of G span G.

Let A and B be a partition of the vertices of G with $|A| \leq |B|$, such that the vertices of A and the vertices of B each form complete subgraphs.

First consider the subcase when $|A| \geq 4k + 1$. The existence of k pairs of disjoint edges between A and B follows from conditions (3) and (4) and a simple induction proof. Denote the j^{th} pair by $\{a_{1j}b_{1j}, a_{2j}b_{2j}\}$. By Corollary F there exists k edge disjoint Hamiltonian paths in each of A and B with $P_j(a_{1j}, a_{2j})$ the j^{th} path in A and $P_j'(b_{1j}, b_{2j})$ the j^{th} path in B. Hence, for each j, $\{P_j(a_{1j}, a_{2j}), a_{1j}b_{1j}, P_j'(b_{1j}, b_{2j}), a_{2j}b_{2j}\}$ determines a Hamiltonian cycle in G, and the k cycles are edge disjoint.

In the second subcase, when $2k + 1 \leq |A| \leq 4k$, each vertex of A is adjacent to at least 2 vertices of B. Theorem G implies the existence of k edge disjoint Hamiltonian paths $P_j(a_{1j}, a_{2j})$, $(1 \leq j \leq k)$ in A, such that the $2k$ endvertices are distinct. For each j, there are disjoint edges $a_{1j}b_{1j}$ and $a_{2j}b_{2j}$ with b_{1j} and b_{2j} in B. Just as in the previous subcase, there are k edge disjoint Hamiltonian paths $P_j'(b_{1j}, b_{2j})$ $(1 \leq j \leq k)$ in B. For each j, the paths $P_j(a_{1j}, a_{2j})$ and $P_j'(b_{1j}, v_{2j})$ along with the edges $a_{1j}b_{1j}$ and $a_{2j}b_{2j}$ determine a Hamiltonian cycle in G. This implies there are k edge disjoint Hamiltonian cycles of G.

In the final subcase, when $m = |A| \leq 2k$, each vertex of A is adjacent to at least $4k - m + 2$ vertices of B. There are $t = \lfloor (m-1)/2 \rfloor$ edge disjoint Hamiltonian cycles of the type described in the previous subcase. After the deletion of the edges of these Hamiltonian cycles, each vertex of A is adjacent to at least $4k - m + 1$ vertices of B. This fact, and a straightforward induction argument, implies there are $k - t$ edge disjoint (and disjoint from the t Hamiltonian cycles just described) graphs R_j, $(1 \leq j \leq k-t)$, where each R_j is the union of disjoint paths. The paths in R_j alternate between vertices of A and B, have their endvertices in B, and contain all of the vertices of A. Since B has large order, repeated application of Theorem E implies that each R_j can be extended to a Hamiltonian cycle of G. Also, this can be accomplished such that all of the cycles, including the t Hamiltonian cycles previously described, are edge disjoint.

Therefore, a contradiction is reached in each of the three subcases by exhibiting k edge disjoint Hamiltonian cycles.

<u>Fact 3</u>: $\delta(G) \geq n/6$.

Suppose Fact 3 fails to hold, and v is a vertex with $d(v) = m < n/6$. Let W be the vertices of G that are nonadjacent to v. Then, $d(w) \geq n/2 + 2C/3$ for $w \in W$, so the vertices of W form a complete graph of order $n - m - 1$ by Fact 1. Also, by Fact 1, any vertex of G of degree at least $m + 4k - 2$ must be adjacent to each vertex of W, and hence must have degree at least $n - m - 1$. Partition the vertices of G into two sets A and B, where A is the set of vertices of degree at most $m + 4k - 3$, and B is the remaining set of vertices. It has already been noted that the vertices of B form a complete graph. Since the union of the neighborhoods of each pair of vertices of A is less than $2n/3$, these vertices also form a complete graph, which contradicts Fact 2.

Fact 4: L contains no cycle of length at least $n - 4k$.

Let $C = v_1v_2...v_mv_1$ be a cycle of maximal length in L, and assume that $m \geq n - 4k$, and that v is a vertex not on C. If $vv_r, vv_t \in L$, then $vv_{r+1}, vv_{t+1}, v_{r+1}v_{t+1} \notin L$ by the maximality of the length of C in L. However, any of the last three edges could be in G. Therefore, since $d(v) \geq n/6$, there is, with no loss of generality, an integer r such that $vv_1, vv_r \in L$, but $vv_2, vv_{r+1}, v_2v_{r+1} \notin G$. In addition, r can be chosen to be small; in particular, $r \leq 24k$.

Assume that $d(v_{r+1}) \geq d(v_2)$, and so $d(v_{r+1}) \geq (n + C)/3$. Let $S = \{v_{j+1} : v_j \in (N'(v) \cup N'(v_2))$ and $r \leq j \leq m\}$. Then, for $v_j \in S$, $v_{r+1}v_j \in L$ implies that L has a cycle of length $m + 1$. This gives that $N'(v_{r+1}) \cap S = \emptyset$, which implies that $2(n + C)/3 - 32k + (n + C)/3 - 30k \leq n$, a contradiction.

Fact 5: If $d(y) \geq n/3$, then there does not exist a $p < q$, such that $xx_p \notin G$ but $xx_q \in L$ (along the path $P = P(x,y)$).

Suppose p and q exists and select q is as small as possible and p is a large as possible subject to p being less than q. Hence, for $p < i < q$, $xx_i \in H$ and $q - p < 2k$. Let $X = \{x_j : x_{j+1} \in (N(x) \cup N(x_q)\}$. If for $x_j \in X$, $yx_j \in L$, and $xx_{j+1}, x_qx_{j+1} \notin H$, then L has a cycle of length at least $n - 2k$. Thus, $|X \cap N(y)| \leq 6k$, which gives the inequality $2(n + C)/3 + (n + C)/3 - 6k \leq n$. This gives a contradiction.

Fact 6: It is not possible that both $d(x), d(y) \geq n/3$.

Suppose $d(x), d(y) \geq n/3$ and let p be the largest integer such that $xx_p \in L$, and let q be the smallest integer such that $yx_q \in L$. By Fact 5, $xx_i \in G$ for each $i \leq p$ and $yx_j \in G$ for each $j \geq q$. Thus, $p \leq q$. With no loss of generality we can assume that x and y have been chosen to minimize $q - p$ in the path P associated with x and y. Let A be the vertices of P which precede x_p and B the vertices of P which follow x_q.

No vertex of A is adjacent in L to a vertex of B, because (since Fact 5 holds) this would give a cycle of length at least $n - 4k$, which is prohibited by Fact 4. However, if $uv \notin G$ for u and v in A, then

$$|N'(u) \cup N'(v)| + d'(y) \geq 2(n + C)/3 + n/3 - 6k > n,$$

which implies that u or v is adjacent to a vertex of B. Since this cannot occur, the vertices of A, and likewise those of B, form a complete graph.

Select t such that $p < t \leq q$. If $t < q$ and x_t is adjacent in L to a vertex of A or B, then the minimality of $q - p$ would be contradicted. If $x_t x_j \notin G$ for $x_j \in A$, then by the same count used in the previous paragraph either x_t or x_j is adjacent in L to some vertex of B. Since this is impossible, $t = q$ and and there are no vertices between x_p and x_q. The same reasoning implies x_q is adjacent to each vertex of B and x_p is adjacent to each vertex of A. This contradicts Fact 2, and completes the proof of Fact 6.

<u>Fact 7</u>: G does not exist.

The vertices of G of degree less than $n/3$ form a complete graph by assumption, and the remaining vertices form a complete graph by Fact 6. This is impossible by Fact 2, which completes the proof of Fact 7, and of Theorem 1.

The proof of Theorem 2 closely parallels the proof of Theorem 1. An edge maximal counterexample G can be chosen.

Thus, associated with each edge $xy \notin G$, there are m edge disjoint perfect matchings $H_1, H_2, \ldots, H_m$ in $G + xy$ with $xy \in H_m$. Also, for each $j < m$, $H_j \cup (H_m - xy)$ is a disjoint union of even cycles and a path $P(x,y)$. When neither x nor y have low degree, arguments similar to those in Theorem 1 give a disjoint union of even cycles to replace the cycles and a path. When there is a vertex of low degree, the graph G can be shown to have two complete subgraphs which span G. In this case, matchings of these subgraphs have to properly patched to give perfect matchings of G. Since the nature of the proof is so similar, it is not included here.

Open Questions

Although the types of conditions listed in both Theorem 1 and Theorem 2 are necessary, not all are sharp. It would be of interest to determine the best possible conditions of this type for both Hamiltonian cycles and perfect matchings.

The neighborhood condition NC used in both Theorem 1 and Theorem 2 is defined for pairs of nonadjacent vertices. A natural generalization is to consider a neighborhood condition NC_t, which considers the union of the neighborhoods of any set of t independent vertices, and determine what is needed to insure Hamiltonian cycles and perfect matchings. This more general neighborhood condition has been considered in [1], [3], and [6].

<u>References</u>

[1] N. Alon, R.J. Faudree, and Z. Furedi, *On a Neighborhood Condition and Cliques in Graphs*, <u>Annals N.Y. Acad. Sci.</u>, (to appear).

[2] M. Behzad, G. Chartrand, and L. Lesniak-Foster, <u>Graphs and Digraphs</u>, Prindle, Weber and Schmidt, Boston, (1979).

[3] R.J. Faudree, R.G. Gould, M.S. Jacobson, and L. Lesniak, *On a Neighborhood Condition Implying the Existence of Disjoint Complete Graphs*, (to appear).

[4] R.J. Faudree, R.G. Gould, M.S. Jacobson, and R.H. Schelp, *Neighborhood Unions and Hamiltonian Properties in Graphs*, <u>J. Comb. Theory</u>, (to appear).

[5] R.J. Faudree, C.C. Rousseau, and R.H. Schelp, *Edge-Disjoint hamiltonian Cycles*, <u>Graph Theory and its Applications to Algorithms and Computer Science</u>, Wiley Interscience, (1985), 231-250.

[6] P. Fraisse, *A New Sufficient Condition for Hamiltonian Graphs*, <u>J. Graph Theory</u>, 10, (1986), 405-409.

[7] L. Hao and Z. Yongjin, *Edge-Disjoint Hamiltonian Cycles in Graphs*, <u>Proceedings of the First China - USA International Conference on Graph Theory</u>, Jinan, China, (to appear).

[8] F. Harary, <u>Graph Theory</u>, Addison Weseley, Reading, Mass. (1969).

[9] O. Ore, *Note on Hamiltonian Cycles*, <u>Amer. Math. Monthly</u>, 67, (1960), 55-56.

[10] O. Ore, *Hamiltonian Connected Graphs*, <u>J. de Math. Pures Appl.</u>, 42, (1963), 21-27.