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ORIGINAL PAPER

Results and Problems on Chorded Cycles: A Survey

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Abstract

A *chord* of a cycle C is an edge between two non-consecutive vertices of the cycle. A cycle C in a graph G is *chorded* if the vertex set of C induces at least one chord. In 1961 Posa formulated a natural question: What conditions imply a graph contains a chorded cycle? In this paper, we survey results and problems that relate to Posa's question on chorded cycles in graphs. These include sufficient conditions for a chorded cycle to exist, or sets of chorded cycles exist, or cycles with multiple chords exist, or chorded cycles with additional properties exist.

Keywords Chord · Cycle · Chorded cycle · Pancyclic

Mathematics Subject Classification 05C38

1 Introduction

The study of cycles in graphs is a rich and an important area. One natural question is to study cycles satisfying certain other conditions. These conditions include properties like sets of independent cycles, cycles containing specified vertices or edges, cycle length, and cycles spanning the vertex set. Over the last few years answers to a new question on cycles with additional properties has gained interest. A *chord* of a cycle is an edge between two non-consecutive vertices of the cycle. We say a cycle is *chorded* if it contains at least one chord, and the cycle is *doubly chorded* if it contains at least two chords, etc.

In 1961 Posa [62] asked a very natural question:

Question 1 What conditions imply a graph contains a chorded cycle?

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Posa (see [62], solution prob. 10.2, p 376) supplied one answer by showing that a graph on n vertices with at least $2n-3$ edges contains a chorded cycle. However, real interest in this problem took years to develop. Recently there has been a considerable increase in results on this question. These new results have shown that Posa's question is an important and natural one, as these results extend the general theory on cycles in graphs, and our overall understanding of substructures in graphs.

In this paper we will survey results that provide answers to Posa's question. We will also look at more particular questions and problems for further study in regard to his question. We will not address areas like chordal graphs, as this topic has already been well studied.

All graphs are simple. For a graph G, denote the vertex and edge sets of G as $V(G)$ and $E(G)$, respectively, Let $d(v)$ denote the degree of the vertex v. We denote by $G=(V,X,Y)$ a bipartite graph with vertex set V and partite sets X and Y. A cycle of length ℓ is called an *ℓ-cycle*. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and $N_G[v]=N(v)\cup\{v\}$. Note that if the graph is clear we will simple use $N(v)$ and $N[v]$. When the graph in question is clear we may simply use $N(u)$. Let H be a subgraph of G, and let $S \subseteq V(G)$. For $u \in V(G)-V(H)$, we denote $N_H(u)=N_G(u)\cap V(H)$ and $d_H(u)=|N_H(u)|$. For $X \subseteq V(G)$, let $d_H(X)=\sum_{x\in X} d_H(x)$. For $u \in V(G)-S$, $N_S(u)=N_G(u)\cap S$. Furthermore, $N_G(S)=\cup_{w\in S}N_G(w)$ and $N_H(S)=N_G(S)\cap V(H)$. Let A, B be two vertex-disjoint subgraphs of G. Then $N_G(A)=N_G(V(A))$ and $N_B(A)=N_G(A)\cap V(B)$. If $S=\{u\}$, then we write $G-u$ for $G-S$. For two disjoint graphs G_1 and G_2, the graph $G_1\cup G_2$ denotes the disjoint union of G_1 and G_2. Further, we denote by $G_1\vee G_2$ the *join* of G_1 and G_2, that is, $G_1\vee G_2$ has vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$ together with all edges from $V(G_1)$ to $V(G_2)$. A 2-*factor* in a graph G is a collection of disjoint cycles whose vertices span $V(G)$. A graph G is *pancyclic* if it contains cycles of each length from 3 to $|V(G)|$ and G is *k-pancyclic* if it contains cycles of each length from k to $|V(G)|$. Let $\sigma_k(G)=\min\{\sum_{i=1}^{k}\deg(x_i)\}$ where the minimum is taken over all independent sets of vertices $x_1,x_2,\dots,x_k$ in G. Let $\sigma_k(G)\ \infty$ if the independence number of G is less than k. When this degree sum condition holds for any $k\geq 2$ we refer to this as an *Ore-type* property. We denote by $\overline{G}$ the complement of the graph G. For terminology and notation not defined here, see [41].

2 Early Development

Early on, Posa's question drew a simple answer due to Czipszer (1963) (see [57], problem 10.2, pp 65) when he showed that $\delta(G)\geq 3$ was sufficient to imply G contains a chorded cycle. This is easy to see by simply considering a longest path in G, say $P=x_1,x_2,\dots,x_k$. Then, besides x_2, the vertex x_1 has at least two more adjacencies to vertices on P. These two edges, along with the path P now contain a chorded cycle (see Figure 1). In fact, Czipszer's proof showed more (see [57], solution prob. 10.2, p 376). He showed that if G has minimum degree at least $k+2$, then G has a cycle with at least k chords, all incident to the same vertex. Still, Posa's question

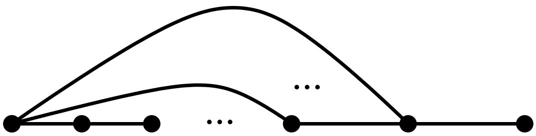

Fig. 1 Minimum degree 3 creates chorded cycle

remained mostly in the background until 1999 when Ali and Staton [1] extended Czipszer's observation.

Theorem 1 [1] *If G is a graph with* $\delta(G) \geq k$, *then G contains a cycle with* $\left\lceil \frac{k(k-2)}{2} \right\rceil$ *chords*.

This gives an immediate improvement of Czipszer's observation.

Corollary 1 *If G is a graph with minimum degree* $\delta(G) \geq 3$, *then G contains a cycle with at least two chords*, *that is*, *a doubly chorded cycle*.

This work helped Posa's question gain more interest. In 2008 Finkel [36] proved a beautiful result on the existence of k vertex-disjoint chorded cycles. It is a natural extension of Czipszer's bound and more.

Theorem 2 [36] *Let* $k \geq 1$ *be an integer*. *If G is a graph of order at least* $4k$ *and* $\delta(G) \geq 3k$, *then G contains k vertex-disjoint chorded cycles*.

The minimum degree condition in Finkel's Theorem is sharp, as can be seen by the graph $K_{3k-1,n-3k+1}$ when $n \geq 6k$.

Theorem 2 can also be considered a natural extension of the famous theorem of Corrádi and Hajnal [20].

Theorem 3 [20] *Let G be a graph of order* $n \geq 3k$ *with* $\delta(G) \geq 2k$, *then G contains k vertex-disjoint cycles*.

A collection of $r+s$ vertex-disjoint cycles, s of them chorded, will be call an (r, s)-*system.* Bialostocki, Finkel, and Gyárfás [7] proposed the following conjecture:

Conjecture 1 *Let* r, s *be nonnegative integers and let G be a graph with* $|V(G)| \geq 3r+4s$ *and minimal degree* $\delta(G) \geq 2r+3s$. *Then G contains an* (r, s)-*system*.

Bialostocki, Finkel, and Gyárfás were able to prove their conjecture for $r=0, s=2$ and for $s=1$. Babu and Diwan [4] and Chiba, Fujita, Gao, and Li [18] independently proved a stronger Ore-type version of the conjecture.

Theorem 4 *Let $r \geq 0, s \geq 0$ be integers and G a graph of order at least $3r+4s$. If $\sigma_2(G) \geq 4r+6s-1$, then G contains an (r, s)-system.*

Corollary 2 *Let $s \geq 1$ be an integer. If G is a graph of order at least $4s$ with $\sigma_2(G) \geq 6s-1$, then G contains s vertex-disjoint chorded cycles, that is, G contains a $(0, s)$-system.*

The degree sum condition cannot be lowered in Theorem 4. This is shown by the complete bipartite graph $K_{2r+3s-1,n-2r-3s+1}$. Hence, in this sense, this theorem is best possible.

Given a degree condition implying some property, graphs with degree one less and failing to have the desired property will be called *extremal graphs* for that relationship. In [59], the extremal graphs for Corollary 2, with minimum degree at least $6k-2$, that fail to have k vertex-disjoint chorded cycles were characterized. See Fig. 2 for the complete tripartite graph $G_2(k)$.

Theorem 5 [59] *For $k \geq 2$, let G be a graph of order $n \geq 4k$ and with $\sigma_2(G) \geq 6k-2$. Then G does not contain k vertex-disjoint chorded cycles if and only if $G \in \{K_{3k-1,n-3k+1}, G_2(k)\}$.*

This idea was extended in [53] where the extremal graphs G with Ore-degree $\sigma_2(G) \geq 6k-3$ were characterized. Extremal graphs for the general case of Theorem 4 were characterized in [60].

Theorem 6 *Given integers $r \geq 0$ and $s \geq 1$ such that $r+s \geq 2$, let $m = 2r+3s-1$ and let G be a graph on $n \geq 3r+4s$ vertices with $\delta(G) \geq m$. Then G is an (r, s)-extremal graph if and only if either:*

1. $n \geq 2m = 4r+6s-2$ *and* $G \cong K_{m,n-m}$;
2. $n = 2m-1 = 4r+6s-3$ *and* $G \cong K_{2m-n,n-m,n-m}$;
3. $s = 1, 3r+4s \leq n \leq 4r+6s-4$ *and* $K_{2m-n,n-m,n-m} \subseteq G \subseteq (\overline{K_{2m-n}} \vee K_{n-m,n-m})$; or
4. $s = 1$, *r is even*, $n = 3r+4s$, *and* $G \cong \overline{K}_{m/2,m/2} \vee \overline{K}_{n-m}$.

This was followed in 2013 by another type of sufficient condition, using a weaker adjacency condition. Let $N(x, y) = N(x) \cup N(y)$ denote the union of the

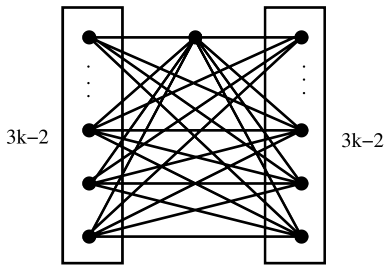

Fig. 2 The graph $G_2(k) = K_{3k-2,3k-2,1}$

neighborhoods of non-adjacent vertices x and y. The following was shown independently in [44] and [38].

Theorem 7 *Let G be a graph of order at least $4k$ such that for any pair of nonadjacent vertices x, y we have that $|N(x,y)| \geq 4k+1$. Then there exist k vertex-disjoint chorded cycles in G.*

The neighborhood union condition in Theorem 7 is sharp, even for $k = 1$. This can be seen by considering the graph tK_3 composed of the disjoint union of t copies of K_3. When t is sufficiently large we have $n \geq 4k$. Further, the neighborhood union of any two independent vertices is 4, but there are no chorded cycles in the graph. If we also impose connectivity, then the graph obtained by appending a K_3 to each vertex of a path (see Figure 3) or cycle still fails. It is not known if higher connectivity changes the outcome.

Problem 1 Can the neighborhood union condition of Theorem 7 be decreased if the graph is at least k-connected for some $k \geq 2$?

Qiao [66] provided a neighborhood union version of Conjecture 1 proposed by Bialostocki, Finkel, and Gyárfás [7].

Theorem 8 [66] *Let $s \geq 0, t \geq 0$ be two integers and G a graph of order $n \geq 3s+4t$. If $|N(u,v)| \geq 3s+4t+1$ for any two nonadjacent vertices u and v, then G contains an (s, t)-system.*

The neighborhood union condition in this theorem is sharp. Let $t \geq 2$ be even. The graph $G = K_{2t+3} \cup K_{2t-1}$ satisfies $|N(u,v)| = 4t$ for all pairs of nonadjacent vertices u and v, but G contains no set of t vertex-disjoint chorded cycles. Again, there is the question of the sharpness of this neighborhood union bound under higher connectivity conditions.

Bialostocki et al. [7] went on to investigate the size ($|E(G)|$) of a graph sufficient to imply the existence of chorded cycles. In particular, they were interested in the existence of two vertex-disjoint chorded cycles. Let $f(n)$ ($g(n)$) be the smallest number of edges in a graph of order n that ensures the existence of two vertex-disjoint cycles, one (both) of them chorded. They show the following.

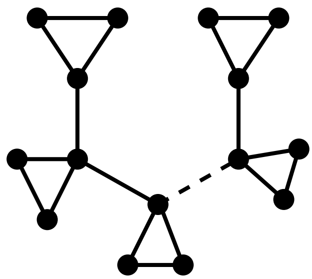

Fig. 3 Traingles appended to the vertices of a path

Theorem 9 *For* $n \geq 10$, $f(n) = 4n - 15$ *and for* $n \geq 12$, $g(n) = 5n - 24$.

This, of course, leaves open the fundamental questions:

Question 2

1. For $k \geq 3$, what size is needed in a graph of order n to imply the existence of k vertex-disjoint chorded cycles?
2. Given $r \geq 0$ and $s \geq 0$, what size is needed in a graph of order n to imply the existence of an (r, s)-system?

There are a number of extremal results that also imply the existence of chorded cycles, although the subgraph obtained is usually of a more general nature. We shall consider only one such example, due to Kawarabayashi [51].

Theorem 10 *Every graph* G *on* $4s$ *vertices with* $\delta(G) \geq \frac{5}{2}s$ *contains* s *disjoint copies of* $K_{1,1,2}$.

3 Extensions of Finkel's Theorem

Degree conditions have often been used to prove results on disjoint cycles. For example, in [31] and [71] it is shown that if G is a graph of order $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k vertex-disjoint cycles. Extending the degree sum condition, in [37] it is shown that if $n \geq 3k + 2$ and $\sigma_3(G) \geq 6k - 2$ $(k \geq 2)$, then G contains k-vertex-disjoint cycles. Building on the degree sum approach, and answering a conjecture from [43], Ma and Yan [58] showed that for integers $t \geq 5$ and $k \geq 2$, if G is a graph of sufficiently large order with $\sigma_t(G) \geq 2kt - t + 1$, then G contains k vertex-disjoint cycles.

As we have just seen, degree conditions lend themselves to a variety of generalizations. We now take a similar approach to degree conditions and vertex-disjoint chorded cycles. Our first example of this is a natural extension of Finkel's Theorem (Theorem 2) and Corollary 2.

Theorem 11 [43] *Let* $k \geq 1$ *be an integer. If* G *is a graph of order at least* $8k + 5$ *with* $\sigma_3(G) \geq 9k - 2$, *then* G *contains* k *vertex-disjoint chorded cycles.*

Again the degree condition of this result is sharp as can be seen by the same example $G = K_{3k-1,n-3k+1}$. Then $\sigma_3(G) = 9k - 3$, but again there do not exist k vertex-disjoint chorded cycles in G.

The pattern of lower bounds on the degree conditions given in Theorem 2, Corollary 2 and Theorem 11 lead naturally to the followig result from [30].

Theorem 12 *Let $k \geq 1$ and $t \geq 1$ be integers. If G is a graph of order $n \geq (10t-1)(k-1)+12t+13$ and $\sigma_t(G) \geq 3kt-t+1$, then G contains k vertex-disjoint chorded cycles. Further, this degree condition is sharp.*

Remark 1 Theorem 12 is sharp with respect to the degree sum condition. Once again the extremal graph is the complete bipartite graph $G = K_{3k-1,n-3k+1}$, for large $n = |V(G)|$. Then $\sigma_t(G) = t(3k-1)$. However, G does not contain k vertex-disjoint chorded cycles, since any chorded cycle must contain at least three vertices from each partite set, in particular, from the $3k-1$ partite set.

4 Multiple Chords

As indicated by the Ali and Staton result (Theorem 1), and that of Czipszer (see [9] p 386) finding a condition that implies a chorded cycle may actually imply more. This also suggests the problem of finding conditions that control the type of chords, when more than one chord is known to exist. In this section we explore results showing the existence of multiple chords in a cycle or collection of cycles. An early result of this type is from [67].

Theorem 13 *Let $k \geq 1$ be an integer and G a graph of order $n \geq 4k$ and minimum degree at least $\left\lceil \frac{7k}{2} \right\rceil$. Then G contains a set of k vertex-disjoint doubly chorded cycles.*

A stronger result was found in [42].

Theorem 14 *Let G be a graph of order $n \geq 6k$ with $\sigma_2(G) \geq 6k-1$. Then G contains k vertex-disjoint doubly chorded cycle.*

Theorem 14 is sharp, as again shown by the bipartite graph $K_{3k-1,n-3k+1}$. This result also has Finkel's Theorem (Theorem 2), Corollary 2, and Theorem 13 as immediate corollaries.

Considering the complete graph K_{k+1}, one can view this graph as a chorded cycle having all possible chords. In fact, it has

$$f(k) = \frac{(k-2)(k+1)}{2}$$

chords. An improvement of Theorem 1 was shown independently in [50] and [45]. Note that this function is taken as $f(k)$ instead of $f(k+1)$ as K_{k+1} is k-regular.

Theorem 15 [45, 50] *Let $k \geq 1$ be an integer. If G is a graph with $\delta(G) \geq k$, then G contains an $f(k) = \frac{(k-2)(k+1)}{2}$-chorded cycle.*

One can view the classic theorem of Hajnal and Szemerédi [46] as producing a highly chorded cycle (clique) covering.

Theorem 16 *Given integers* $s, k \geq 1$ *and a graph* G *of order* $s(k+1)$ *with* $\delta(G) \geq sk$, *there exist* s *vertex-disjoint copies of* K_{k+1} *in* G.

One can think of a graph with $f(k)$ chords as a *loose clique*, in the sense that it contains all the chords of K_{k+1}, but is not as confined (order wise) as the clique. A relaxation of the conditions and conclusion of Theorem 16 was conjectured in [45].

Conjecture 2 *Given integers* $s, k \geq 1$ *and a graph* G *of order* $n \geq s(k+1)$ *with* $\delta(G) \geq sk$, *there exist* s *vertex-disjoint cycles in* G, *each with at least* $f(k)$ *chords*.

In [45] Conjecture 2 was shown to hold when s, k and n are sufficiently large.

Theorem 17 [45] *There exist* s_0 *and* k_0 *so that if* $s \geq s_0$ *and* $k \geq k_0$, *then there exists an* $n_0 = n_0(s, k)$ *so that if* $\delta(G) \geq sk$ *and* $|V(G)| > n_0$, *then* G *contains* s *vertex-disjoint* $f(k)$*-chorded cycles*.

One potential strengthening of Theorem 17 would be to require a certain number of crossing chords, or chords incident to the same vertex, or parallel chords (neither crossing or all incident to the same vertex). The complete graph K_{k+1} has $\binom{k+1}{4}$ pairs of crossing chords, so it would be natural to ask for the same in a strengthening of Conjecture 2. It is easy to show that if $\delta(G) \geq 3$, then G contains a cycle with a pair of crossing chords, but it is unknown what happens when $\delta(G)$ is larger.

Providing a stronger conclusion to Conjecture 1, Balister, Li, and Schelp [5] showed the following.

Theorem 18 *If* G *is a graph on at least* $3r + 4s$ *vertices with mininum degree at least* $2r + 3s$, *then* G *contains* $r + s$ *vertex-disjoint cycles*, *where each of* s *of these cycles either contain two chords*, *or are of order* 4 *and contain one chord*.

They further suggest that it is likely that all s chorded cycles should be doubly chorded cycles. In addition, the following two theorems were also shown in [5]. The second theorem broadens Czipszer's result for minimum degree three.

Theorem 19 *Suppose* G *is a graph with minimum degree* $\delta(G) \geq 2$. *Suppose further that if there is more than one vertex of degree two in* G, *then the degree* 2 *vertices of* G *induce a path in* G. *Then* G *contains a cycle with at least two non-incident chords*.

Theorem 20 *Suppose* G *contains a cycle* C *and every vertex in* $S = G - C$ *has degree at least* 3 *in* G *and* $S \neq \emptyset$. *Then either* G *contains a cycle shorter than* C, *or* G *contains a cycle with two chords*.

Average degree was also used in [45] to show the following.

Theorem 21 *Let α denote the positive root of*

$$g(x) = 2x(x-2) - (d+1)(d-2).$$

Let $k = \left\lceil \sqrt{\frac{d(d-1)}{2}} \right\rceil$ *denote the largest integer strictly less than* α. *Then*:

a. *If G has average degree at least* $2k$, *then G contains a* $\frac{(d+1)(d-2)}{2}$*-chorded cycle.*
b. *There exist graphs with average degree* $2k - o(1)$ *with no* $\frac{(d+1)(d-2)}{2}$*-chorded cycles.*

Considering edge density with respect to the Ore-type degree sum condition, the following was obtained.

Theorem 22 [15] *Let k be a positive integer and let G be a graph of order* $n \geq 4k$. *If* $\sigma_2(G) \geq n$, *then G can be partitioned into k cycles.*

This result was extended to chorded cycles for sufficiently large graphs in [19].

Theorem 23 *Let k and c be positive integers, and let G be a graph of order* $n \geq f(k, c)$. *If* $\sigma_2(G) \geq n$, *then G can be partitioned into k cycles with at least c chords each.*

Given positive integers n and k, let $g_k(n)$ denote the maximum number of edges of a graph of order n that does not contain a cycle with k chords incident to a vertex on the cycle. Bollobás conjectured (see [9], problem 13 p 398) that there exists a function $n(k)$ such that $g_k(n) = (k+1)n - (k+1)^2$ for all $n \geq n(k)$. The complete bipartite graph $K_{k+1,n-k-1}$ does not contain a cycle with k chords incident to a vertex. Thus, $g_k(n) \geq (k+1)n - (k+1)^2$. Jiang [49] proved Bollobás's conjecture by showing that $n(k) \leq 3k + 3$.

5 Chorded Cycles Containing Specified Elements

In this section we will see that conditions sufficient to imply known cycle results often also imply stronger chorded cycle results. This, in fact, will be the underlying theme for much of the rest of the paper. The following pairs of theorems demonstrates this fact. All concern placing elements on cycles and all results provide cycles not much larger than the number of elements placed on them.

Theorem 24 [29] *Let* $k \geq 1$ *be an integer and let G be a graph of order* $n \geq 6k - 3$. *If* $\delta(G) \geq \frac{n}{2}$, *then for any k distinct vertices* $v_1, v_2, \ldots, v_k$ *in G, there exist k vertex-disjoint cycles* $C_1, C_2, \ldots, C_k$ *such that* $v_i \in V(C_i)$ *and* $3 \leq |V(C_i)| \leq 5$ *for all* $1 \leq i \leq k$.

Theorem 25 [21] *Let G be a graph of order $n \geq 16k - 5$ for an integer $k \geq 1$. If $\delta(G) \geq \frac{n}{2}$, then for any k distinct vertices $v_1, v_2, \ldots, v_k$ in G, there exist k vertex-disjoint chorded cycles $C_1, C_2, \ldots, C_k$ such that $v_i \in V(C_i)$ and $4 \leq |V(C_i)| \leq 6$ for all $1 \leq i \leq k$.*

Theorem 26 [27] *Let G be a graph of order n and for $2 \leq k \leq n/4$, let $e_1, e_2, \ldots, e_k$ be any k independent edges in G. If $\delta(G) \geq \frac{n}{2} + k - 1$, then there exist k vertex-disjoint cycles $C_1, C_2, \ldots, C_k$ such that $e_i \in E(C_i)$ and $3 \leq |V(C_i)| \leq 4$ for all $1 \leq i \leq k$. Furthermore, G contains k disjoint cycles $D_1, D_2, \ldots, D_k$ such that $e_i \in E(D_i)$ for all $1 \leq i \leq k$ and $V(G) = \cup_{i=1}^{k} V(D_i)$.*

Theorem 27 [21] *Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 18k - 3$. If $\delta(G) \geq \frac{n}{2} + k - 1$, then for any k independent edges $e_1, e_2, \ldots, e_k$ in G, there exist k vertex-disjoint doubly chorded cycles $D_1, D_2, \ldots, D_k$ such that $e_i \in E(D_i)$ and $4 \leq |V(D_i)| \leq 6$ for all $1 \leq i \leq k$.*

Remark 2 The minimum degree condition is sharp in the following sense. Let $H = K_{n/2-k+1} \vee K_{2k-2} \vee K_{n/2-k+1}$. Now consider the graph G obtained by adding an edge e between the two copies of $K_{n/2-k+1}$. Take $k - 1$ independent edges in K_{2k-2} and e as the specified k independent edges. Then $\delta(G) = n/2 + k - 2$, but there is no doubly chorded cycle containing e as a cycle edge without using vertices in K_{2k-2}. But then, an independent set of edges spanning the K_{2k-2} along with e cannot be placed on k independent cycles.

Theorem 27 was then further extended. Here for paths $P_1, P_2, \ldots, P_k$, the path P_i has order $r_i \geq 2$. Further let $r = \sum_{i=1}^{k} r_i$.

Theorem 28 [21] *Let $k \geq 1$ be an integer. Let G be a graph of order $n \geq 16k + r - 3$ with $\delta(G) \geq \frac{n}{2} + r - k - 1$, and let $P_1, P_2, \ldots, P_k$ be k independent paths in G. Then there exist k disjoint doubly chorded cycles $D_1, D_2, \ldots, D_k$ such that D_i contains P_i as a path along the cycle and $r_i + 2 \leq |V(D_i)| \leq r_i + 4$ for all $1 \leq i \leq k$.*

Another minimum degree condition implies a strong condition on a 2-factor.

Theorem 29 [21] *Let G be a graph of order $n \geq 4k$ where $k \geq 1$ is an integer. Let $\delta(G) \geq \frac{n+k}{2}$, and $e_1, e_2, \ldots, e_k$ be any k independent edges in G. Then if G contains k vertex-disjoint chorded cycles $C_1, C_2, \ldots, C_k$ such that e_i is a cycle edge of C_i for all $1 \leq i \leq k$, then there exist k vertex-disjoint chorded cycles $D_1, D_2, \ldots, D_k$ such that e_i is a cycle edge of D_i for all $1 \leq i \leq k$ and $V(G) = \cup_{i=1}^{k} V(D_i)$.*

Remark 3 Since a chorded cycle has order at least 4, the condition $n \geq 4k$ is clearly necessary. The minimum degree condition is also sharp in the following sense. Let $G = K_{p+k-1} \vee \bar{K}_p$ with $n \geq 5k + 1$, $p = (n - k + 1)/2 \geq 2k + 1$ and $n \not\equiv k \ (mod\ 2)$. Take k independent edges in K_{p+k-1}. Then $\delta(G) = (n + k - 1)/2$ and G does contain

k disjoint chorded cylces with the specified cycle edges. However, the cycle system cannot be extended to one that spans $V(G)$.

We now have the following corollaries.

Corollary 3 *Let G be a graph of order $n \geq 4k$ for an integer $k \geq 1$ with $\delta(G) \geq (n+k)/2$. Suppose that G contains k vertex-disjoint chorded cycles $C_1, C_2, \dots, C_k$. Then there exist k vertex-disjoint chorded cycles $D_1, D_2, \dots, D_k$ such that $V(G) = \cup_{i=1}^{k} V(D_i)$.*

Corollary 4 *For an integer $k \geq 1$, let G be a graph of order $n \geq 4k$ with $\delta(G) \geq (n+k)/2$. Let $v_1, v_2, \dots, v_k$ be any k distinct vertices in G. Suppose that G contains k vertex-disjoint chorded cycles $C_1, C_2, \dots, C_k$ such that $v_i \in V(C_i)$ for all $1 \leq i \leq k$. Then there exist k disjoint chorded cycles $D_1, D_2, \dots, D_k$ such that $v_i \in V(D_i)$ for all $1 \leq i \leq k$ and $V(G) = \cup_{i=1}^{k} V(D_i)$.*

6 Chorded Weak Pancyclicity

Recall that the *circumference* of a graph is the length of a longest cycle in the graph. A graph is *weakly k-pancyclic* if it contains cycles of each length from k to the circumference of G. Further, G is *chorded weakly k-pancyclic* if G contains a chorded cycle of each length from k to the circumference of G.

Denote by $\kappa(G)$ the *connectivity* of G, that is, the minimum number of vertices whose removal separates G into two or more components. Denote by $\alpha(G)$ the *vertex independence number* of G, that is, the maximum number of mutually nonadjacent vertices in G.

Amar et al. [3] conjectured that if $\alpha(G) \leq \kappa(G)$ and G is not bipartite, then G has cycles of every length from 4 to $|V(G)|$. Lou [56] considered this conjecture and proved the following.

Theorem 30 *Let G be a triangle-free graph of order $n \geq 4$ with $\alpha(G) \leq \kappa(G)$. Then G is 4-pancyclic, or $G = K_{\frac{n}{2},\frac{n}{2}}$, or $G = C_5$.*

This result was extended to chorded cycles in [22].

Theorem 31 *Let G be a triangle-free graph of order $n \geq 13$ with $\alpha(G) \leq \kappa(G)$. Then G is chorded weakly 8-pancyclic, or $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Note that since G is triangle-free, there cannot be a chorded C_4 or C_5 in G, and thus, this result is best possible

In his thesis, Brandt [13] showed the following result. Note that the *circumference* of a graph is the length of a longest cycle in the graph.

Theorem 32 *Let $G \neq C_5$ be a nonbipartite triangle-free graph of order n. If $\delta(G) > n/3$, then G is weakly pancyclic with girth 4 and circumference $min\{2(n - \alpha(G)), n\}$.*

Brandt, Faudree, and Goddard [14] provided another result on weakly pancyclic graphs, removing the triangle free condition of the previous result.

Theorem 33 *Every nonbipartite graph G of order n with minimum degree $\delta(G) \geq (n + 2)/3$ is weakly pancyclic of girth 3 or 4.*

In [22], each of these last two theorems were extended to chorded cycle results as follows.

Theorem 34 *Let G be a nonbipartite triangle-free graph of order $n \geq 13$. If $\delta(G) \geq \frac{n+1}{3}$, then G is chorded weakly 6-pancyclic with circumference $min\{2(n - \alpha(G)), n\}$.*

Theorem 35 *Every nonbipartite graph G of order $n \geq 13$ with minimum degree $\delta(G) \geq (n + 2)/3$ is chorded weakly 6-pancyclic.*

7 Meta-Conjectures

A graph of order $n \geq 3$ is said to be *pancyclic* if it contains a cycle of each length from 3 to n. In the early 1970's Bondy ([11, 12]) noticed a tie between some conditions implying a graph is hamiltonian and those same conditions implying the stronger property of being pancyclic. He stated his well-known meta-conjecture.

Bondy's Meta-Conjecture: Almost all sufficient conditions for the existence of a hamiltonian cycle make a graph pancyclic, possibly with a small number of well-described families of exceptional graphs.

At the time of this meta-conjecture most hamiltonian results were based on degree conditions which forced sufficient edge density to provide the hamiltonian cycle. Bondy supported his meta-conjecture with the following extension of Ore's [61] classic theorem.

Theorem 36 [11] *If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is pancyclic or $G = K_{n/2,n/2}$.*

Bondy further supported his meta-conjecture with the following result.

Theorem 37 [12] *Let G be a hamiltonian graph of order n. If $|E(G)| \geq \frac{n^2}{4}$, then either G is pancyclic or $G = K_{n/2,n/2}$.*

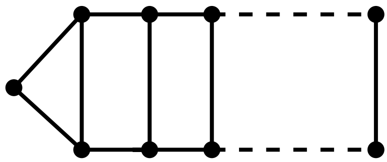

Fig. 4 Infinite family of pancyclic, but not chorded pancyclic graphs

The results we have seen so far led Cream et al. [23] to extend Bondy's Meta-Conjecture farther. We say a graph of order n is *chorded pancyclic* if it contains a chorded cycle of length k for each $4 \leq k \leq n$. Note that by default, a chorded 4-cycle contains a 3-cycle, so the graph must be pancyclic. We also note that being pancyclic is not enough to imply the graph is also chorded pancyclic. The infinite family of graphs of Figure 4 demonstrates this fact.

CGH Meta-Conjecture: [23] Almost any condition on a graph that implies it is hamiltonian also implies it is chorded pancyclic. There may be a small number of simple families of exceptional graphs or some small order exceptions.

This meta-conjecture was supported by the following theorem from [23] which also extends Theorem 36. Here $H_1 \square H_2$ denotes the standard cartesian product of the graphs H_1 and H_2.

Theorem 38 [23] *Let G be a graph of order* $n \geq 4$. *If* $\sigma_2(G) \geq n$, *then G is chorded pancyclic, or* $G = K_{n/2,n/2}$, *or* $G = K_3 \square K_2$. *(see Fig. 5.)*

Further evidence of the CGH meta-conjecture was given in [17].

Theorem 39 [17] *A hamiltonian graph G of order* $n \geq 4$ *with* $|E(G)| \geq \frac{n^2}{4}$ *is chorded pancyclic unless* $G = K_{n/2,n/2}$ *or* $G = K_3 \square K_2$.

Note that the graph $2K_{(n-1)/2} \vee K_1$ serves as a sharpness example for the degree condition of Theorem 36 and Theorem 38.

Theorem 39 is a corollary of Theorem 37 and the following.

Theorem 40 [17] *Let G be a graph of order n with* $|E(G)| \geq \frac{n^2}{4}$ *and let k be a positive integer. If G contains a k-cycle, then G contains a chorded k-cycle unless* $k = 4$ *and G is either* $K_{n/2,n/2}$ *or* $K_3 \square K_2$.

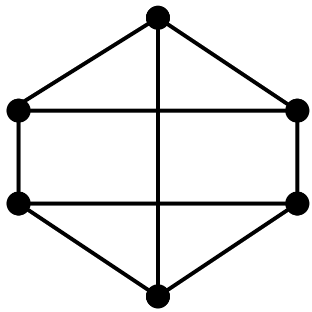

Fig. 5 Small order exception to Theorem 38

This result suggests that the existence of a chorded cycle in a dense graph is independent of hamiltonicity and pancyclicity. In this setting we may be able to obtain a refined density condition. For example, if we assume a chorded 5-cycle exists in G, the bipartite graphs are ruled out and are no longer a barrier to lowering the edge density.

A result using a lesser edge density is due to Bollobás and Thomason [10]. Here $c(G)$ is the *circumference* of G, that is, the length of the longest cycle.

Theorem 41 *A non-bipartite graph G of order n with at least $\left\lfloor \frac{n^2}{4} \right\rfloor - n + 59$ edges contains a k-cycle for every integer k with $4 \le k \le c(G)$.*

Similarly, Chen et al. [17] showed the following for slighty larger cycles.

Theorem 42 *Let $k \ge 8$ be an integer and G a graph of order n with $|E(G)| \ge \frac{n^2}{4} - n + 16$. If G contains a k-cycle then G contains a chorded k-cycle.*

It was assumed $k \ge 8$ in the above theorem. It is not know if the result holds for $k = 6$ or 7. But it is known that it fails for $k = 5$. Let n be an even integer with $n \ge 30$. Let H be a copy of $K_{n/2,n/2}$. Select a distinct pair of vertices u, v in one partite set and a vertex w from the other partite set. Delete the edge vw and delete all edges incident with u except the edge uw. Now insert the edge uv. The resulting graph H^* (see Fig. 6) satisfies the edge condition and contains 5-cycles, but contains no chorded 5-cycles.

For $k = 4$, no subgraph of the complete balanced bipartite graph contains a chorded 4-cycle. Moreover, even if we restrict to non-bipartite graphs we can find graphs with no chorded 4-cycles, for example, adding the edge vw to H^* produces a non-bipartite graph with sufficient edges and no chorded 4-cycles. In some ways we should expect 4-cycles to be the hardest to chord, as they are the smallest possibilty.

Now using Theorems 41 and 42 the following was obtained.

Corollary 5 [17] *A non-bipartie graph G of order n with at least $\left\lfloor \frac{n^2}{4} \right\rfloor - n + 59$ edges contains a chorded k-cycle for each k with $8 \le k \le c(G)$.*

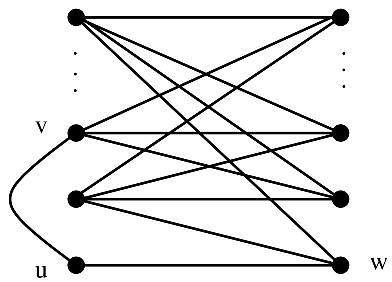

Fig. 6 The graph H^*

For long cycles Thomasssen [64, 65] has a well-known conjecture which he posed as a graduate student and was later published in [2].

Conjecture 3 *Every longest cycle in a* 3*-connected graph is a chorded cycle.*

Although the conjecture remains unsoved in general, many results related to this conjecture have been shown, including by Thomassen himself [64], who showed the conjecture holds if G is cubic. This is a bit surprising, since one would think more edges would help make the conjecture easier to prove. Other results related to this conjecture include the following. Li and Zhang [54] verified the conjecture for graphs with minimum degree at least four that are embedded in the projective plane and they showed in [55] that the conjecture holds when G is 4-connected and embedded in a torus or a Klein bottle. Zhang [72] showed the conjecture holds for cubic planar graphs or planar graphs with minimum degree at least four. Kawarabayashi et al. [52] showed the conjecture holds for locally 4-connected planar graphs, and Birmelé [8] verified it for every 3-connected graph with no $K_{3,3}$-minor.

Many of these results involve sparse graphs. Considering dense graphs, Harvey [47] conjectured the following (which he showed would be best possible).

Conjecture 4 *Let G be a graph of order n. If* $\delta(G) > \sqrt{n} - 1$ *then any cycle of maximum order in G contains a chord.*

Recall, $c(G)$ denotes the circumference of G, so let $c'(G)$ denote the length of a longest chordless cycle. Clearly, $c'(G) \leq c(G)$. Using this, Harvey reframed his conjecture as:

Conjecture 5 *If G is* 3*-connected, then* $c'(G) \leq c(G) - 1$.

Harvey then proved the following result where the case $k = 1$ comes close to solving his first conjecture.

Theorem 43 *Let k be a positive integer and G be a graph of order n. If* $\delta(G) \geq \frac{\sqrt{17}+3}{2\sqrt{2}}\sqrt{n} + 5k - 4$, *then* $c'(G) \leq c(G) - k$.

Harvey went on to conjecture more by adding a connectivity condition.

Conjecture 6 *Let k be a positive integer. If G is a* 2*-connected graph and* $\delta(G) \geq \lceil k/2 + 2 \rceil$, *then* $c'(G) \leq c(G) - k$.

8 More Extensions of Cycle Results

In this section we continue our development of chorded cycle results based on known cycle results. Here we need some more definitions. A graph G of order n is called *vertex pancyclic* (*edge pancyclic*) if each vertex (edge) is contained on a cycle of each possible length from 3 to n. Extending this idea to chorded cycles we say G is *chorded vertex pancyclic* (*chorded edge pancyclic*) if each vertex (edge) is contained on a chorded cycle of each possible length from 4 to n. Similarly, G is *chorded vertex k-pancyclic* (*chorded edge k-pancyclic*) if each vertex (edge) of G lies on a chorded cycle of every length from k to n.

8.1 Vertex Pancyclic Extensions

In this section we examine extensions of vertex pancyclic theorems to chorded vertex pancyclic results. We begin with a result from Hendry [48].

Theorem 44 *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n+1}{2}$, then G is vertex pancyclic.*

In [24] Hendry's Theorem was extended to chorded vertex pancyclic graphs.

Theorem 45 *If G is a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+1}{2}$, then G is chorded vertex pancyclic.*

The above minimum degree condition is actually strong enough to imply a bit more as we see in the next theorem from [24].

Theorem 46 *Let G be a graph of order $n \geq 5$ with $\delta(G) \geq \frac{n+1}{2}$. Then for every $k \geq 5$ and any vertex $x \in V(G)$, there is a doubly chorded k-cycle in G containing x, that is, G is doubly chorded 5-pancyclic.*

Remark 4 To see that the graph of the previous theorem need not be doubly chorded 4-pancyclic, let $n \cong 3(mod\, 4)$ and consider the graph $K_{\frac{n+1}{2},\frac{n-1}{2}}$ along with a perfect matching in the larger partite set. This graph is $\left(\frac{n+1}{2}\right)$-regular. However, no vertex lies in a K_4, hence there are no doubly chorded 4-cycles in this graph.

Another result of Randerath et al. [68] is the following:

Theorem 47 *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$. Then G is vertex 4-pancyclic.*

The next result from [24] determines the σ_2 condition for chorded vertex pancyclic graphs.

Theorem 48 *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$. Then G is chorded vertex 5-pancyclic.*

Remark 5 In Theorem 48, the degree sum condition is sharp. The balanced complete bipartite graph, $K_{\frac{n}{2},\frac{n}{2}}$ has $\sigma_2 = n$ but it does not contain any odd cycles. Thus, it is not pancyclic, so clearly it is not chorded vertex pancyclic either.

Remark 6 Theorem 48 is also sharp in terms of vertex 5-pancyclicity. Consider the graph $G = (K_{n-1} - E(K_c))$, where $c \geq 3$, with an additional vertex v adjacent to each vertex of the K_c whose edges were removed. Then v clearly lies on no chorded 4-cycle. Further, the $\sigma_2(G)$ condition is realized by v and any vertex in $V(G) - N[v]$ as $c + n - 2 \geq n + 1$ as long as $c \geq 3$. Thus, G need not be chorded vertex pancyclic under this degree sum condition.

Remark 6 shows that $\sigma_2(G) \geq n + c$ for any constant c will fail to imply the graph is vertex pancyclic when n is sufficiently large. In [68], a sharp minimum degree sum condition implying the existence of vertex pancyclic graphs was determined.

Theorem 49 *Let G be a graph of order $n \geq 3$ such that $\sigma_2(G) \geq \left\lceil \frac{4n}{3} \right\rceil - 1$. Then G is vertex pancyclic.*

Let G be a graph of order n with minimum degree δ. Then G is a member of the class $H_{\delta,n}$ if there exists a vertex w of minimum degree δ such that $N_G(w)$ induces an independent set and every vertex $u \in N_G(w)$ is adjacent to every vertex $x \in V(G) - N_G[w]$. The unique $G \in H_{\delta,n}$ with the added property that $V(G) - N_G[w]$ induces a complete graph is denoted by $H^c_{\delta,n}$. The graph $H^c_{n/3+1,n}$ shows the sharpness of the degree condition in Theorem 49.

The next result extends Theorem 49 to chorded vertex pancyclic graphs.

Theorem 50 [24] *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \left\lceil \frac{4n}{3} \right\rceil - 1$. Then G is chorded vertex pancyclic.*

Definition 1 A graph G of order n is called (h, k)**-pancyclic** if every set of h vertices in G is contained in a cycle of every length from k to n. Hence, $(1, k)$-pancyclic is just vertex k-pancyclic and $(0, 3)$-pancyclic is just pancyclic.

Theorem 51 [34] *Let $2 \leq k$, $2k \leq m$ and $m < n$ be integers and let G be a graph of order n. If $\delta(G) \geq \left\lfloor \frac{n+2}{2} \right\rfloor$, then G is (k, m)-pancyclic.*

This was extended in [24].

Theorem 52 *Let $k \geq 2$ and $n > 2k$ be integers, and let G be a graph of order n with $\delta(G) \geq \left\lfloor \frac{n+2}{2} \right\rfloor$. Then G is chorded $(k, 2k+1)$-pancyclic.*

Remark 7 In order to see that $2k + 1$ is sharp when $k = 2$ consider a graph G from Remark 4. Take a vertex x from one edge that was added to the larger partite set and a vertex y from a different such edge. Then x and y together lie on no chorded 4-cycles.

In [68], the extremal σ_2 bound was studied for vertex 4-pancyclic graphs.

Theorem 53 *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n$. Then G is vertex 4-pancyclic unless n is even and $G = K_{n/2,n/2}$.*

In [34] a stronger σ_2 bound was given for G to be (k, m)-pancyclic when $k \geq 2$.

Theorem 54 [34] *Let $k \geq 2$ and $n > 2k$ be integers. If G is a graph of order n with $\sigma_2(G) \geq 2\left\lfloor \frac{n}{2} \right\rfloor + 1$, then G is $(k, 2k)$-pancyclic.*

Theorem 54 was further extended in [24].

Theorem 55 *Let $k \geq 2$ and $n > 2k$ be integers. If G is a graph of order n with $\sigma_2(G) \geq 2\left\lfloor \frac{n+1}{2} \right\rfloor + 1$, then*

(1) *If $k \geq 4$, then G is chorded $(k, 2k)$-pancyclic,*
(2) *If $k = 2$ or 3, then G is chorded $(k, 2k + 1)$-pancyclic.*

Next, consider what happens for chorded cycles when we reduce the bound on σ_2.

Theorem 56 [24] *Let G be a graph of order $n \geq 4$, and let x be any specified vertex of G. If $\sigma_2(G) \geq n$, then one of the following statements holds.*

(i) *G is chorded vertex pancyclic.*
(ii) *$\overline{K}_{n/2} \vee \overline{K}_{n/2} \subseteq G \subseteq \overline{K}_{n/2} \vee (K_1 \cup F)$ (n is even), where F is a spanning subgraph of $K_{n/2-1}$, satifying the following conditions:*

- if $E(F) = \emptyset$, then $x = v$ for any $v \in V(G)$,
- if $E(F) \neq \emptyset$, then $x \in V(K_1)$, or $x = v$ such that $\deg_F(v) = 0$ for $v \in V(F)$.

(iii) *G is a spanning subgraph of $H = B \vee x \vee \overline{K}_a \vee (K_1 \vee K_c \vee K_d)$, ($|V(H)| = n, a \geq 2, c \geq 1, 0 \leq d \leq a - 2$) with all the edges of $B \vee (K_c \cup K_d)$, where B is a graph of order $b \geq 0$ with $|E(B)| \leq 1$ satifying the following conditions:*

- $h = h_1 + h_2 \leq 1$, where $h_1 = |E(B)|$ and $h_2 = |E(\overline{K}_a, B)|$,
- if $z_1z_2 \in E(B)$ for $z_1, z_2 \in V(B)$, then $N_{K_c \cup K_d}(z_1) \cap N_{K_c \cup K_d}(z_2) = \emptyset$,
- if $mz \in E(G)$ for $m \in V(\overline{K}_a)$ and $z \in V(B)$, then $N_{K_c \cup K_d}(m) \cap N_{K_c \cup K_d}(z) = \emptyset$.

8.2 Edge Pancyclic Extensions

A natural variation of vertex pancyclic graphs is that of edge pancyclic graphs. In [68], a sharp minimum degree condition was established for edge pancyclic graphs. The graph $K_{n/2,n/2}$ shows we cannot reduce this minimum degree by one.

Theorem 57 [68] *If G is graph of order n with $\delta(G) \geq \frac{n+2}{2}$, then G is edge pancyclic.*

Our next result is the natural extension of Theorem 57.

Theorem 58 [24] *If G is a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+2}{2}$, then G is chorded edge pancyclic.*

In [33], the idea of edge pancyclic graphs was extended to containing paths.

Definition 2 If G is a graph of order n, we say G is (P, m)-**pancyclic** if any path $P = P_k$ is contained on a cycle of every length from m to n.

Definition 3 If G is a graph of order n, we say G is **chorded** (P, m)-**pancyclic** if any path $P = P_k$ is contained on a chorded cycle of every length from m to n.

The next result follows easily from Theorem 58.

Corollary 6 *Given a fixed integer k, let G be a graph of order $n \geq k+2$ containing a path $P = P_k$, $k \geq 2$, and with $\delta(G) \geq \frac{n}{2} + k - 1$. Then G is chorded $(P, k+2)$-pancyclic.*

Theorem 59 [33] *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n+1$, then G is edge 3-pancyclic.*

The extension to chorded cycles was proven in [24], along with a further extension to chorded cycles containing paths.

Theorem 60 [24] *Let G be a graph of order $n \geq 5$. If $\sigma_2(G) \geq n+1$, then G is chorded edge 5-pancyclic.*

Corollary 7 [24] *Let $k \geq 2$ be an integer and let G be a graph of order $n \geq k+3$. If $\sigma_2(G) \geq n + 2k - 3$, then the graph G is chorded $(P_k, k+3)$-pancyclic.*

9 Bipartite and Multipartite Graphs

Over the years many cycle results have been determined by imposing some sufficient conditions on a bipartite graph. Thus, it is natural that such conditions would now be considered for implying chorded cycles. We begin with the following from [70].

Theorem 61 *Let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$ where k is a positive integer. If $\delta(G) \geq 2k + 1$, then G contains k vertex-disjoint 6-cycles, each with at least two chords.*

This rather special result was generalized to larger graphs in [39].

Theorem 62 *Let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = |V_2| \geq 3k$ where k is a positive integer. If $\delta(G) \geq 2k + 1$, then G contains k vertex-disjoint cycles such that each has at least two chords.*

In [70], Wang conjectured a bipartie version of the classic result of Corrádi and Hajnal (Theorem 3) [20].

Conjecture 7 *For integers $s \geq 2$ and $k \geq 1$, let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = sk$. If $\delta(G) \geq (s-1)k + 1$, then G contains k vertex-disjoint subgraphs isomorphic to $K_{s,s}$.*

If the order of G is sufficiently large, then a result a bit stronger then Conjecture 7 holds.

Theorem 63 [73] *For each $s \geq 2$ there exists a k_0 such that the following holds for all $k \geq k_0$. Let $G = (V_1, V_2, E)$ be a bipartite graph with $|V_1| = |V_2| = n = sk$ such that $\delta(G) \geq n/2 + s - 1$ if k is even, and $\delta(G) \geq \frac{n+3s}{2} - 2$ if k is odd. Then G contains k disjoint subgraphs isomorphic to $K_{s,s}$. That is, G contains k disjoint $2s$-cycles with $s^2 - 2s$ chords each.*

Following the ideas in [7] on the size of the graph, Gao and Wang [40] studied size in bipartite graphs. They showed the following.

Theorem 64 *Let $G = (V_1, V_2, E)$ be a bipartite graph with $3 \leq m = |V_1| \leq |V_2| = n$. If $|E(G)| \geq 2(n-2) + 2m$, then G contains a chorded cycle.*

Theorem 65 *Let $G = (V_1, V_2, E)$ be a bipartite graph with $3k \leq m = |V_1| \leq |V_2| = n$ where k is a positive integer. If $|E(G)| \geq (4k-2)(n-2) + 2m$, then G contains k independent chorded cycles.*

Again considering size, and using Theorem 39, Ferrero and Lesniak [35] showed the following.

Theorem 66 *For any integer $k \geq 3$ and $p \geq 1$, a balanced k-partite graph of order kp with at least*

$$\frac{(k^2-k)p^2-2p(k-1)+4}{2}$$

edges is chorded pancyclic.

10 Many Cycles of the Same Length

Thomassen [63] conjectured a graph with minimum degree at least $2k$ contains k vertex-disjoint cycles of the same length. Egawa [28] confirmed this conjecture. Verstraëte [69] gave a simpler proof of the conjecture. His result is stated below.

Theorem 67 *Let k be a natural number. Then there exists a positive integer n_k such that if G is a graph of order at least n_k and minimum degree at least $2k$, then G contains k vertex-disjoint cycles of the same length.*

By combining this theorem and Finkel's Theorem (Theorem 2), the following was shown in [16].

Theorem 68 *For every natural number k, there exist positive integers n_k and n'_k such that for every graph G with order n and minimum degree at least $3k+8$, the following holds:*

(*i*) *if $n \geq n_k$, then G contains k vertex-disjoint chorded cycles of the same length; and*
(ii) *if $n \geq n'_k$, then G contains k vertex-disjoint isomorphic chorded cycles of the same length.*

They also proposed that the $3k+8$ bound could be reduced to $3k$, which is best possible as shown by $K_{3k-1,n-3k+1}$.

In [16] the following was also shown.

Theorem 69 *Let G be a multigraph of order n and minimum degree at least 5. Then G contains a chorded cycle of length at most $300\log_2 n$*

In the paper it is shown that minimum degree 5 is best possible. The factor 300 is not best possible, however the log n term is best possible, since when $r \geq 10$, Dahan [26] showed there are infinitely many $(r+1)$-regular graphs of order n with girth larger than $\log_r n$.

For simple graphs the following was conjectured in [16].

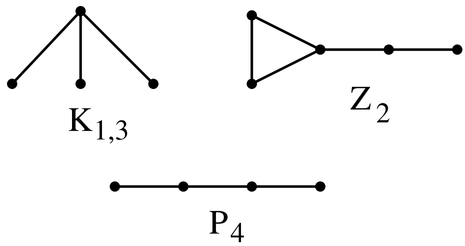

Fig. 7 Common forbidden graphs

Conjecture 8 *Let G be a graph of order n and minimum degree at least* 3. *Then G contains a chorded cycles of length at most* $\alpha \log_2 n$, *where* $\alpha > 0$ *is a universal constant.*

11 Forbidden Subgraphs

We say a graph G is *H-free* if there is no induced copy of H as a subgraph in G. If G is H-free, then we say H is *forbidden.* This idea easily extends to families of forbidden subgraphs.

The use of forbidden subgraphs to obtain cycle results has a very long and well developed history. In this section we extend these ideas to chorded cycles.

Forbidden pairs for pancyclic graphs were characterized in [32]. Here Z_i denotes a triangle with the end vertex of a P_{i+1} attached to one vertex (see Fig. 7 for Z_2).

Theorem 70 *Let* $R, S \neq P_3$ *be connected graphs and let* $G \neq C_n$ *be a* 2*-connected graph of order* $n \geq 10$. *Then G is* $\{R, S\}$*-free implies G is pancyclic if and only if* $R = K_{1,3}$ *and S one of* P_4, P_5, P_6, Z_1, *or* Z_2.

In [25] these ideas were studied for chorded cycles.

Theorem 71 [25] *Let G be a* 2*-connected graph of order* $n \geq 10$. *If G is* $\{K_{1,3}, Z_2\}$ *-free, then* $G = C_n$ *or G is chorded pancyclic.*

Theorem 72 [25] *Let G be a* 2*-connected graph of order* $n \geq 5$. *If G is* $\{K_{1,3}, P_4\}$ *-free, then G is chorded pancyclic.*

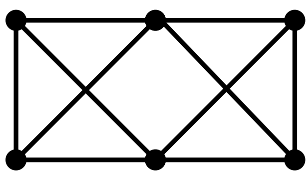

Fig. 8 A $\{K_{1,3}, P_4\}$-free graph with no doubly chorded 4-cycle

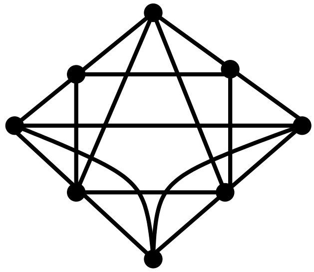

Fig. 9 A $\{K_{1,3}, P_5\}$-free graph with no doubly chorded 4-cycle

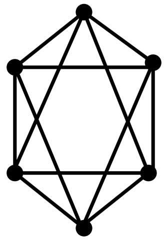

Fig. 10 A $\{K_{1,3}, Z_1\}$-free graph with no doubly chorded 4-cycle

Theorem 73 [25] *Let G be a 2-connected graph of order $n \geq 8$. If G is $\{K_{1,3}, P_5\}$-free, then G is chorded pancyclic.*

Theorem 74 [25] *Let G be a 2-connected graph of order $n \geq 13$. If G is $\{K_{1,3}, P_6\}$-free, then G is chorded pancyclic.*

These results were extended in [6].

Theorem 75 [6] *Let G be 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n \geq 7$. Then G is doubly chorded pancyclic. The order is also sharp (see Fig. 8).*

Note that the graph of Fig. 8 is a 2-connected $\{K_{1,3}, P_4\}$-free graph of order 6 which contains no doubly chorded 4-cycle. Thus, Theorem 75 is sharp with respect to order.

Theorem 76 [6] *Let G be a 2-connected, $\{K_{1,3}, P_5\}$-free graph of order $n \geq 9$. Then G is doubly chorded pancyclic. The order is also sharp (see Fig. 9).*

Theorem 77 [6] *Let $G \neq C_n$ be a 2-conneted graph. If*

1. *G has order $n \geq 7$ and is $\{K_{1,3}, Z_1\}$-free, or*
2. *G has order $n \geq 10$ and is $\{K_{1,3}, Z_2\}$-free, then G is doubly chorded pancyclic.*

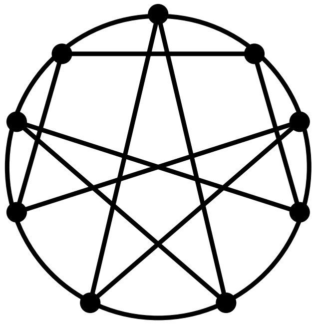

Fig. 11 A $\{K_{1,3}, Z_2\}$-free graph with no doubly chorded 4-cycle

That this result is sharp with respect to the orders is shown by the graphs of Figs. 10 and 11.

12 Final Comments

There are two areas where conditions sufficient to imply the existence of cycles have been found, but as of this date, no chorded cycles results have been discovered. Hence, we pose the following two Posa-like questions.

Question 3 What spectral conditions imply the existence of a chorded cycle in a graph?

Question 4 What conditions in a hypergraph imply the existence of a chorded cycle?

Finally, we have seen numerous examples to justify the following general chorded cycle meta-conjecture.

Meta-Conjecture: Almost any condition that implies a result on cycles in a graph also implies a similar result on chorded cycles. There may be some simple families exceptional graphs, or small order exceptions.

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