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ARTICLE

On the saturation spectrum of odd cycles

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Abstract

Given a graph H, we say that a graph G is H-saturated if H is not a subgraph of G, but the addition of any new edge to G creates at least one copy of H. In this paper we determine all pairs (n, m) for which there is a C_5-saturated graph on n vertices and m edges. In addition, we determine all but $O(nk)$ possible sizes for n-vertex H-saturated graphs when H is an odd cycle C_{2k+1} for $k \geq 3$.

KEYWORDS

extremal number, odd cycle, saturation number, saturation spectrum

1 | INTRODUCTION

In this paper, all graphs are simple, that is, they do not contain loops or multiple edges. A graph is *H-saturated* if it does not contain a subgraph isomorphic to H, but adding an edge between any pair of nonadjacent vertices creates such a subgraph. For example, the complete bipartite graph $K_{p,q}$ is C_{2k+1}-saturated as long as $p, q \geq k + 1$. The *saturation number* of H, denoted by $\mathrm{sat}(n, H)$, is the minimum size $|E(G)|$ of an n-vertex H-saturated graph G, whereas the maximum size is the *extremal number* $\mathrm{ex}(n, H)$. The complete graph K_n is trivially the only H-saturated graph on n vertices for $n < |V(H)|$, so that we will usually assume that $n \geq |V(H)|$.

Since it is known that nonbipartite graphs of order n have extremal numbers that are quadratic in n [7] and that all graphs have saturation numbers that are linear in n [12], a natural question is what values of $|E(G)|$ can be the size of an H-saturated graph G of order n. The set of all such values is called the *saturation spectrum* of H. The saturation spectrum was first investigated for the triangle K_3 by Barefoot et al. [3]. Later saturation spectrums were investigated for K_4 [1], for K_t, $t \geq 3$ [2], and for $K_4 - e$ [8].

In this paper we completely determine the saturation spectrum of C_5 and we provide a range of values for the size of a C_{2k+1}-saturated graph where $k \geq 3$. In Section 2 we provide several constructions for odd cycle-saturated graphs. In Section 3 we use these constructions to determine the saturation spectrum of C_5. We now define some basic terminology needed to state our main results. For terms not defined here see [11].

We let $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G, respectively. We will consistently use n for the order of G, that is, $|V(G)|$, and m for the size of G, that is, $|E(G)|$. We use K_n to denote the complete graph of order n, and $K_{a_1,a_2,\ldots,a_s}$ to denote the complete s-partite graph with s parts of orders $a_1, a_2, \ldots, a_s$, respectively. To be able to describe H-saturated graphs that have cut-vertices we say that *identifying* G_1 at $v_1 \in V(G_1)$ with G_2 at $v_2 \in V(G_2)$ results in the graph $G = G_1 \cdot G_2$ (with identified vertex v) obtained from G_1 and G_2 by adding the edge v_1v_2 and contracting v_1v_2 to the new vertex v. The graph $G_1 \cdot G_2$ has $|V(G_1)| + |V(G_2)| - 1$ vertices and $|E(G_1)| + |E(G_2)|$ edges. Furthermore, we let $T(n) = K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ be the balanced complete bipartite graph on n vertices, and let $t(n) = \lfloor n^2/4\rfloor$ denote its number of edges.

For $n \geq 4k - 1$, Corollary 5.4 in Chap. 3 of Bollobás [4] yields that $\mathrm{ex}(n, C_{2k+1}) = t(n)$ and that equality is only achieved by $T(n)$ for n sufficiently large (see [9]). The primary focus of this paper is to determine the *saturation spectrum* of C_5, that is, given n we want to find all values m for which there are C_5-saturated graphs on n vertices and m edges. The case $k = 2$ in Bollobás' result and a case analysis for $n \leq 7$ yields the upper bound on the spectrum. In fact, Füredi and Gunderson [9] determined all extremal graphs for odd cycles. The results for C_5 are given in the next theorem.

Theorem 1. *The extremal number* $\mathrm{ex}(5, C_5) = 7 = t(5) + 1$ *is achieved only by* $K_{1,1,3}$ *and* $K_4 \cdot K_2$. *For* $n \geq 6$ *we have* $\mathrm{ex}(n, C_5) = t(n) = \lfloor n^2/4\rfloor$, *and for* $n \geq 8$ *equality is only achieved by* $T(n)$. *For* $n = 7$ *there is also* $K_4 \cdot K_4$, *and for* $n = 6$ *there are also* $K_{1,1,4}$ *and* $K_4 \cdot K_3$.

Chen [5, 6] exactly determined the lower bound on the spectrum for C_5 and she also characterized all extremal examples.

Theorem 2. *If* $n \geq 5$, *then* $\mathrm{sat}(n, C_5) = \left\lceil \frac{10(n-1)}{7} \right\rceil - \varepsilon$, *where* $\varepsilon = 1$ *for* $n \in \{11, 12, 13, 14, 16, 18, 20\}$ *and* $\varepsilon = 0$ *otherwise.*

We completely determine the saturation spectrum of C_5. For $5 \leq n \leq 8$ this can be found in Table 1 in Section 3, but our main results are the following:

Theorem 3. *If* $n \geq 9$, *then there is a* C_5*-saturated graph* G *on* n *vertices and* m *edges if and only if* $\mathrm{sat}(n, C_5) \leq m \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 6$ *or* $G = K_{p,q}$ *or* $G = K_3 \cdot K_{p,q}$ *for some* $p, q \geq 3$.

Theorem 4. *If* $k \geq 3$ *and* $n \geq 6k - 3$, *then there is a* C_{2k+1}*-saturated graph* G *on* n *vertices and* m *edges if* $\frac{k+1}{2}n - k \leq m \leq \left\lfloor \frac{(n-4k+5)^2}{4} \right\rfloor + \binom{2k+1}{2} - 6$.

Theorem 4 settles the saturation spectrum for given C_{2k+1} and n for all but $O(nk)$ of the $\binom{n}{2}$ possible values of m, since Füredi and Gunderson [9] proved that $ex(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$ when $n \geq 4k - 2$. For large enough n the lower bound on m in Theorem 4 may well be replaced by $\mathrm{sat}(n, C_{2k+1})$. One problem is that the precise value of $\mathrm{sat}(n, C_{2k+1})$ is unknown for $k \geq 3$. However, we remark that the conjectured optimal C_{2k+1}-saturated graphs constructed by Füredi and Kim [10] have $n + \frac{n}{2k-3} + O(k^2)$ edges, and can probably be used in a similar fashion as Proposition 12 to improve the lower bound on m, since the vertex b_1 in this construction can be seen to be k-suitable in the sense of the definition at the start of Section 2. It is also worth noting that very recently two papers concerning the saturation number of even cycles have appeared. Lan et al. [13] provided bounds on the saturation number of C_6 that are close to tight. In Ma et al. [14], the saturation number for the family of all cycles of length at least 6 is found.

2 | GENERAL CONSTRUCTIONS

Constructions with close to $\mathrm{sat}(n, C_{2k+1})$ edges can be obtained by identifying vertex-disjoint graphs at suitable vertices. For this purpose a vertex v in a graph G is *k-suitable* if for every vertex $u \neq v$ there is a u, v-path of length k, as well as a u, v-path of length at most $2k - 1$ that is of parity different from k.

Lemma 5. *Let G_1, G_2 be C_{2k+1}-saturated graphs with k-suitable vertices v_1, v_2, respectively.*

1. *If we identify G_1 at v_1 with G_2 at v_2, then we obtain a C_{2k+1}-saturated graph and the identified vertex v is k-suitable.*
2. *If we identify G_1 at v_1 with $K_{p,q}$ (for $p, q \geq k + 1$) at any vertex, then we obtain a C_{2k+1}-saturated graph.*

Proof. Observe that the new graphs G we obtain are still C_{2k+1}-free since v is a cut-vertex and a C_{2k+1} would need to be contained in G_1 or G_2 (respectively, $K_{p,q}$). Also the identified vertex v is still k-suitable, since the desired paths can be found in G_1 or G_2 as v_1, v_2 are k-suitable. To see that G is C_{2k+1}-saturated it suffices to add an edge u_1u_2 where $u_1 \in V(G_1) - \{v\}$ and $u_2 \notin V(G_1)$. In part 1 combining the v, u_i-paths of length k with u_1u_2, we get a $(2k + 1)$-cycle. For part 2, suppose that v_2 is in the part of order p in $K_{p,q}$. If k is even and u_2 is also in that part, then by the same argument as for part 1, we get a C_{2k+1} in $G + u_1u_2$. If u_2 is in the other part of $K_{p,q}$ and if we have a u_1, v-path of odd length ℓ in G_1, then together with u_1u_2 and a u_2, v-path of length $2k - \ell$ we again get a C_{2k+1}. A similar argument works when k is odd, but with the roles of the parts interchanged. □

We begin with our first general construction.

Definition 6. For $c \geq 1$ and positive integers $n_0, n_1, \ldots, n_c$, let $H(n_0, n_1, \ldots, n_c)$ be the graph obtained from $c + 1$ cliques $V_0, V_1, \ldots, V_c$ with $|V_i| = n_i$ by making every vertex in V_0 adjacent to a fixed vertex $v_i \in V_i$ for all $1 \leq i \leq c$. This graph has $n = n_0 + n_1 + \cdots + n_c$ vertices and $\binom{n_0}{2} + \binom{n_1}{2} + \cdots + \binom{n_c}{2} + cn_0$ edges (see, e.g., Figure 1).

Proposition 7. *Let m, n, k be integers with $k \geq 2$ and $\frac{k+1}{2}n - k \leq m$. Then there is a C_{2k+1}-saturated graph $H(k, n_1, n_2, \ldots, n_c)$ on n vertices and m edges with a k-suitable vertex if*

1. *$k = 2$, $n \geq 7$, and $m \leq 2n - 3$, or*

2. $k \geq 3$, $n \geq 3k + 1$, and $m \leq k(n - k) - \binom{k-1}{2}$.

Proof. $H(k, n_1, n_2, \ldots, n_c)$ is clearly C_{2k+1}-free as long as $n_1, \ldots, n_c \leq 2k$, and every vertex in $V_0 = \{x_1, \ldots, x_k\}$ is k-suitable. Moreover, if $c \geq k + 1$ then it is C_{2k+1}-saturated: Adding the edge $v_1 v_{k+1}$ we get the $(2k + 1)$-cycle $v_1 x_1 v_2 x_2 \ldots v_k x_k v_{k+1} v_1$. Similarly, adding the edge xy with $x \in V_1 - v_1$ or $x = x_1$ and $y \in V_k - v_k$ we get $x v_1 x_2 v_2 \ldots x_k v_k y x$. Finally, adding $v_1 y$ with $y \in V_k - v_k$ yields $v_1 x_1 x_2 v_2 x_3 v_3 \ldots x_k v_k y v_1$.

Let q, r be integers with $\binom{k}{2} + k(n - k) - m = q(k - 1) - r$ and $0 \leq r \leq k - 2$. Now $q(k - 1) - r \leq \binom{k}{2} + k(n - k) - \left(\frac{k+1}{2}n - k\right) = (k - 1)\frac{n-k}{2}$ implies that $q \leq n - 2k - 1$: If $k = 2$, then $r = 0$ and this follows directly from $n \geq 7$. For $k \geq 3$ we get $q \leq \left\lceil \frac{n-k}{2} \right\rceil \leq n - 2k - 1$, so when $n - k$ is even this follows from $n \geq 3k + 1$, and when $n - k$ is odd this follows from $n \geq 3k + 3$. In any case $c = n - k - q \geq k + 1$. Moreover, for $k \geq 3$ we get $q(k - 1) - r = \binom{k}{2} + k(n - k) - m \geq \binom{k}{2} + \binom{k-1}{2} = (k - 1)^2$ and for $k = 2$ that $q = 1 + 2(n - 2) - m \geq 0$, so that in either case $q \geq 0$.

Suppose first that $r = 0$ and consider $H(k, 2, \ldots, 2, 1, \ldots, 1)$ with q cliques of order 2, and $c - q = n - k - 2q \geq 0$ cliques of order 1. This indeed has n vertices and $\binom{k}{2} + q\binom{2}{2} + (n - k - 2q)\binom{1}{2} + k(n - k - q) = \binom{k}{2} + k(n - k) - q(k - 1) = m$ edges. When $k = 2$ this suffices since we must have $r = 0$. So we may now assume that $k \geq 3$ and $r > 0$, so that $q(k - 1) - r \geq (k - 1)^2$ implies $q \geq k$.

Replacing two cliques of order s with a clique of order $s - 1$ and $s + 1$ increases m by 1, so starting from $r = 0$ it suffices to show that we can make r such replacements starting with $k(\leq q)$ cliques of order 2. Let $b = \left\lfloor \frac{k}{8} \right\rfloor$ and $k = 8b + w$.

Recall that $r \leq k - 2$ and hence $r \leq 8b + w - 2$. Thus, finally, we must increase m over the range of values of r up to $k - 2$. This includes allowing w to range up to 7.

Since we have at least $8b$ cliques of order $s = 2$, this addresses the case $r \leq 4b$. Replacing pairs of the resulting $4b$ cliques of order $s = 3$ covers the case $r \leq 6b$. Replacing pairs of the resulting $2b$ 4-cliques and 2-cliques, respectively, again increases m by b each for a total of $r \leq 8b$. Replacing pairs of the $2b$ resulting 3-cliques we get $r \leq 9b$. Observe that $9b \geq k - 2 \geq r$, unless $b = 0$ (and thus $w = k \geq 3$) or $w \geq 4$. Thus it suffices to be able to make $w - 2$ additional replacements for $3 \leq w \leq 7$, using the $w = k - 8b$ 2-cliques we have not used so far. If $w = 3$, then the single replacement $(2, 2) \rightarrow (3, 1)$ suffices. For $w = 4, 5$ we have $(2, 2, 2, 2) \rightarrow (3, 2, 2, 1) \rightarrow (3, 3, 1, 1) \rightarrow (4, 2, 1, 1)$. For $w = 6, 7$ we can extend this by making one or two additional changes $(2, 2) \rightarrow (3, 1)$. □

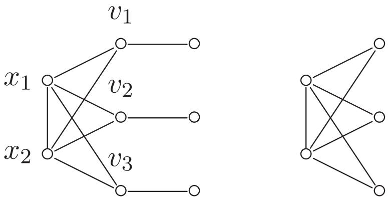

FIGURE 1 $H(2, 2, 2, 2)$ and $H(2, 1, 1, 1) = K_{1,1,3}$.

For the range when $m \geq kn - O(k^2)$ we have another family of constructions.

Definition 8. The C_r *blow up* $C_r(n_1, n_2, ..., n_r)$ is a graph whose vertices are partitioned into r parts V_i with $|V_i| = n_i \geq 1$, in which two vertices are adjacent if and only if they are in consecutive parts V_i, V_{i+1}, where subscripts are taken modulo r. If we put an asterisk on an entry n_i, then the corresponding part V_i is a clique rather than an independent set. (See Figure 2.) The graph $C_r(n_1, n_2, ..., n_r) + ij$ is the graph obtained from $C_r(n_1, n_2, ..., n_r)$ by adding every edge between V_i and V_j.

Proposition 9. *Let p, q, r, k be integers with $p, q, r \geq 2$ and $k \geq 3$.*

1. $C_6(1, 1, 2^*, 1, p, q)$ *is a C_5-saturated graph on $p + q + 5$ vertices and $(p + 1)(q + 1) + 5$ edges.*
2. $C_7(1, 1, 2, 2, r, p, q)$ *is a C_5-saturated graph on $p + q + r + 6$ vertices and $(p + 2)(r + q) - q + 7$ edges.*
3. $C_7(1, 1, 1, 2, r, p, q) + 13$ *is a C_5-saturated graph on $p + q + r + 5$ vertices and $(p + 2)(r + q) - q + 5$ edges.*
4. $C_{2k+2}(2, r, p, q, 1, (2k - 2)^*, 1, ..., 1)$ *is a C_{2k+1}-saturated graph on $p + q + r + 4k - 3$ vertices and $(p + 2)(q + r) - q - 4 + \binom{2k+1}{2}$ edges when $r \geq p \geq k - 2$, except if $p = k - 2 = 2$.*

Proof. The edge and vertex-counts are obvious, as is the fact that these blow-ups are C_{2k+1}-free. So it remains to show that they are saturated.

To see that the C_6-blow up is C_5-saturated, observe that adding an edge from V_i to V_{i+2} we get a C_5 by taking a vertex from each part other than V_{i+1}. Adding an edge from V_i to V_{i+3} we get a C_5 by using both vertices from V_2. Adding an edge in V_5 or V_6 we obtain a C_5 through the vertex in V_1 and V_4, respectively.

To see that the C_7-blow up is C_5-saturated, observe that adding an edge from V_i to V_{i+3} we get a C_5 by taking a vertex from each part other than V_{i+1}, V_{i+2}. Adding an edge from V_i to V_{i+2} we get a C_5 by using one vertex from V_{i+1} and one more from V_{i+2} (or V_i) and one from V_{i+3} (or V_{i-1}, respectively). Adding an edge inside a part V_i of order at least 2 we obtain a C_5 with two vertices in V_{i+1} (or V_{i-1}) and one in V_{i+2} (respectively, V_{i-2}).

The exact same proof works to see that the modified C_7-blow-up is C_5-saturated, except that we note that we cannot add the edge between V_1 and V_3, since it is already present. It remains to check the last graph for $k \geq 3$.

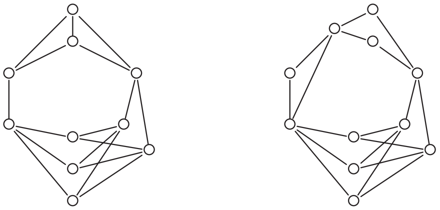

FIGURE 2 $C_6(1, 1, 2^*, 1, 2, 3)$ and $C_7(1, 1, 1, 2, 1, 2, 3) + 13$.

Observe that adding an edge from V_i to V_{i+2} we get a C_{2k+1} by taking a vertex from each part other than V_{i+1}. Adding an edge from V_i to V_{i+j} with $2 < j < 2k$ such that V_6 is among $V_i, \ldots, V_{i+j}$ we get a C_{2k+1} by using one vertex each from $V_i, \ldots, V_{i+j}$ except that we use $2k + 1 - j$ vertices from V_6.

It remains to consider the case when we add an edge xy with $x, y \in V_i$ for $1 \le i \le 4$. If $k = 3$, then since $p, q, r \geq 2$ we get a 7-cycle through a vertex in V_{i+3} (when $i \le 2$) or V_{i-3} (when $i \geq 3$), respectively. So we may assume that $k \geq 4$. Observe that $q \geq 2$, $r \geq p \geq 2$ and $q + r \geq p + 2 \geq k$ implies that we can find a $(2k + 2)$-cycle $v_1 v_2 \ldots v_{2k+2} v_1$ with $V_1 = \{v_1, v_3\}$, $V_{2k+2} = \{v_2\}$, $V_5 = \{v_7\}$, $V_4 \supseteq \{v_6, v_8\}$, $V_2 \supseteq \{v_4, v_{10}, v_{12}, \ldots, v_{2k+2}\}$, and $V_3 \supseteq \{v_5, v_9, v_{11}, \ldots, v_{2k+1}\}$. Thus if $i = 1$, then $xy = v_1 v_3$ and the cycle can be shortened to a $(2k + 1)$-cycle by omitting v_2. Similarly if $i = 4$, then we may assume that $xy = v_6 v_8$ to yield the $(2k + 1)$-cycle. If $i = 3$, then we may assume $xy = v_9 v_{11}$ (and thus omit v_{10}) as long as $2k + 1 \geq 11$. Otherwise if $p \geq k - 1$, then we may instead let $v_7 \in V_3$ and thus $xy = v_5 v_7$. Hence the only remaining case is $2 \geq k - 2 \geq p \geq 2$, but the case $p = k - 2 = 2$ is excluded. Finally for $i = 2$, in the case $k \geq 5$ we may assume that $xy = v_{2k} v_{2k+2}$, since $r \geq p \geq k - 2 \geq 3$. In the case when $k = 4$ and $r \geq 3$ we may let $v_2 \in V_2$ and $xy = v_2 v_4$. If $k = 4$ and $r = 2$, then $2 = r \geq p \geq 2$ and again there is nothing to show. □

The following technical lemma, whose proof closely follows Theorem 3 from [3], will help us use these constructions to cover large ranges of m.

Lemma 10. *Let k, y, z be integers with $k \geq 2$, $z \geq 6$, $z \geq 2k - 2$, $y \geq z(k - 1) - (k - 2)^2$, and $3z - 4 \le y \le t(z + 2) - 2$. Then we can find integers $p, q, r \geq 2$ with $p + q + r = z$, $(p + 2)(q + r) - q = y$, $r \geq p$, and $p + 2 \geq k$.*

Proof. Let $f(x) = x(z - x - 2)$ and let p be the smallest integer with $p \geq 2$, $p \geq k - 2$, and $f(p) \geq y - 2z + 2$. This p exists as long as $p^* = \frac{z-2}{2} \geq k - 2$, $p^* \geq 2$, and $f(p^*) \geq y - 2z + 2$, which is the case since $z \geq 2k - 2$, $z \geq 6$, and $y \le t(z + 2) - 2$.

Then let $q = f(p) - y + 2z \geq 2$, and $r = z - p - q$. Thus

$$\begin{aligned}(p + 2)(r + q) - q &= (p + 2)(z - p) - (f(p) - y + 2z)\\ &= 2(z - p) + 2p + y - 2z = y.\end{aligned}$$

It remains to verify $r \geq p$. Indeed, if $f(p - 1) \le y - 2z + 1$, then $q = f(p) - (y - 2z + 1) + 1 \le f(p) - f(p - 1) + 1 = z - 2p$ and thus $r = z - p - q \geq z - p - (z - 2p) = p$. If $f(p - 1) \geq y - 2z + 2$, then by definition $p = 2$ or $p = k - 2$. Thus $y - 2z + 2 \le f(p - 1) = f(1) = z - 3$ or $y - 2z + 2 \le f(k - 3) = (k - 3)(z - k + 1)$, which contradicts our assumptions that $y \geq 3z - 4$ and $y \geq z(k - 1) - (k - 2)^2$. □

The results now allow us to prove Theorem 4.

Proof of Theorem 4. Let $k \geq 3$ and $n \geq 6k - 3$. If $\frac{k+1}{2}n - k \le m \le k(n - k) - \binom{k-1}{2}$, then Proposition 7.2 shows that the desired C_{2k+1}-saturated graph on n vertices and m edges exists.

Now let $z = n - 4k + 3 \geq 2k \geq 6$ and $y = m + 4 - \binom{2k+1}{2} \leq t(n - 4k + 5) - 2 = t(z + 2) - 2$. Then

$$\begin{aligned}
y &\geq k(n-k) - \binom{k-1}{2} + 1 + 4 - \binom{2k+1}{2} = kn - \tfrac{k}{2}(7k-1) + 4 \\
&\geq 3n + (k-3)(6k-3) - \tfrac{k}{2}(7k-1) + 4 \geq 3n - 12k + 5 = 3z - 4 \quad \text{and} \\
y &\geq kn - \tfrac{k}{2}(7k-1) + 4 \geq n(k-1) + (6k-3) - 3.5k^2 + 0.5k + 4 \\
&\geq n(k-1) - 5k^2 + 11k - 7 = z(k-1) - (k-2)^2.
\end{aligned}$$

Thus by Lemma 10 we can find integers $p, q, r \geq 2$ with $r \geq p \geq k - 2$, $p + q + r = n - 4k + 3$, and $(p + 2)(q + r) - q = m + 4 - \binom{2k+1}{2}$. Proposition 9.4 yields the desired graph, except when $p = k - 2 = 2$. But in that case $q + r = n - 15$ and $m = 4(n - 15) - q - 4 + \binom{9}{2} = 4n - 28 - q \leq 4n - 30 \leq k(n - k) - \binom{k-1}{2}$, so this is covered by the first paragraph. ☐

3 | THE SATURATION SPECTRUM FOR C_5

The graphs K_3 and K_4 are C_5-saturated graphs in which every vertex is 2-suitable. Thus $T(n-2) \cdot K_3$ yields a C_5-saturated graph on n vertices and $t(n-2) + 3$ edges. We will prove in Theorem 15 that this is the maximum for all nonbipartite graphs on $n \geq 10$ vertices. For small values of n a case analysis/computer search yields the following.

Remark 11. For $n = 7$ the maximum number of edges for a nonbipartite C_5-free graph is 12, achieved only by $K_4 \cdot K_4$. For $n = 5, 6, 8$ the maximum is $2n - 3$, achieved (not only) by $K_{1,1,n-2}$. For $n = 9$ the maximum is $2 \cdot 9 - 3 = 15 = t(7) + 3$ and it is achieved (not only) by $K_3 \cdot T(7)$ and $K_{1,1,7}$. For $n = 10$ the maximum is uniquely achieved by $K_3 \cdot T(8)$.

Theorem 1, Theorem 2, Remark 11, and Proposition 7 now allow us to describe the C_5-saturation spectrum for $5 \leq n \leq 9$. For simplicity we will let $H_{n,p}$ denote the n-vertex graph $H(2, 2, ..., 2, 1, ..., 1)$ such that exactly $p + 1 \leq n/2$ of the cliques have order 2. Thus $H_{n,0} = K_{1,1,n-2}$, and $H_{n,p}$ has $2n - 3 - p$ edges. Table 1 exhibits a C_5-saturated graph for every possible number of edges, except for $K_{3,6}$ and $K_{4,5}$ for $n = 9$. Observe that for $n = 5, 6$ we have

TABLE 1 Saturation spectrum of C_5 for $5 \leq n \leq 9$ and $m \leq \mathrm{ex}(8, C_5) = 16$.

$n\backslash m$	6	7	8	9	10	11	12	13	14	15	16
5	$K_3 \cdot K_3$	$H_{5,0}$	–	–	–	–	–	–	–	–	–
6	–	–	$H_{6,1}$	$H_{6,0}$	–	–	–	–	–	–	–
7	–	–	–	$H_{7,2}$	$H_{7,1}$	$H_{7,0}$	$K_4 \cdot K_4$	–	–	–	–
8	–	–	–	–	$H_{8,3}$	$H_{8,2}$	$H_{8,1}$	$H_{8,0}$	–	$K_{3,5}$	$K_{4,4}$
9	–	–	–	–	–	–	$H_{9,3}$	$H_{9,2}$	$H_{9,1}$	$H_{9,0}$	–

$\mathrm{ex}(n, C_5) = 2n - 3$ and $\mathrm{ex}(7, C_5) = 12 = 2 \cdot 7 - 2$, so the spectrum for $n = 5, 6, 7$ is gap-free. For $n = 8, 9$ gaps occur in the spectrum as Remark 11 implies that there are no C_5-free nonbipartite graphs on 8 vertices and 14 edges (and none on 9 vertices and 16 edges).

More generally we get the following:

Proposition 12. *If* $n \geq 5$ *and* $\left\lceil \frac{10(n-1)}{7} \right\rceil \leq m \leq 2n - 3$, *then there is a* C_5*-saturated graph* $G_{n,m}$ *on* n *vertices and* m *edges with a 2-suitable vertex.*

Proof. For $n = 5, 6$ the four required graphs are found in Table 1. If $\left\lceil \frac{3n}{2} \right\rceil - 2 \leq m \leq 2n - 3$, then the result follows from Proposition 7. It can be checked that if $7 \leq n \leq 13$, then $\left\lceil \frac{3n}{2} \right\rceil - 2 = \left\lceil \frac{10(n-1)}{7} \right\rceil$, so that we may now assume that $n \geq 14$ and $m \leq \frac{3n}{2} - 2$. If we let $n' = n - 7$ and $m' = m - 10$, then $\left\lceil \frac{10(n'-1)}{7} \right\rceil = \left\lceil \frac{10(n-1)}{7} \right\rceil - 10 \leq m - 10 = m'$ and $m' = m - 10 \leq \frac{3n}{2} - 2 - 10 = \frac{3n'-3}{2} \leq 2n' - 3$ since $n' \geq 3$. Thus we obtain the desired graph if we identify $G_{n',m'}$ and $G_{8,10} = H_{8,3}$ at a 2-suitable vertex. □

This result enables us to give all constructions we will need to prove Theorem 3.

Theorem 13. *There is a* C_5*-saturated graph on* $n \geq 5$ *vertices and* m *edges for every* m *with* $\mathrm{sat}(n, C_5) \leq m \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 6 = t(n-3) + 6$.

Proof. Recall that $\mathrm{sat}(n, C_5) \geq \left\lceil \frac{10(n-1)}{7} \right\rceil - 1$, so that Proposition 12 settles the case $m \leq 2n - 3$.

If $2n - 2 \leq m \leq 3n - 16$, then let $k = m - 2n + 7 \geq 3$ and observe that $n - k - 2 = 3n - m - 9 \geq 5$. Thus, the graph obtained by attaching $K_{3,k}$ at a 2-suitable vertex of $H_{n-k-2,0}$ is a C_5-saturated graph on n vertices and $e(K_{3,k}) + e(H_{n-k-2,0}) = 3k + 2(n - 2 - k) - 3 = 2n + k - 7 = m$ edges.

If $m = 3n - 15 \geq 2n - 2$, then $n \geq 13$ and $C_7(1, 1, 2, 2, 2, 2, n - 10)$ is the desired graph.

For $m \geq t(n-3) + 4$ we have the following: $K_4 \cdot T(n-3)$ has $m = t(n-3) + 6$ edges. For odd $n = 2s + 3$ with $s \geq 4$ we have $C_6(1, 1, 2^*, 1, s-1, s-1)$ for $m = s^2 + 5 = t(n-3) + 5$ and $C_6(1, 1, 2^*, 1, s-2, s)$ for $m = s^2 + 4 = t(n-3) + 4$. For even $n = 2s + 2$ with $s \geq 4$ we have $C_6(1, 1, 2^*, 1, s-2, s-1)$ for $m = s(s-1) + 5 = t(n-3) + 5$, and for $s > 4$ we have $K_4 \cdot K_{s-2,s+1}$ for $m = (s-2)(s+1) + 6 = s(s-1) + 4 = t(n-3) + 4$. For $n = 2 \cdot 4 + 2 = 10$ the remaining value of $m = t(7) + 4 = 16 < 2(10) - 3$ is given by Proposition 12.

It remains to consider the case $3n - 15 < m \leq t(n-3) + 3$, and therefore $n \geq 12$. In this case there is a C_5-saturated graph $C_7(1, 1, 1, 2, r, p, q) + 13$ on n vertices and m edges, since Lemma 10 with $k = 2$, $z = n - 5 \geq 6$, and $y = m - 5$ guarantees the appropriate choices of p, q, r since $m - 5 \geq 3n - 19 = 3z - 4 \geq z$ and $m - 5 \leq t(n-3) - 2 = t(z+2) - 2$. □

The following results will simplify some of the computations needed to prove Theorem 3.

Lemma 14. *For positive integers x, n with $x < n$ we have*

1. $t(n+2) = t(n) + n + 1$.
2. $t(n+1) = t(n) + \lceil n/2 \rceil$.
3. $t(x) + t(n-x) \geq t(x+1) + t(n-x-1)$ *if and only if* $x < \frac{n}{2}$ *or* $x = \frac{n}{2}$ *is even.*

Proof. The first part is an easy computation, and the second part follows since $T(n+1)$ is obtained from $T(n)$ by adding a vertex to the part of order $\lfloor n/2 \rfloor$. The last part follows since moving a vertex from the larger part of $T(n-x)$ to the smaller part of $T(x)$ we gain exactly $\left\lceil \frac{x}{2} \right\rceil - \left\lceil \frac{n-x}{2} \right\rceil$ edges. This quantity is positive precisely when $x \geq \frac{n}{2}$ unless $x = \frac{n}{2}$ is even. □

This is useful in the proof of Theorem 15, which combines with Remark 11 and Theorem 13 to prove Theorem 3.

Theorem 15. *The following statements hold for every nonbipartite C_5-saturated graph G on $n \geq 10$ vertices and m edges.*

1. *If G has a cut-vertex, then $m \leq t(n-2) + 3$. If also $m \geq t(n-3) + 7$, then $G = K_3 \cdot K_{p,n-2-p}$ for $n - 2 - p \geq p \geq 3$.*
2. *If G is 2-connected, then $m \leq t(n-3) + 6$.*
3. *$m \leq t(n-2) + 3$ with equality only for $G = K_3 \cdot T(n-2)$.*

Proof. We prove these statements simultaneously by induction on n. The base cases $n = 10$ follow from Remark 11, since $t(10-2) + 3 = 19$ and $t(10-3) + 6 = 18$. So suppose now that $n > 10$.

Statement 1: Suppose that $G = G_1 \cup G_2$ for the C_5-saturated graphs G_1, G_2 with $V(G_1) \cap V(G_2) = \{v\}$. Let $s = |V(G_1)| \leq |V(G_2)| = n - s + 1$. Observe that for $s \geq 5$ we have $n - s + 1 \geq 6$ and thus by Theorem 1 and Lemma 14 we have $m \leq t(n-s+1) + t(s) + 1 \leq t(n-4) + t(5) + 1 = t(n-3) - \left\lceil \frac{n-4}{2} \right\rceil + 7 \leq t(n-3) + 3$.

So we may assume that $2 \leq s \leq 4$ and that $G_1 = K_s$.

Suppose now that G_2 is bipartite. Thus, $G_2 = K_{p,n-s+1-p}$ and $s > 2$, since otherwise G is bipartite. If $s = 4$, then $m = e(G_1) + e(G_2) \leq 6 + t(n-3)$. So $s = 3$ and we have the desired graph G where $p \geq 3$ in order for G_2 to be C_5-saturated.

Thus we may assume that G_2 is not bipartite. If $n - s + 1 \geq 9$, then by Statement 3 and Remark 11 we get $m = e(G_1) + e(G_2) \leq \binom{s}{2} + t(n-s-1) + 3$ and the result follows. Thus $8 \geq n - s + 1 \geq 11 - 4 + 1 = 8$ and equality must hold. So $e(G_2) \leq 2 \cdot 8 - 3 = 13$ and the result still follows since $m \leq 6 + 13 = 19 = t(11-3) + 3$.

Statement 2: Suppose first that G has a vertex-cut $\{x_1, x_2\}$, and that $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{x_1, x_2\}$. We choose this cut such that $|V(G_1)|$ is minimal. Thus $s = |V(G_1)| \leq |V(G_2)| = n + 2 - s$, for $s \geq 3$.

If $6 \le s \le \frac{n}{2} + 1$, then $m \le e(G_1) + e(G_2) \le t(s) + t(n + 2 - s) \le t(6) + t(n - 4) \le t(n - 3) + 6$. If $s = 5$, then $m \le t(5) + 1 + t(n - 3) = t(n - 3) + 7$ and equality can only hold if $G_2 = T(n - 3)$, and $G_1 = K_{1,1,3}$ or $K_4 \cdot K_2$, and x_1, x_2 are nonadjacent. However, in these cases it is easy to see that G contains a C_5. So we may now assume that $s < 5$.

If for some $z \in V(G_1) - \{x_1, x_2\}$ we have that $G - z$ is nonbipartite, then inductively by Statement 3, $m \le e(G - z) + s - 1 \le t((n - 1) - 2) + 3 + s - 1 \le t(n - 3) + 6$. (Observe that while an induced subgraph of a C_5-saturated graph need not be C_5-saturated, it must be C_5-free, and thus be a spanning subgraph of a C_5-saturated graph.)

Now suppose that for every $z \in V(G_1) - \{x_1, x_2\}$ we have that $G - z$ is bipartite. If $s = 3$, then x_1, x_2 are adjacent to $z \in V(G_1) - \{x_1, x_2\}$, because G is 2-connected. Since $G - z$ is bipartite, but G is nonbipartite, then x_1, x_2 must be in different parts X, Y of $G - z$. Moreover, x_1, x_2 are adjacent since G is C_5-saturated. If $Y' = N(x_1) - z$ and $X' = N(x_2) - z$, then there is no edge between $X' - x_1$ and $Y' - x_2$ since those would yield 5-cycles through z. However, all edges between $X - X'$ and $Y - x_2$ (and between $Y - Y'$ and $X - x_1$) are present since adding in such an edge would not create a 5-cycle. Thus $G - z$ is isomorphic to $C_6(1, 1, p, q, r, u)$, where $p, q, r, u \ge 1$ since otherwise at least one of x_1 or x_2 would be a cut-vertex of G. Moreover $p, u \ge 2$, since otherwise we could add an edge between z and X' or Y' without creating a C_5. Thus for $a = p + r$ and $u + q = n - 3 - a$ we get

$$\begin{aligned} e(G) &= 2 + 1 + p + pq + qr + ru + u \\ &= (p + r)(u + q) - (p - 1)(u - 1) + 4 \\ &\le a(n - 3 - a) - 1 + 4 \le t(n - 3) + 3. \end{aligned}$$

If $s = 4$ then x_1, x_2 are adjacent to both $z_1, z_2 \in V(G_1) - \{x_1, x_2\}$, because G is 2-connected and otherwise we are back in case $s = 3$. Since $G - z_1$ is bipartite, z_2 is in one part, Y, and x_1, x_2 are in the other part X and thus they are nonadjacent. Since G is not bipartite it now follows that z_1 is adjacent to z_2, so that $G_2 = K_4 - x_1x_2$. If $Y_i = N(x_i) \cap Y$, then $Y_i \neq \varnothing$ since otherwise x_{3-i} would be a cut-vertex. Moreover, $Y_1 \cap Y_2 = \{z_2\}$, since otherwise there is a C_5 through z_1, z_2. Also $Y_1 \cup Y_2 = Y$, because otherwise we could add the edge yx_1 for $y \in Y - Y_1 \cup Y_2$ without creating a C_5, since such a C_5 would need to include z_1 or z_2 and x_2 as well. Let $X' = X - \{x_1, x_2\}$ and observe that there is no edge missing between X' and Y since otherwise such an edge could be added without creating a C_5. Thus $G = C_6(1, 2^*, 1, |Y_2 - z_2|, |X'|, |Y_1 - z_2|)$ has

$$m = e(G_1) + e(G_2) = 5 + (|X| - 1)|Y| \le 5 + t(n - 3).$$

We may now assume that G is 3-connected. If we can find a vertex v of degree at most $\left\lceil \frac{n}{2} \right\rceil - 2$ such that $G - v$ is nonbipartite (and thus contained in a nonbipartite 2-connected C_5-saturated graph on $n - 1$ vertices), then by Statement 2 for $n - 1$ we get $m \le e(G - v) + d(v) \le t(n - 4) + 6 + \left\lceil \frac{n - 4}{2} \right\rceil = t(n - 3) + 6$. So suppose every vertex v such that $G - v$ is nonbipartite has at least $\frac{n}{2} - 1$ neighbors.

Let C_k be an odd cycle of minimum length k in G and let $V(C_k) = S$. If $k > 3$, then $k \ge 7$ and no vertex $v \in \overline{S} = V(G) - S$ can have more than 2 neighbors in S or we would find a shorter odd cycle. Thus $e(S, \overline{S}) \le 2(n - k)$ and it follows from Lemma 14 and

Theorem 1 that $e(G) = e(C_k) + e(S, \overline{S}) + e(G - S) \le k + 2(n - k) + t(n - k) + 2 = t(n - k + 2) + n + 1 \le t(n - 5) + n + 1 = t(n - 3)$, as desired. So we can assume that G contains triangles.

Next consider a $K_{1,1,m}$ subgraph of G where m is as large as possible. Since G contains triangles $m \ge 1$ and we let $Z = \{z_1, z_2, \ldots, z_m\}$ be the part of size m, and $S = V(K_{1,1,m}) = \{x, y\} \cup Z$. Since G is C_5-free, Z must be an independent set in G except if $m = 2$ and S induces a K_4. Furthermore, every vertex in $\overline{S}$ has at most 1 neighbor in S since otherwise we contradict the maximality of m or we can find a C_5-subgraph.

Suppose first that $m = 1$. So $S = \{x, y, z_1\}$ is the vertex set of a triangle in G such that every vertex in $\overline{S}$ has at most one neighbor in S. Moreover since G is 3-connected, every vertex $v \in S$ has a distinct neighbor $v' \in \overline{S}$, and these neighbors satisfy $d(v') \ge \frac{n}{2} - 1$, since $G - v'$ is not bipartite. The vertices x', y', z_1' are also pairwise nonadjacent and share no common neighbors, since otherwise there would be a C_5. Thus $n \ge |N(x')| + |N(y')| + |N(z_1')| + 3 \ge 3\left(\frac{n}{2} - 1\right) + 3 = 3n/2$, a contradiction.

If $m \ge 2$, then every vertex in Z has degree at least $\frac{n}{2} - 1$. Thus $n - (m + 2) \ge e(Z, \overline{S}) \ge m(\frac{n}{2} - 1 - 2)$ or equivalently $2n - 4 \ge m(n - 4)$. But then $n > 10$ implies that $m \le 2$, so that it remains to consider the case $m = 2$.

If S induces a K_4, then every vertex in $\overline{S}$ has at most one neighbor in S, but every vertex in S has degree at least $\left\lceil \frac{n}{2} \right\rceil - 1$. Thus $n - 4 \ge e(S, \bar{S}) \ge 4\left(\left\lceil \frac{n}{2} \right\rceil - 1 - 3\right)$, which is impossible for $n > 10$ except when $n = 12$. Even when $n = 12$ we can only have equality if $\overline{S}$ induces a graph on 8 vertices and at least $\frac{1}{2} \cdot 8\left(\left\lceil \frac{n}{2} \right\rceil - 1 - 1\right) = 16$ edges. The only such graph that is C_5-free is $T(8) = K_{4,4}$, but it is easy to see that this must form a C_5 together with the K_4, since each vertex in K_4 must be adjacent to 2 unique vertices in $K_{4,4}$.

So we may now assume that $Z = \{z_1, z_2\}$ is an independent set. In this case $n - 4 \ge e(S, \overline{S}) \ge (d(x) - 3) + (d(y) - 3) + 2(\frac{n}{2} - 1 - 2)$, or equivalently $d(x) + d(y) \le 8$. Since without loss of generality $d(x) \le d(y)$, it follows that $d(x) \le 4 < \frac{n}{2} - 1$. Hence $G - x$ is bipartite with parts X, Y such that $y \in Y$ and $Z \subseteq X$. Since every vertex in $\overline{S}$ has at most one neighbor in $S \supseteq \{z_1, z_2\}$ it follows that each vertex in $Y - y$ has at most $|X| - 1$ neighbors in X. Moreover, $|Y| \ge (d(z_1) - 2) + (d(z_2) - 2) + 1 = d(z_1) + d(z_2) - 3 \ge 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) - 3 \ge n - 5$ and $|X| = n - 1 - |Y| \le 4$. Thus

$$\begin{aligned} e(G) &= e(X, Y) + d(x) \le (|X| - 1)|Y| + 1 + 4 \le 3(n - 5) + 5 \\ &= t(n - 3) - t(n - 9) + 8. \end{aligned}$$

This yields the desired inequality when $t(n - 9) \ge 2$, or equivalently $n \ge 12$. Since $n > 10$ it only remains to consider the case $n = 11$. But even in that case we have $e(G) \le t(n - 3) + 6$, unless $|X| = 4$ and every vertex in Y is adjacent to every vertex in X except one of z_1, z_2. So if we let $x' \in X - Z$ and $y' \in Y - y$, then $y'z_i \in E(G)$ for some i and thus $y'z_ixyx'y'$ is a C_5, a final contradiction.

Statement 3: This follows directly from Statements 1 and 2. □

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