Discrete Mathematics 313 (2013) 2772–2777

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Precise location of vertices on Hamiltonian cycles

Ralph J. Faudree [a,*], Ron J. Gould [b]

[a] University of Memphis, Memphis, TN 38152, United States

[b] Emory University, Atlanta, GA 30033, United States

ARTICLE INFO

Article history:
Received 22 January 2013
Received in revised form 12 May 2013
Accepted 7 August 2013
Available online 4 September 2013

Keywords:
Hamiltonian cycle
Minimum degree
Vertex location

ABSTRACT

Given $k \geq 2$ fixed positive integers $p_1, p_2, \ldots, p_{k-1} \geq 2$, and k vertices $\{x_1, x_2, \ldots, x_k\}$, let G be a simple graph of sufficiently large order n. It is proved that if $\delta(G) \geq (n + 2k - 2)/2$, then there is a Hamiltonian cycle C of G containing the vertices in order such that the distance along C is $d_C(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k - 1$. Also, let $\{(x_i, y_i)|1 \leq i \leq k\}$ be a set of k disjoint pairs of vertices and a graph of sufficiently large graph n and $p_1, p_2, \ldots, p_k \geq 2$ for $k \geq 2$ fixed positive integers. It will be proved that if $\delta(G) \geq (n + 3k - 1)/2$, then there are k vertex disjoint paths $P_i(x_i, y_i)$ of length p_i for $1 \leq i \leq k$.

1. Introduction

Given an ordered set of vertices $S = \{x_1, x_2, \ldots, x_k\}$ in a graph, there are a series of results giving minimum degree conditions that imply the existence of a Hamiltonian cycle such that the vertices in S are located in order on the cycle with restrictions on the distance between consecutive vertices of S. Examples include results by Kaneko and Yoshimoto [6], Sárközy and Selkow [9], and Faudree, Gould, Jacobson, and Magnant [3]. In each of these results the distances on the Hamiltonian cycle was close, by not precise relative to the predetermined objective. In the case of pairs of vertices, there are results in which the distance between the vertices is precise by Faudree and Li [5] and Faudree, Lehel, and Yoshimoto [4]. The objective is to replace 2 by a fixed number $k \geq 3$ of vertices that can be placed on a Hamiltonian cycle at precise predetermined distances. However, in this case the distances will not be a positive fraction of the order of the Hamiltonian cycle, as was true in some of the previous cases.

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. The connectivity of a graph G will be denoted by $\kappa(G)$ and the independence number by $\alpha(G)$. A path (or cycle) with an ordered set of vertices $\{x_1, x_2, \ldots, x_k\}$ will be denoted by $(x_1, x_2, \ldots, x_k)$ (or $(x_1, x_2, \ldots, x_k, x_1)$). If x_i is a vertex of a cycle (path) then x_i^+ will denote the successor x_{i+1}, and if S is a set of its vertices, then S^+ will denote the set of all successors of the vertices of S. The set S^- of predecessors is defined similarly. Given a cycle C containing vertices x and y, let $d_C(x, y)$ denote the distance between x and y on the cycle C. The set of all adjacencies of a vertex $v \in G$ in $S \subset G$ will be denoted by $N_S(v)$, and we set $d_S(v) = |N_S(v)|$.

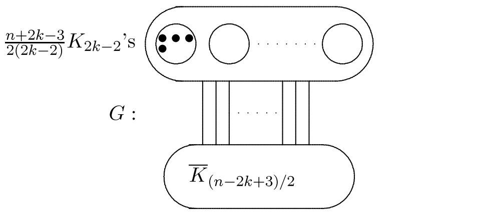
A graph G of order n is *panconnected* if between each pair of vertices x and y of G there is a path $P_i(x, y)$ of length i for each $d_G(x, y) \leq i < n$. Williamson [10] gave a minimum degree condition that implies a graph is panconnected.

Theorem 1 ([10]). *If G is a graph of order n with $\delta(G) \geq n/2 + 1$, then for any $2 \leq k \leq n - 1$ and for any vertices x and y, G has a path from x to y of length k.* □

We prove the following, which in some sense generalizes the result of Williamson [10].

* Corresponding author.
E-mail address: rfaudree@memphis.edu (R.J. Faudree).

0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.08.012



$$G = \overline{K}_{(n-2k+3)/2} + ((n+2k-3)/2(2k-2))(K_{2k-2})$$
$$\delta(G) = (n+2k-3)/2$$

Fig. 1. Panconnected example.

Theorem 2. *Let $\{p_1, p_2, \ldots, p_{k-1}\}$ be a set of $k-1$ integers each at least 2, and $\{x_1, x_2, \ldots, x_k\}$ an ordered set of k vertices of a graph G of order n. If*

$$\delta(G) \geq (n+2k-2)/2,$$

then, there is an $n_0 = n_0(k, p_1, p_2, \ldots, p_{k-1})$ such that if $n \geq n_0$ there is a path P such that $d_P(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k-1$.

The graph G in Fig. 1 implies that the degree condition in Theorem 2 is sharp, since if k vertices are all in one of the complete graphs K_{2k-2} of G, there does not exist $k-1$ paths of length 3 between consecutive pairs of these vertices.

The following conjecture of Enomoto [8] created research interest in placing vertices at precise distances on Hamiltonian cycles.

Conjecture 1 (*[8]*). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2 + 1$, then for any pair of vertices x, y in G, there is a Hamiltonian cycle C of G such that $d_C(x, y) = \lfloor n/2 \rfloor$.*

The following result of Faudree, Lehel, and Yoshimoto [4], which was motivated by the conjecture of Enomoto [8], deals with locating a pair of vertices at a precise distance on a Hamiltonian cycle.

Theorem 3 (*[4]*). *Let $k \geq 2$ be a fixed positive integer. If G is a graph of order $n \geq 6k$ and $\delta(G) \geq (n+2)/2$, then for any vertices x and y, G has a Hamiltonian cycle C such that $d_C(x, y) = k$.*

This result along with Theorem 2 will be generalized in some sense in the following theorem, which will be proved.

Theorem 4. *Let $\{p_1, p_2, \ldots, p_{k-1}\}$ be a set of $k-1$ integers and $\{x_1, x_2, \ldots, x_k\}$ a fixed set of k ordered vertices in a graph G of order n. If*

$$\delta(G) \geq (n+2k-2)/2,$$

then, there is a $n_0 = n_0(k, p_1, p_2, \ldots, p_{k-1})$ such that if $n \geq n_0$, there is a Hamiltonian cycle C such that $d_C(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k-1$.

Just as in the case of Theorem 2, Fig. 1 verifies that the degree condition of Theorem 4 is sharp.

For $k \geq 1$ a graph G is *k-linked*, if given any set of k disjoint pairs of vertices $\{(x_i, y_i) \mid 1 \leq i \leq k\}$, there exist k vertex disjoint paths $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_k(x_k, y_k)$ between the k pairs of vertices.

The next theorem to be proved is a natural companion of Theorem 2.

Theorem 5. *Let $\{(x_i, y_i) \mid 1 \leq i \leq k\}$ be a set of k disjoint pairs of vertices in a graph G of order n, and let $p_1, p_2, \ldots, p_k \geq 2$ and $k \geq 2$ be fixed positive integers. If*

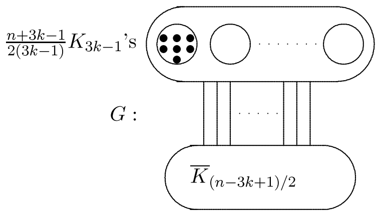
$$\delta(G) \geq (n+3k-1)/2,$$

then, there is an $n_0 = n_0(k, p_1, p_2, \ldots, p_k)$ such that if $n \geq n_0$, there are k vertex disjoint paths $P_i(x_i, y_i)$ of length p_i for $1 \leq i \leq k$.

The graph G in Fig. 2 that follows implies that the degree condition in Theorem 5 is sharp, since if the $2k$ vertices in the linkage are all in one of the K_{3k-1} complete graphs of G, there does not exist k paths of length 3 between the k pairs of these vertices.

2. Proofs

Proof of Theorem 2. Let $\{x_1, x_2, \ldots, x_k\}$ denote the k vertices and $p_1, p_2, \ldots, p_{k-1}$ the $k-1$ integers each at least 2 with $p = (\sum_{i=1}^{k-1} p_i) + 1$. The proof will be by double induction, first on k and then on p. In the case when $k = 2$, the result is just Theorem 1. The smallest value of p is $2(k-1)+1$. Observe that each x_i has at least $(n+2k-2)/2 - (k-1) = n/2$



$$G = \overline{K}_{(n-3k+1)/2} + ((n+3k-1)/2(3k-1))(K_{3k-1})$$
$$\delta(G) = (n+3k-1)/2$$

Fig. 2. Linkage example.

adjacencies in $G - \{x_1, x_2, \ldots, x_k\}$. Thus, each x_i and x_{i+1} have at least k common adjacencies in $G - \{x_1, x_2, \ldots, x_k\}$. Thus, there is a path $P(x_1, x_k)$ in which $d_P(x_i, x_{i+1}) = 2$, so the theorem is true when $p = 2(k-1) + 1$.

Case 1: Suppose $p_1 = 2$. Since $\delta(G) \geq (n + 2k - 2)/2$, there is a vertex $y \notin \{x_1, x_2, \ldots, x_k\}$ such that $x_1y, yx_2 \in G$. Consider the graph $G' = G - \{x_1, y\}$. Then G' has $n' = n - 2$ vertices and $\delta(G') \geq (n + 2k - 2)/2 - 2 = (n' + 2(k-1) - 2)/2$. Thus, by induction on k there is a path $P'(x_2, x_k)$ such that $d_{P'}(x_i, x_{i+1}) = p_i$ for $i \geq 2$. Thus, clearly there is a path $P(x_1, x_k)$ in G obtained from P' by adjoining the path (x_1, y, x_2). Hence we can assume that $p_1 > 2$.

Case 2: Suppose $p_1 > 2$. By induction we can assume there is a path $P' = P'(x_1, x_k)$ with $p-1$ vertices such that $d_{P'}(x_1, x_2) = p_1 - 1$ and $d_{P'}(x_i, x_{i+1}) = p_i$ for $i \geq 2$. Let $x_1 = y_1, y_2, \ldots, y_{p_1} = x_2$ be the path from x_1 to x_2. Consider the neighborhoods $N_{G-P'}(y_i)$ for $1 \leq i \leq p_1$ and observe that if $N_i = N_{G-P'}(y_i)$, then $|N_i| \geq (n + 2k - 2)/2 - (p - 2) \geq (n + 2k + 2)/2 - p$. If $N_i \cap N_{i+1} \neq \emptyset$, then the path from x_1 to x_2 can be lengthened by 1, giving the required path. Hence, we can assume that $N_i \cap N_{i+1} = \emptyset$. This implies that $|N_i \cap N_{i+2}| \geq (n + 6k - 4p + 4)/2$, and also there are no edges in $N_i \cap N_{i+2}$, since this would allow the path from x_1 to x_2 to be lengthened. This implies there is a nearly complete bipartite graph between $N_i \cap N_{i+2}$ and N_{i+1}. More generally, we can define the two sets N_o and N_e such that the intersection of the adjacencies of y_i for i odd outside of P' are in N_o and the intersection of the adjacencies of y_i for i even outside of P' are in N_e. Also, there are no edges in N_o or N_e and there is a nearly complete bipartite graph between N_o and N_e. For each vertex $y \in N_1$, there is a path (x_1, z, y) from x_1 to y avoiding the vertices $\{x_2, x_3, \ldots, x_k\}$. One of the following two subcases will occur. There will be at least p such paths that are disjoint except for the initial vertex x_1, or there will be at least p such paths of the form (x_1, z, y) where the x_1 and z are fixed and the y are different. Using the dense bipartite graph between N_o and N_e this implies in the first subcase that there are more than p paths of length p_1 from x_1 to x_2 whose interior vertices are disjoint. In the second case this implies there are more than p paths of length p_1 from x_1 to x_2 whose interior vertices are disjoint except for z. Consider the graph $G' = G - \{x_1\}$ in the first subcase and $G' = G - \{x_1, z\}$ in the second subcase. By the induction assumption on p there is a path $P''(x_2, x_k)$ with the desired properties. Since P'' has less than p vertices, one of the vertex disjoint paths from x_1 to x_2 will be disjoint from P'' and so can be appended to P'' to get the required path $P(x_1, x_k)$. This completes the proof of Theorem 2. □

The following proof has the same structure as the proof of Theorem 2, and so less detail will be given in this proof because of the similarity.

Proof of Theorem 5. Let $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}$ denote the k pairs of vertices and $p_1, p_2, \ldots, p_k$ the k integers each at least 2 with $p = (\sum_{i=1}^{k} p_i) + k$. The proof will be by double induction, first on k and then on p. In the case when $k = 1$, the result is just Theorem 1. The smallest value of p is $3k$. Observe that each x_i and y_i has at least $(n + 3k - 1)/2 - 2(k-1) - 1 = (n - k + 1)/2$ adjacencies in $G - \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$. Thus, each x_i and y_i have at least k common adjacencies in $G - \{x_1, y_1, x_2, y_2 \cdots, x_k, y_k\}$. Hence, there is system of vertex disjoint paths $P(x_1, y_1), P(x_2, y_2), \ldots, P(x_k, y_k)$ each of length 2, $1 \leq i \leq k$. Thus, the theorem is true when $p = 3k$.

Case 1: Suppose $p_1 = 2$. Since $\delta(G) \geq (n + 3k - 1)/2$, there is a vertex $z \notin \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$ such that $x_1z, zy_1 \in G$. Consider the graph $G' = G - \{x_1, z, y_1\}$. Then G' has $n' = n - 3$ vertices and $\delta(G') \geq (n + 3k - 1)/2 - 3 = (n' - 3(k-1) - 1)/2$. Thus, by induction on k there is a system of paths $P' = \{P(x_2, y_2), P(x_3, y_3), \ldots, P(x_k, y_k)\}$ such that $d_{P'}(x_i, y_i) = p_i$ for $i \geq 2$. Thus, clearly the required path system in G is obtained from P' by adding the path (x_1, z, y_1) to the system. Hence we can assume that $p_1 > 2$.

Case 2: Suppose $p_1 > 2$. By the induction assumption we can assume there is a path system P' with $p - 1$ vertices, such that the respective path lengths are $p_1 - 1, p_2, \ldots, p_k$. Let $x_1 = w_1, w_2, \ldots, w_{p_1} = y_1$ be the path from x_1 to y_1. Consider the neighborhoods $N_{G-P'}(w_i)$ for $1 \leq i \leq p_1$ and observe that if $N_i = N_{G-P'}(w_i)$, then $|N_i| \geq (n + 3k - 1)/2 - (p - 2) \geq (n + 3k + 3)/2 - p$. If $N_i \cap N_{i+1} \neq \emptyset$, then the path from x_1 to y_1 can be lengthened by 1, giving the required path from x_1 to y_1. Hence, we can assume that $N_i \cap N_{i+1} = \emptyset$. This implies that $|N_i \cap N_{i+2}| \geq (n + 9k + 7)/2 - 2p$, and also there are no edges in $N_i \cap N_{i+2}$, since this would allow the path from x_1 to x_2 to be lengthened by one. This implies there is a nearly complete bipartite graph between $N_i \cap N_{i+2}$ and N_{i+1}. More generally, there are two sets N_o and N_e such that all the adjacencies of y_i for i odd outside of P' are in N_o and all the adjacencies of y_i for i even outside of P' are in N_e. Also, there are no edges in N_o or N_e and there is a nearly complete bipartite graph between N_o and N_e. For each vertex $w \in N_1$, there is a path (x_1, z, w) from

x_1 to w which is disjoint from $\{y_1, x_2, y_2, \ldots, x_k, y_k\}$. One of the following two subcases will occur. There will be at least p such paths that are disjoint except for the initial vertex x_1, or there will be at least p such paths of the form (x_1, z, w) where the x_1 and z are fixed and the w are different. Using the dense bipartite graph between N_o and N_e this implies in the first subcase that there are more than p paths of length p_1 from x_1 to y_1 whose interior vertices are disjoint. In the second case this implies there are more than p paths of length p_1 from x_1 to y_1 whose interior vertices are disjoint except for z. Consider the graph $G' = G - \{x_1, y_1\}$ in the first subcase and $G' = G - \{x_1, z, y_1\}$ in the second subcase. By the induction assumption applied to G' there is a path system P'' containing $k-1$ vertex disjoint paths from x_i to y_i for $2 \le i \le k$ of the required lengths. Since P'' has less than p vertices, one of the vertex disjoint paths from x_1 to y_1 will be disjoint from P'' and so can be added to P'' to get the required path system. This completes the proof of Theorem 5. □

Before giving the proof of Theorem 4, some additional results needed in the proof will be stated and some proved.

We will use a classical result from Nash-Williams [7] on dominating cycles. A cycle C is called a *dominating cycle* in G if $G - C$ is an independent set.

Theorem 6 ([7]). *Let G be a 2-connected graph on n vertices with $\delta(G) \ge (n+2)/3$. Then, every longest cycle of G is a dominating cycle.* □

The minimum degree condition $\delta(G) \ge (n+2k-2)/2$ in a graph of order n forces a relationship between the connectivity and the independence number, which the following result gives.

Lemma 1. *If G is a graph of order n with $\delta(G) \ge (n+2k-2)/2 + 1$, then $\kappa(G) \ge \alpha(G)$.*

Proof of Lemma 1. Let $\kappa(G) = s$, and let S be a minimum cut set of G, so that $|S| = s$. Let H_1 and H_2 be the components of $G - S$, with h_1 and h_2 vertices, respectively. Let H_i^* be the subgraph spanned by $H_i \cup S$, for $i = 1, 2$. For $i = 1$ or 2, any independent set in H_i^* with a vertex in H_i will have at most $h_i + s - ((n+2k-2)/2)$ vertices. Hence, any independent set in G containing a vertex in H_1 or H_2 will have at most $h_1 + h_2 + 2s - (2(n+2k-2)/2) \le s$ vertices. Since S cannot contain an independent set with more than s vertices, $\alpha(G) \le s = \kappa(G)$ follows. □

In the following proof given a fixed k vertices $x_1, x_2, \ldots, x_k$ and fixed integers $p_1, p_2, \ldots, p_{k-1}$ each at least 2, then a path $P = P(x_1, x_k)$ containing $\{x_1, x_2, \ldots, x_k\}$ in order such that $d_P(x_i, x_{i+1}) = p_i$ will be called a *good path*. A cycle containing a good path will be called a *good cycle* of G. A good path will contain $p = (\sum_{i=1}^{k-1} p_i) + 1$ vertices.

Proof of Theorem 4. Let $x_1, x_2, \ldots, x_k$ be a fixed set of k vertices of G, $p_1, p_2, \ldots, p_{k-1}$ be a fixed set of $k-1$ integers each at least 2, and $p = (\sum_{i=1}^{k-1} p_i) + 1$.

Since $\delta(G) \ge (n+2k-2)/2$, $\kappa(G) \ge 2k$. If $\kappa(G) = 2k$, then it follows immediately that $G = S + (K_{(n-2k)/2} \cup K_{(n-2k)/2})$, where $S \subseteq K_{2k}$. With n being sufficiently large, it is straightforward to construct the required Hamiltonian cycle that is good. Hence, we will assume from this point on that $\kappa(G) \ge 2k+1$.

We will first prove the following claim.

Claim 1. *There is a good path P in G such that $\kappa(G - P) \ge 2$.*

Theorem 2 implies there is a good path P with $p+1$ vertices. If $\kappa(G) \ge p+3$, then, $\kappa(G-P) \ge 2$, which verifies the claim. Thus, we can assume that $2k+1 \le \kappa(G) \le p+2$, and so G has a minimum cutset S of order s, $2k+1 \le s \le p+2$. Let H_1, H_2 be the connected components of $G-S$, and so $\delta(H_1), \delta(H_2) \ge (n+2k-2)/2 - s$ and $(n+2k)/2 - s \le |H_1| \le |H_2| \le (n-2k)/2$. Since p is fixed and n is sufficiently large, this implies that both H_1 and H_2 are nearly complete graphs. Also, there is a matching with s edges between S and each of the H_i. Also, each of the vertices of S will have a large number of adjacencies in either H_1 or H_2 or possibly both. In building the $k-1$ subpaths that make up the good path P in G, there are several cases to consider: both endvertices are in some H_i, endvertices are in different H_i, or at least one of the endvertices is in S. However, in each of these cases it is straightforward to show that the subpath can be constructed using at most one vertex of S in the interior of the path. Also, one can avoid a fixed neighbor of a vertex in S in building these paths, since H_1 and H_2 are so dense. Thus, a good path P can be constructed using at most $s' \le k + (k-1)$ vertices of S. Thus, $G - P$ would contain two nearly complete graphs with $(s - s') \ge 2$ disjoint paths of length 2 between the two nearly complete graphs. Hence, $\kappa(G - P) \ge 2$. This proves Claim 1. □

Consider the graph $H = G - P$, where P is good path with $p+1$ vertices. Since H is 2-connected and $\delta(H) \ge (n+2k-2)/2 - p = (n+2k-2p-2)/2$, by Dirac's Theorem [2] H has a cycle C of length at least $n+2k-2p-2$. Also, by theorem [7] this cycle is a dominating cycle. Thus, $G - P - C$ is an independent set with at most $p - 2k + 3$ vertices.

Claim 2. *There is a good cycle in G of length at least $n + 2k - p - 2$.*

Consider the case when x_1 and x_k of the path P have no neighbors in the independent set $G - C$, and so $d_C(x_i) \ge (n+2k-2p-2)/2$, for $i = 1, k$. If there exist a neighbor of x_1 and a neighbor of x_k which are consecutive on C, then P and C join into a good cycle of length at least $n+2k-p-2$ that misses the independent set $G-C$, and thus is dominating. If a neighbor of x_1 and a neighbor of x_k are never consecutive on C, then a good cycle can be formed by selecting a neighbor of x_k closest to a neighbor of x_1 on C, which will yield a cycle of length at least $2(n+2k-2p-2)/2+p = n+2k-p-2$. If x_1

or x_k has an adjacency in $G-C$, say x_1' or x_k', then this longer path P' with endvertices x_1' or x_k', can be used as in the previous argument to insert P' into C to obtain a good cycle of the same length or longer. This completes the proof of Claim 2. □

Claim 3. *If C is a longest good cycle and it has length at least $n+2k-p-2$, then C is a dominating good cycle.*

Let $C = P \cup Q$ be a good cycle of maximum length $m \geq n+2k-p-2$, where P is the good path on $p+1$ vertices from x_1 to x_k, and Q is the path from x_k to x_1 with $m-p+1$ vertices. Assume that $H = G-C$ is not independent. Let u and v be endvertices of a longest path in H. By the maximality of C, neither u nor v can be adjacent to consecutive vertices of Q, and also any adjacency of u on Q implies that v is not adjacent to any vertex of Q within a distance 2 of this adjacency. Each vertex of H has at least $(n+2k-2)/2-(n-m)+1$ adjacencies in C, and so $d_C(v) \geq m-n/2+k$. Also, $d_Q(u) \geq m-n/2+k-p+2$. If $uz \in G$ for $z \in Q$, then $vz^+, vz^{++} \notin G$ except for two vertices at the end of the path Q, since this would result in a longer cycle than C'. This implies that $d_C(v) \leq m-2(m-n/2+k-p)$, which gives the following inequality:

$$m-n/2+k \leq d_C(v) \leq m-2(m-n/2+k-p).$$

This results in $m \leq 3n/4-3k/2+p$, which is in contradiction to the fact that $m \geq n+2k-p-2$, since p and k are fixed and n is sufficiently large. Thus, we can assume that H is an independent set. This completes the proof of Claim 3. □

Claim 4. *If P is a good path in a graph G with $\alpha(G) \geq (n+2k-2p-2)/2$ and $\delta(G) \geq (n+2k-2)/2$, then P can be inserted into a dominating good cycle of G.*

Since $\alpha(G) \geq (n+2k-2p-2)/2$ and $\delta(G) \geq (n+2k-2)/2$, Lemma 1 implies that $\kappa(G) \geq \alpha(G) \geq p+3$. Therefore, $G-P$ is 2-connected. Therefore, by Claims 1–3, P can be inserted into a good cycle of length at least $n+2k-p-2$. This completes the proof of Claim 4. □

By Claims 1–3, G has a dominating good cycle $C = P \cup Q$ of maximum length $m \geq n+2k-p-2$, where P is a good path from x_1 to x_k, Q is a path from x_k to x_1, and $H = G-C$ is an independent set. Given any $w \in H$, the maximality of C implies that w cannot be adjacent to two consecutive vertices of Q. Moreover, $A(w) = N^+_{Q-x_1}(w) \cup \{w\}$ is an independent set, since any adjacency within $A(w)$ would result in a longer good cycle including w.

Observe that every $w' \in H - A(w)$ has at most one adjacency in $A(w)$, for otherwise a good cycle could be formed including w and w'. Therefore each w' can either be added to $A(w)$ or can replace its only neighbor in $A(w)$. In this way we obtain an independent set $A(H)$ containing H such that $|A(H)| \geq |A(w)| = |N_{Q-x_1}(w)|+1 \geq (d_G(w)-d_{P-x_k}(w))+1 \geq (n+2k-2p-2)/2$.

For $w \in H$, let $U(w) = N^+_Q(w) \cap N^-_Q(w)$. If $u \in U(w)$, then w is interchangeable with u to obtain a good cycle C' that includes w and excludes u. This C' is dominating, provided $H' = (H-w)+u$ is independent. However, any edge between u and $H-w$ results in a cycle C' of the same maximum length that is not dominating, contradicting Claim 3. Thus we conclude that $U(w) \subseteq A(H)$. Since $d_Q(w) \geq (n+2k-2)/2-p+2 = (n+2k-2p+2)/2$, and $|Q| \leq n-p$, we have $|U(w)| \geq 3(d_Q(w)-1)-|Q| \geq 3((n+2k-2p+2)/2)-(n-p) = (n+6k+6-4p)/2$. Consequently there are more than $(n+6k+6-4p)/2$ vertices of C that might play the role of a given $w \in H$ in the independent set $A(H)$. For a given maximum length dominating good cycle C, let $A = A(C)$ be an independent set of maximum order containing $H = G-C$.

Our next objective is to find a good path P^* that contains as many vertices of A as is possible. We say that P^* is saturated by A. Recall that the path P^* consists of $k-1$ subpaths P_i from x_i to x_{i+1} of length p_i for $1 \leq i \leq k-1$. Thus, actually the objective is to saturate each of these subpaths, since the vertices in $\{x_1, x_2, \ldots, x_k\}$ are fixed. In selecting the path P_i of length p_i from x_i to x_{i+1} there are 3 cases to be considered: A contains 0, 1 or 2 vertices of $\{x_i, x_{i+1}\}$. Observe that any pair of vertices of G has at least $2k-2$ common adjacencies, a pair of non-adjacent vertices has at least $2k$ common adjacencies, and the vertices in A have nearly $n/2$ common adjacencies. Also, observe that for any $s+1 < p_i$ vertices of A, there is a path P' with $2s+1$ vertices, such that vertices of P' alternate between A and $\overline{A}$ starting with and ending with predetermined vertices of A. This is a consequence of the fact that each pair of vertices of A have nearly $n/2$ common adjacencies. To obtain a path P' with $2s+2$ vertices such that s of the vertices are in A and s are in $\overline{A}$, some edge with both vertices in $\overline{A}$ can be inserted in the path between the neighborhoods of 2 vertices of A. If x (or y) is not in A, then x (or y) is adjacent to a vertex in A. Observe also that any parity issues that arise in the paths can be handled, since each pair of vertices in G has at least $2k-2$ common adjacencies. Using these observations, the following can easily be verified for each of the paths P_i.

Case (1) $x_i, x_{i+1} \in A$: If p_i is odd, then P_i can be chosen such that $|P_i \cap A| \geq (p_i+1)/2$, and if p_i is even, $|P_i \cap A| \geq (p_i+2)/2$.
Case (2) $x_i \in A$, $x_{i+1} \notin A$: If p_i is odd, then P_i can be chosen such that $|P_i \cap A| \geq (p_i+1)/2$, and if p_i is even, $|P_i \cap A| \geq p_i/2$.
Case (3) $x_i, x_{i+1} \notin A$: If p_i is odd, then P_i can be chosen such that $|P_i \cap A| \geq (p_i-1)/2$, and if p_i is even, $|P_i \cap A| \geq p_i/2$.

Thus, the good path P^* will contain as many as $\sum_{i=1}^{k-1}(p_i-1)/2 = (p-1)/2-(k-1)/2 = (p-k)/2$ vertices in A. Hence any vertex in A will have at most $(p+k+2)/2$ adjacencies in P^*.

Let C^* be a maximum length dominating good cycle containing P^*, that is given by Claim 4. Set $C^* = P^* \cup Q^*$ and $H^* = G-C^*$. Assume that $|C^*| = m < n$. Consider the case when there is a $w \in H^* \cap A$. Observe that $d_{P^*}(w) \leq (p+k+2)/2$, and $d_{Q^*}(w) \leq (m-p)/2$. This gives the following inequality

$$(n+2k-2)/2 \leq d_G(w) = (p+k+2)/2 + (m-p)/2 \leq (n-1+k+2)/2,$$

a contradiction. Thus, $w \notin A$. Let $B = A(C^*)$ be a maximum independent set containing H^* which also has at least $(n+2k-2p-2)/2$ vertices. However, since there are at least $(n+6k+6-4p)/2$ vertices that can play the role of w, there are at least $(n+6k+6-4p)/2$ vertices of B that are not in A. Thus, $|A\cup B| \geq (n+2k-2p-2)/2+(n+6k+6-4p)/2 = n+4k-3p+2$. Hence, if $u \in A \cap B$, then $d_G(u) \leq 3p-4k-2$, a contradiction. Thus, we can conclude that $A \cap B = \emptyset$.

Our next step is to find a good path P^{**} that contains as many vertices of $A \cup B$ as possible. The paths are obtained in the same way as the paths that were saturated in A, except the paths in P^{**} will mainly alternate between A and B with some vertices in $G-(A\cup B)$. As before, let C^{**} be a maximum length dominating good cycle containing P^{**} that is given by Claim 4. Set $C^{**} = P^{**} \cup Q^{**}$, $H^{**} = G - C^{**}$ and assume that $w \in H^{**} \cap (A \cup B)$. A repeat of the previous argument with P^* and C^* implies that $w \notin A \cup B$. Thus, there is a disjoint independent set which we will denote by D relative to C^{**}. The set D corresponds to the disjoint independent sets A and B relative to C and C^* respectively. Each of these sets has at least $(n+2k-2p-2)/2$ vertices, which implies $3(n+2k-2p-2)/2 \leq n$, or equivalently $n \leq 2p-2k+2$, a contradiction. This completes the proof of Theorem 4. □

3. Questions

There are many natural open questions left from these results. However, the major one is the following:

Question 1. *In Theorems 2, 4 and 5 can the condition that n is sufficiently large be removed or at least reduced?*

Acknowledgments

The authors would like to thank the referees for their careful reading of the manuscript and their helpful comments.

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