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Ramsey numbers in rainbow triangle free colorings

Ralph J. Faudree

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
U.S.A.

Ronald J. Gould

Department of Mathematics and Computer Science
Emory University
Atlanta, GA 30322
U.S.A.

Michael S. Jacobson

Department of Mathematics
University of Colorado Denver
Denver, CO 80217
U.S.A.

Colton Magnant

Department of Mathematics
Lehigh University
Bethlehem, PA, 18015
U.S.A.

Abstract

Given a graph G, we consider the problem of finding the minimum number n such that any k edge colored complete graph on n vertices contains either a three colored triangle or a monochromatic copy of the graph G. This number is found precisely for a C_4 and all trees on at most 6 vertices and bounds are provided for general paths.

1 Introduction

In this paper, we consider coloring the edges of complete graphs with k colors. We restrict our attention to those colorings containing no triangle in which each edge receives a distinct color. Such a triangle will be called rainbow. Of course, every 2-coloring contains no rainbow triangle so we will generally suppose $k \geq 3$.

The concept of avoiding rainbow triangles dates back to 1967 when Gallai [11] proved the following result, which was restated in [14] in terms of edge coloring.

Theorem 1 *Any rainbow triangle free coloring can be obtained by substituting rainbow triangle free colorings of complete graphs into vertices of 2-colored nontrivial complete graphs.*

In particular, this implies that every rainbow triangle free coloring of a complete graph can be partitioned into at least 2 parts such that there are at most 2 colors on the edges between the parts and between each pair of parts, there is only one color on the edges. Due to this result, rainbow triangle free colorings have been termed *Gallai colorings*.

As noted, we will restrict our attention to Gallai colorings of complete graphs. However, as opposed to finding global structure, we would like to find more local structure. More specifically, for a given graph G, we find the minimum n such that any Gallai coloring of K_n contains a monochromatic copy of G. This problem can be viewed as a restricted Ramsey problem. When $k = 2$, this reduces to the classical 2 color Ramsey problem so our results will generalize known 2-color Ramsey results (see [17] for a dynamic survey of known Ramsey results).

In general, we define the k colored *Gallai-Ramsey numbers*, denoted by $gr_k(H : G_1, G_2, \ldots, G_k)$, to be the minimum integer n such that any rainbow H free coloring of K_n contains a copy of G_i in color i for some i. Since we will mainly consider only a single monochromatic graph G, we also define $gr_k(H : G)$ to be the minimum integer n such that any rainbow H free coloring of K_n contains a monochromatic copy of G.

For some closely related results, see [1, 12, 13, 14]. Most of these works are concerned mainly with global structures of graphs that are free of rainbow structures but some local results are also mentioned. For example, the following result was first proven in [7] in a different context but was examined in more depth in both [1] and [13]. Also included in [1] is a complete characterization of all extremal graphs for this result.

Theorem 2

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \textit{for } k \textit{ even.} \\ 2 \cdot 5^{(k-1)/2} + 1 & \textit{otherwise.} \end{cases}$$

Similar ideas have been examined under the titles of anti-Ramsey theory (see [2, 3, 4, 16]) and rainbow (or constrained) Ramsey theory (see [9, 15, 18]). See [10] for a survey of this work in these areas. In anti-Ramsey theory, for a given n, the goal is to find the smallest k such that any coloring of K_n using at least k colors contains some desired rainbow structure. In rainbow Ramsey theory, the goal is simply to find the minimum n such that any coloring (with any number of colors) of K_n contains either a monochromatic or a rainbow coloring of a desired graph. Both of these questions are clearly related to Gallai-Ramsey theory but not enough for results in one area to imply strong results in another.

We will now provide some results for Gallai colorings which will be useful in our proofs. For the sake of the first result, a *broom* is a graph consisting of a path $P = v_1, \ldots, v_t$ and a star centered at v_t. See Figures 3, 4, and 7 for examples. The collection of vertices $v_1, \ldots, v_{t-1}$ is called the *handle* of the broom. Notice a star is a broom with a trivial handle and a path is a broom with a trivial star. All other notation in this paper can be found in [6]. The following result comes from [14].

Theorem 3 *In every rainbow triangle free coloring of a complete graph, there is a monochromatic spanning broom.*

Our next proposition constructs a coloring which will serve as a lower bound for many of our results. Let $c(G)$ denote the edge cover number (the number of vertices needed to cover all the edges of G).

Proposition 1 *Given a graph G on n vertices, with edge cover number $c = c(G)$ and an integer k, there exists a rainbow triangle free k-coloring of the edges of the complete graph on $n - 1 + (c - 1)(k - 1)$ vertices with no monochromatic copy of G.*

Proof: Consider a complete graph on $n - 1$ vertices colored entirely with color 1. This graph certainly contains no rainbow triangle or monochromatic copy of G. To this, we join a copy of K_{c-1} and all new edges are colored with color 2. To the resulting graph, we join a copy of K_{c-1} and all new edges are colored with color 3. This process is repeated $k - 3$ more times resulting in a k-coloring of the complete graph on $n - 1 + (c - 1)(k - 1)$ vertices (see Figure 1) which contains no rainbow triangle or monochromatic copy of G. □

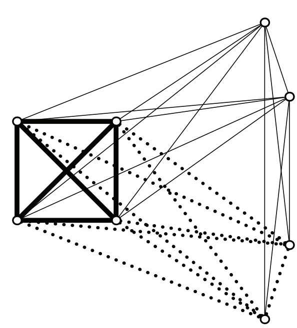

Figure 1: Construction for $k = 3$, $n = 5$ and $c = 3$.

For an upper bound, the following fact shows that classical Ramsey numbers trivially give an upper bound on the Gallai-Ramsey number. Certainly if a k-edge colored complete graph on n vertices contains a monochromatic copy of H, then it must contain either a rainbow copy of G or a monochromatic copy of H. Formally, this is stated as follows.

Fact 1 *For any graphs G and H,*

$$gr_k(G : H) \leq r_k(H).$$

Concerning monochromatic stars, the following result was proven in [14].

Theorem 4 *For $k \geq 3$,*

$$gr_k(K_3 : K_{1,n}) = \frac{5n-3}{2}.$$

The lower bound comes from the following example. Consider 5 copies of $K_{(n-1)/2}$ in color 1 labeled $A_0, A_1, \ldots, A_4$. The edges between A_i and A_j are colored with color 2 when $j - i \equiv 1$ or $4 \mod 5$ and color 3 when $j - i \equiv 2$ or $3 \mod 5$ (see Figure 2).

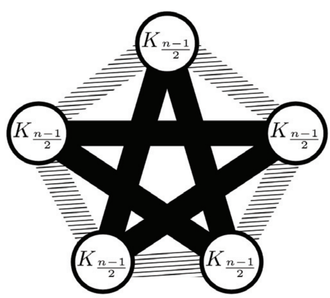

Figure 2: Construction.

Notice that this number is considerably lower than the regular k-color Ramsey number for a monochromatic $K_{1,n}$ which is approximately $k(n-1)$ [5].

On the other hand, the problem of finding monochromatic matchings provides exactly the same result as in classical Ramsey theory. Given a collection of integers $c_1, \ldots, c_t$ in decreasing order, the classical Ramsey number [17] for matchings satisfies:

$$r_t(c_1K_2, c_2K_2, \ldots, c_tK_2) = \sum_{i=1}^{t}(c_i - 1) + c_1 + 1.$$

By Fact 1, we get

$$gr_t(K_3 : c_1K_2, c_2K_2, \ldots, c_tK_2) \leq r_t(c_1K_2, c_2K_2, \ldots, c_tK_2)$$

but in this case, the canonical lower bound example for matchings (see Proposition 1) is also free of rainbow triangles. Therefore we get the following equation

$$gr_t(K_3 : c_1K_2, c_2K_2, \ldots, c_tK_2) = r_t(c_1K_2, c_2K_2, \ldots, c_tK_2).$$

2 Cycles

For odd cycles C_{2n+1}, consider 4 copies of the complete graph K_{2n}, say $G_1, \ldots, G_4$ with all edges colored 1. Color all edges from G_1 to G_2 and from G_3 to G_4 with color 2 and all remaining edges with color 3. This graph has no monochromatic C_{2n+1} and there are also no rainbow triangles so we trivially get

$$gr_3(K_3 : C_{2n+1}) > 8n$$

for all $n \geq 1$. Also, since $gr_3(K_3 : H) \leq r_3(H)$, we get

$$gr_3(K_3 : C_5) = 17$$

$$gr_3(K_3 : C_7) = 25.$$

It is also known that $r_3(C_{2n+1}) = 8n + 1$ for n sufficiently large [8] so with the above lower bound, we know

$$gr_3(K_3 : C_{2n+1}) = 8n + 1$$

for n sufficiently large.

We now consider the 4-cycle. In comparison to the classical Ramsey number $r_k(C_4) = k^2 + O(k)$ for C_4 (see [17]), the following result shows that the Gallai-Ramsey number is quite different.

Theorem 5

$$gr_k(K_3 : C_4) = k + 4.$$

Proof: For the lower bound, consider the following construction. Partition K_5 into two edge-disjoint 5-cycles. Color one such 5-cycle with color 1 and the other with color 2. To this, we repeatedly add vertices v_i with all edges to v_i having color $i + 2$ for $i = 1, \ldots, k - 2$. This coloring certainly contains no rainbow triangle or monochromatic C_4 and has order $k + 3$.

Let k' be the number of color classes which may contain adjacent edges. The proof of the upper bound is by induction on k'. The base case of this induction is a coloring of K_5 with only one color class (suppose color 1) possibly containing adjacent edges. In order to avoid a C_4, color 1 must induce a subgraph of either a triangle with two pendants or a C_5. Since all other classes induce at most a matching, it is easy to see that this graph contains a rainbow triangle.

Inductively suppose true for all $k' < k$ and consider $k' = k$. Let G be a coloring of $K_{k'+4}$ with at most k' color classes containing adjacent edges and suppose G contains no rainbow triangle or monochromatic C_4.

By Theorem 1, there exists a partition of $V(G)$ into parts such that between each pair of parts there is exactly one color and between the parts in general, there are at most two colors. If there exists a part A of order at least 2, then, in order to avoid a monochromatic C_4, each other part must have order 1 and there can be at most 2 such *small* parts. Let v be the vertex of a small part and suppose v has edges of color

1 to A. Since G contains no monochromatic C_4, the removal of v leaves the graph with at most a matching in color 1. The resulting graph has at most $k' - 1$ color classes containing adjacent edges and order $k' + 3$, hence we may apply induction on k'.

If there exists no part of order at least 2, that means that every part has order 1 and we have a 2-coloring of K_n. Since $r(C_4, C_4) = 6$, we know that $n \leq 5$. Hence we find that $n = 5$ and $k' = 1$ and this is precisely the base case presented above. □

3 Paths

In general, the following corollary to Proposition 1 provides a lower bound on the Gallai-Ramsey number for paths.

Corollary 6

$$gr_k(K_3 : P_t) \geq \left\lfloor \frac{t-2}{2} \right\rfloor k + \left\lceil \frac{t}{2} \right\rceil + 1.$$

In particular, the following three results concern short paths.

Theorem 7

$$gr_k(K_3 : P_4) = k + 3.$$

Proof: The lower bound follows immediately from Corollary 6. Suppose we are given a complete graph K_{k+3} colored with k colors. By Theorem 3, there exists a spanning monochromatic broom. If the handle has 2 vertices, then there exists a monochromatic P_4 so the broom must be a spanning star.

Suppose the spanning star has color 1. Since $k + 3 \geq 4$, there is no extra edge of color 1 between pendants of the star. Hence, we may remove the center of the star, thereby removing all edges of color 1. The resulting graph contains at most $k - 1$ colors and has order $k + 2$, hence we may apply induction on k. The base case of this induction is a 1-coloring of K_4 which clearly contains a monochromatic P_4. □

Theorem 8

$$gr_k(K_3 : P_5) = k + 4.$$

Proof: The lower bound is given by the construction in Corollary 6. For the upper bound, we let k' denote the number of colors which may be used on more than one edge. Consider a coloring of $K_{k'+4}$ with at most k' colors inducing more than a single edge.

We know, by Theorem 3, that there exists a spanning monochromatic broom (suppose color 1). Choose such a broom with the longest handle. Since we assume there is no monochromatic P_5, the handle of the broom must contain at most two edges. In order to avoid a monochromatic P_5 and by the choice of the broom with the longest handle, there can be no extra edges of color 1 outside those used in the broom. If we remove the center of the star of the broom, we are left with at most a single edge of color 1. The resulting graph contains at most $k' - 1$ colors with more than one edge and has order $k' + 3$, hence we may apply induction on k'.

The base case is when $k' = 0$ and $|G| = 4$. This is a rainbow coloring of K_4 which clearly contains a rainbow triangle. □

Theorem 9 *For $k \geq 3$,*

$$gr_k(K_3 : P_6) = 2k + 4.$$

Proof: The lower bound is given by Corollary 6. Let k' be the number of color classes which may contain adjacent edges in our coloring. The proof is by induction on k'. Consider a coloring of $K_2k' + 4$ in which at most k' colors have adjacent edges.

By Theorem 3, there exists a spanning monochromatic (suppose color 1) broom. Choose such a broom with the longest handle. If the chosen broom is a star, then by the choice of the broom, there are no edges of color 1 between pendants of the star. The removal of the center of the star removes all edges of color 1. The resulting graph contains at most $k' - 1$ colors and has order $2k' + 3$, so we may apply induction on k'.

Suppose the handle of the broom contains two vertices and let v be the center of the star. By the choice of the broom, there are no edges from the handle of the broom to the pendants of the star. Suppose there exists a pendant w of the star such that w has edges to at least 2 other pendants of the star. Since our coloring is free of a monochromatic P_6, there can be at most one such vertex w among the pendants. Hence, the removal of v and w leaves at most a matching in color 1. Then we may apply induction on k'.

Finally suppose the handle of the broom contains three vertices u_1, u_2 and u_3 with u_3 adjacent to the center of the star v. Since the graph is monochromatic P_6-free, there must be no edges of color 1 from u_1 or u_3 to the pendants of the star. Also, there can be no edges between pendants of the star. Hence, the removal of u_2 and v removes all edges of color 1 from the graph allowing us to again apply induction on k'.

The base case of this induction is when $k' = 1$ and we consider a coloring of K_6 in which only one color induces more than a matching. This graph clearly contains either a monochromatic P_6 or a rainbow triangle. □

We now provide a general upper bound for $gr_k(K_3 : P_j)$. This upper bound shows that the lower bound of Corollary 6 is asymptotically sharp.

Theorem 10 *Given $\epsilon > 0$, there exists a positive integer t_ϵ such that for all $t \geq t_\epsilon$ and $k \geq t$,*

$$gr_k(K_3 : P_t) \leq \left(\frac{1}{2} + \epsilon\right) tk.$$

Proof: Given $\epsilon > 0$, choose $t = \lceil \frac{4}{\epsilon^2} \rceil$ and suppose $k = t$ (although the same argument holds for $k \geq t$). Also let $j = \lceil \frac{\epsilon t}{2} - 2 \rceil$. Suppose there exists a k-coloring of $G = K_n$ for $n = \lceil (\frac{1}{2} + \epsilon) tk \rceil$ with no rainbow triangle or monochromatic path of order t. By Theorem 3, there exists a monochromatic spanning broom. Since there is no path of order t, we know the star must have at least $n - t + 2$ pendants.

By the above argument, there exists a star with at least $n - t + 3$ leaves. If we remove the center vertex of this star from the graph, by the same argument, there must exist another star with at least $n - t + 2$ leaves. If this process is repeated $(\frac{t}{2} + j - 1)k + 1$ times, by the pigeon hole principle, we can be sure to find $\frac{t}{2} + j$ stars (call this set $\mathscr{S}$) all of one color (suppose color 1). One may easily verify that the smallest such star has at least

$$n - \left[\left(\frac{t}{2} + j - 1\right) k + 1\right] - t + 3 \geq \frac{n}{j+2} + 1 + \frac{t}{2} + j \tag{1}$$

leaves. Inequality (1) shows that even among the smallest of the stars, in any choice of $j + 2$ of the stars there must exist two which intersect in a leaf which is not the center of another star. Let S be the graph with vertex set consisting of the stars of $\mathscr{S}$. Edges appear in S when two stars share a leaf. As noted above, every set of $j + 2$ vertices in S contains an edge.

We form paths in S by the following process. Over the $j + 2$ stars of smallest degree, choose a pair of intersecting stars of smallest total degree such that the intersection vertex is not the center of one of the stars of color 1. We next remove one of the paired stars from consideration and repeat the process with the added condition that the intersection vertex has not already been the intersection vertex of two previous stars. Each time we remove a star from consideration, we leave only one end of a path (in S) of stars in the considered set.

One may easily see the above process yields a collection of at most $j+1$ paths in S (including trivial single-star paths). Consider the path in G constructed by alternating center vertices and intersection vertices along the paths of S. This produces at most $j+1$ path segments all of color 1. In order to link these together, we observe that between any pair of vertex disjoint stars of color 1, if there are no edges of color 1, then all edges between the pair of stars are a single color (suppose color 2). Since each star has order at least $n - \lceil (\frac{t}{2} + j - 1)k + 1 \rceil - t + 3 \geq \frac{t}{2}$, this complete bipartite graph in color 2 would contain the desired path.

There must, therefore, exist an edge of color 1 between every pair of stars in $\mathscr{S}$. This edge could be between the center vertices or it could be between the leaf vertices, but we may suppose it does not fall between a center and a leaf as this would be another intersection of stars that we could have counted above.

Let S_1 and S_2 be end stars of two different star paths. There must, by the above argument, exist an edge between S_1 and S_2. This edge links the two paths of G, which were generated from the star paths into one longer path. For each star path of length p in S, there exists a monochromatic path in G of length $2p - 1$ (as we may not assume the alternating path generation process continues into the leaves of the end stars). The linking of the paths might add no vertices to the paths (as the linking edge may go from the center of one star to the center of the other). Finally the end stars of the final (linked) path will extend the path by 2 by using leaves. This process creates a path in G of color 1 having order at least

$$2\left(\frac{t}{2} + j\right) - (j + 1) + 2 \geq t.$$

This completes the proof of Theorem 10. □

4 Trees

In our final section we find the Gallai-Ramsey number for all trees on at most 6 vertices. All stars and paths are covered in previous sections so we first consider the tree which we denote by T_1, consisting of a $K_{1,3}$ with an extra edge off one of the leaves of the star (see Figure 3).

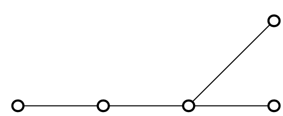

Figure 3: The tree T_1.

Theorem 11

$$gr_k(K_3 : T_1) = k + 4.$$

Proof: The lower bound follows from Proposition 1. For the upper bound, we proceed by induction on the number of colors k. The base case is the 1-coloring of a K_5 which clearly contains a T_1. Similarly, any 2-coloring of K_6 must contain a monochromatic T_1 so suppose $k \geq 3$. Consider a k coloring of the edges of K_{k+4}. By Theorem 3, there exists a monochromatic spanning broom. Without loss of generality, assume this broom is in color 1.

If this broom is not a star or a path, then it contains T_1 so first suppose the broom is just a star. Let v be the center of the star. Since there are at least $(k+4) - 1 \geq 4$ edges in this star, if there exists another edge of color 1 between pendant vertices of the star, this structure contains T_1. Hence, we may now remove the center vertex v to be left with a $k - 1$ coloring of K_{k+3} which, by induction, contains a monochromatic copy of T_1.

Now suppose the broom is a hamiltonian path. Label the vertices of the path as $v_1, v_2, \ldots, v_n$. Consider any edge from v_1 to the interior of the path. In order to avoid a T_1 in color 1, this edge must have a different color (suppose color 2). In order to avoid a rainbow triangle, all other edges from v_1 to interior vertices $v_3, \dots v_{n-1}$ on this path must be of color 2.

If any other edges of color 2 are adjacent to the edges of color 2 already constructed, this would create a T_1 in color 2. Since this graph is rainbow triangle free, the edge $v_2 v_{n-1}$ must have color either 1 or 2. Either situation forms a monochromatic T_1 which is a contradiction. □

The tree T_2 is defined to be the tree on 6 vertices consisting of a path of order 5 and an extra edge from the second vertex along the path to the free vertex (see Figure 4).

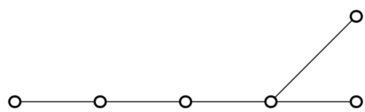

Figure 4: The tree T_2.

Theorem 12

$$gr_k(K_3 : T_2) = k + 5.$$

Proof: The lower bound follows from Proposition 1. The proof of the upper bound is by induction on the number k' of colors inducing more than a matching in G. Consider a coloring of $K_{k'+5}$ with at most k' colors inducing more than a matching and suppose this coloring has no rainbow triangle or monochromatic T_2. By Theorem 3, there exists a monochromatic spanning broom. Choose such a broom with the longest handle. Suppose this broom is in color 1.

First suppose this broom is a star. If there are two adjacent edges in color 1 within the pendant vertices of the star, then there would exist a T_2 in color 1. Hence, we may remove the center vertex of the star and leave behind at most a matching in color 1 and proceed by induction on k'. The base case of this induction is the case when $k' = 0$. By Theorem 3, it is impossible to color K_5 with independent edges of each color and avoid a rainbow triangle.

If the spanning broom has a handle of exactly 2 edges, one may easily show that the removal of the center vertex of the star, again leaves at most a matching in color 1 in the graph. Again we proceed by induction on k'.

If the spanning broom has a handle of at least 3 edges, then in order to avoid a monochromatic T_2, this broom must be a spanning path. First suppose $k' \geq 2$ so the order of our graph is $n \geq 7$. Label the vertices of the graph as $v_1, v_2, \dots, v_n$ along the spanning path. Let $A = \{v_1, v_2, v_3, v_4\}$ and let $B = \{v_{n-1}, v_n\}$. If there are no edges of color 1 between the sets A and B, then, in order to avoid a rainbow triangle, all edges from A to B are have a single color and the graph induced on this color class contains T_2.

Therefore there must exist an edge of color 1 between A and B. In order to avoid a monochromatic T_2, the only edge that could possibly be of color 1 is the edge between the ends of the path. This forms a hamiltonian cycle of color 1. In order to avoid a monochromatic T_2 no chord of the cycle may have color 1. To avoid a rainbow triangle, all chords of the cycle must be of a single color but since $n \geq 7$, this graph contains a monochromatic T_2.

Finally we may assume $k' = 1$, $n = 6$ and the spanning broom is a path. Again label the vertices of the path with $v_1, \ldots, v_6$. Consider the set of vertices $A = \{v_1, v_3, v_5\}$. In order to avoid a monochromatic T_2, there can be no edge of color 1 among the vertices of A. Conversely, to avoid a rainbow triangle, two of these edges must be the same color. Since all colors other than color 1 induce at most a matching, this is a contradiction completing the proof. □

The tree T_3 is defined to be the tree on 6 vertices consisting of a path of order 4 with extra pendant edges off the second and third vertices of this path (see Figure 5).

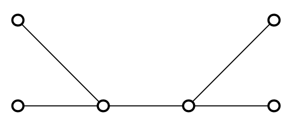

Figure 5: The tree T_3.

Theorem 13

$$gr_k(K_3 : T_3) = 2k + 4.$$

Proof: The lower bound comes from an example similar to the proof of Proposition 1. Again construction is inductive. Suppose there exists a coloring of K_{2k+1} with $k-1$ colors and no rainbow triangle or monochromatic T_3. We then add two vertices u and v where the edge uv has color 1 and all edges from u and v to the rest of the graph are of color k. One may easily see this produces a k-coloring of K_{2k+3} with no rainbow triangle or monochromatic T_3. For the base case, the complete graph on 5 vertices with a single color certainly contains no rainbow triangle or monochromatic T_3.

The upper bound also follows as before. Let k' be the number of color classes which may contain adjacent edges. We consider a coloring of the complete graph on $n = 2k' + 4$ vertices. If there exist two vertices which cover all but a matching in color i for $1 \leq i \leq k'$, we may remove these vertices and proceed by induction on k'. Now suppose otherwise.

The base case of this induction is a K_4 which is colored by only matchings of different colors. By Theorem 3, this is a contradiction.

Claim 1 *There exists no monochromatic path of order* 7 *in* G.

Proof of Claim 1: Suppose there exists a path of order 7 in color 1 and label the vertices of the path with $v_1, v_2, \ldots, v_7$. The edge v_2v_6 must not have color 1 so suppose it has color 2. Since G is rainbow triangle free, the edges v_2v_5 and v_6v_3 must also have color 2. If both v_2 and v_6 had another edge of color 2 to distinct vertices, the color 2 would contain a T_3. Conversely, if one of v_2 or v_6 (suppose v_6) has no more edges of color 2, the edges v_6v_1 and v_6v_4 must have color 1 to avoid a rainbow triangle, thereby forcing a T_3 in color 1 centered at v_4 and v_6.

Hence we may assume each of v_2 and v_6 have exactly one more edge of color 2 and these edges go to the same vertex. If this vertex is not v_4, then v_2v_4 and v_6v_4 have color 1 which forms a T_3 in color 1. Hence, v_2v_4 and v_6v_4 have color 2. To avoid rainbow triangles, the edges v_1v_6 and v_2v_7 must have color 1.

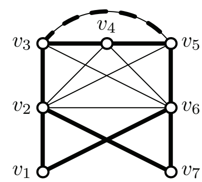

Figure 6: K_7 with a monochromatic P_7.

Now consider the edge v_3v_5. To avoid a rainbow triangle, this edge must have either color 1 or 2 (see Figure 6). If it has color 1, the vertices v_5 and v_6 may serve as centers of the stars in a T_3 of color 1. If the edge v_3v_5 has color 2, then one may easily verify that the vertices $A = \{v_2, \ldots, v_6\}$ must have no more edges of color 2. Also, if the graph in color 2 induced on $G \setminus A$ in color 2 contains a component C with more than one edge, the complete bipartite graph between A and C must be monochromatic and must contain a T_3. Therefore we may remove the vertices v_2 and v_6 to be left with at most a matching of color 2 and proceed by induction on k'.

$\square_{Claim\ 1}$

By Theorem 3, there exists a monochromatic (suppose color 1) spanning broom. Choose such a broom with the longest handle. Since there is no monochromatic path of order at least 7, the handle of the path contains at most 4 vertices (excluding the center of the star).

First suppose the broom is only a star. Aside from the center vertex, each vertex has degree at most 2 in color 1 as any more would create a monochromatic T_3. Therefore removal of the center vertex leaves at most a matching of color 1 at which point we proceed by induction.

Suppose the handle contains 2 vertices v_1 and v_2. Certainly there are no edges from v_1 or v_2 to the pendant vertices of the star in color 1 as this would allow for a longer handle. Within the star, as above, there are no adjacent edges of color 1. Therefore the removal of the center vertex of the star again leaves at most a matching of color 1 and we proceed by induction.

Suppose the handle contains 3 vertices v_1, v_2 and v_3 in order with v_3 adjacent to the center of the star. Clearly there are no edges from the vertices v_1 or v_3 to the pendant vertices of the star in color 1 as this would allow a longer broom handle. Again there is at most a matching within the pendant vertices of the star so we may remove the center of the star and v_2 to leave at most a matching of color 1 and proceed by induction.

Finally suppose the handle contains 4 vertices $v_1, \ldots, v_4$ again in order. Certainly there are no edges of color 1 from the vertices v_1 or v_4 to the pendant vertices of the star as these would make a longer path. Also there is no more than a matching of color 1 within the vertices of the star. First suppose both v_2 and v_3 send an edge of color 1 to the pendant vertices of the star. If these edges go to distinct vertices in the star, then v_2 and v_3 form the centers of a monochromatic copy of T_3 so there must exist a single vertex v in the star with an edge to each of v_2 and v_3. Then v and the center of the star form the two centers of the stars of a monochromatic copy of T_3. Hence, only one of v_2 or v_3 may have edges of color 1 to the pendant vertices of the star (suppose v_3). We may now remove v_3 and the center of the star to leave at most a matching of color 1 and proceed by induction. This completes the proof. □

The tree T_4 is defined to be the tree on 6 vertices consisting of a path of order 4 with two pendant edges off the second vertex (see Figure 7).

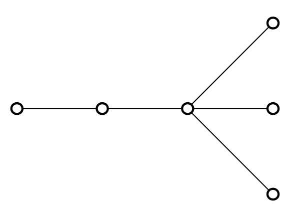

Figure 7: The tree T_4.

Theorem 14

$$gr_k(K_3, T_4) = k + 5.$$

Proof: The lower bound is given by Proposition 1. It is known that $r(T_4, T_4) = 7$ (see [17]). Suppose $k \geq 3$ and consider a k-coloring of $G = K_n$ for $n = k + 5$. By Theorem 1, there exists a decomposition of the graph into blocks such that all edges between blocks have one of two colors (suppose colors 1 and 2) and all edges between any pair of blocks have a single color.

Since $r(T_4, T_4) = 7$, there can be at most 6 blocks in this decomposition. Suppose the order of the largest block B is 2. If there are at least 4 vertices in $G \setminus B$ which send edges of color i to B, this forms a $K_{2,4}$ in color i which contains T_4. Thus, assume this is not the case. That means there are at most three vertices (call this set

A_i) in $G \setminus B$ which send edges of color i to B. Since $n \geq 8$, we find that $n = 8$ and there are exactly three vertices in A_i for each $i = 1, 2$. To avoid a rainbow triangle, all edges from A_1 to A_2 must have color 1 or 2. An easy argument shows there must be either a vertex in A_1 with two edges of color 1 to A_2 or a vertex in A_2 with two edges of color 2 to A_1. In either case, this implies the existence of a T_4 in that color.

If the order of the largest block B is 3, an identical argument to the above follows so suppose the order of the largest block B is at least 4. Since there is a single color between each pair of blocks, all other blocks are single vertices. Otherwise we would have a monochromatic $K_{4,2}$, which contains a T_4. If there are at least 3 vertices in $G \setminus B$, then there are two vertices which have the same colored edges to B. This implies that color induces a $K_{2,4}$ which contains T_4.

Therefore, there are at most two vertices u_1, u_2 in $G \setminus B$. Without loss of generality, suppose u_i has color i to B. To avoid a rainbow triangle, the edge u_1u_2 must have color 1 or color 2. This implies there is a spanning star of one color (suppose color 1). Since $n \geq 8$, there can be no other edge of color 1 as this would create a T_4. Therefore we may remove the center vertex of the star, eliminate all edges of color 1 and apply induction on k. This completes the proof. □

The tree T_5 is defined to be the tree on 6 vertices consisting of a path of order 5 with a pendant edge off the third vertex of the path (see Figure 8).

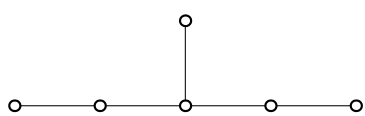

Figure 8: The tree T_5.

Theorem 15

$$gr_k(K_3, T_5) = 2k + 4.$$

Proof: The lower bound follows from Proposition 1. Consider a k-coloring of $G = K_n$ for $n = 2k + 4$. Suppose there exists a monochromatic (suppose color 1) path of order at least 7. Label the first 7 vertices of the path as $v_1, v_2, \ldots, v_7$. Let $A = \{v_1, v_2, v_3\}$ and let $B = \{v_5, v_6, v_7\}$. In order to avoid a monochromatic T_5, all edges between A and B with the exception of v_1v_7 must not be in color 1. In order to avoid a rainbow triangle, these edges must all be of a single color (suppose color 2). The graph induced on color 2 contains a copy of $K_{3,3}^-$ which contains a T_5. Hence, there exists no monochromatic path of order 7.

By Theorem 3, there exists a monochromatic (suppose color 1) spanning broom. Choose such a broom with the longest handle. If this broom was simply a star, we may remove the center of the star, thereby removing all edges of color 1, and proceed by induction. Hence, there must exist at least 2 vertices in the handle of the broom (not including the center of the star). Suppose there are either 2 or 3 vertices in the handle. Label the vertices of the handle v_1, v_2 and possibly v_3 with the vertex of highest label being adjacent to the center of the star. Clearly there can be no edges from v_1 or the vertex of highest label in the handle vertices to the pendant vertices of the star as this would allow for a longer handle. Therefore we may remove v_2 and the center of the star and thereby remove all edges of color 1 to proceed by induction.

Finally suppose there are exactly 4 vertices v_1, v_2, v_3 and v_4 in the handle of the broom (as any more would force a path of order 7). Again v_1 and v_4 must not have any edges of color 1 to the pendant vertices of the star as this would allow a longer handle. Also there must not be any edges of color 1 from v_3 to the pendant vertices of the star as this would create a copy of T_5. Therefore we may again remove the center of the star and v_2, this time leaving behind at most one edge of color 1.

For the base case of this induction, one must simply observe that there is no rainbow triangle-free coloring of K_4 such that each color class induces only a single edge. This completes the proof. □

This completes our treatment of all trees on at most 6 vertices.

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