Critical Graphs for Subpancyclicity of 3-Connected Claw-Free Graphs

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Abstract: Let $\mathcal{F}_k$ be the family of graphs G such that all sufficiently large k-connected claw-free graphs which contain no induced copies of G are subpancyclic. We show that for every $k \geq 3$ the family $\mathcal{F}_1 k$ is infinite and make the first step toward the complete characterization of the family $\mathcal{F}_3$.

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1. INTRODUCTION

A graph is *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$, *the claw*, as an induced subgraph. Claw-free graphs have been widely studied because of their special structural properties. In particular, the following type of problem has been considered by many authors:

Given a property $\mathcal{A}$ characterize the set $\overline{\mathcal{F}}_k(\mathcal{A})$ of all graphs F such that each k-connected claw-free graph containing no induced copies of F has $\mathcal{A}$.

Note that $\overline{\mathcal{F}}_k(A)$ can be equivalently defined as the set of graphs which appear as induced subgraphs in all k-connected claw-free graphs without property $\mathcal{A}$. Thus, if there exists at least one k-connected claw-free graph without property $\mathcal{A}$, then the family $\overline{\mathcal{F}}_k(A)$ is finite and is determined by a finite set of 'critical' graphs without $\mathcal{A}$. For instance, if $\mathcal{H}$ is the property that a graph is hamiltonian, then $\overline{\mathcal{F}}_2(\mathcal{H})$ consists of the path P_6, two other graphs (which we denote below by $F(1,1,1)$ and $F(0,1,2)$, respectively) and all their induced subgraphs (see [4] and references therein). Furthermore, $\overline{\mathcal{F}}_2(\mathcal{H})$ is determined by three critical graphs. At this moment we have only a partial characterization of the family $\overline{\mathcal{F}}_3(\mathcal{H}) \supseteq \overline{\mathcal{F}}_2(\mathcal{H})$ (see, for instance, [6]). Finally, it is believed that each claw-free 4-connected graph is hamiltonian. If this is the case, then $\overline{\mathcal{F}}_4(\mathcal{H})$ contains all graphs.

Claw-free graphs used as 'critical' graphs are typically very small. On the other hand, if there exists F such that there are only finitely many claw-free k-connected graphs without property $\mathcal{A}$ which contain no induced copy of F, then we might also say that prohibiting F, basically, forces $\mathcal{A}$. This is the case with a graph we denote below as $F(0,0,3)$; although it does not belong to $\overline{\mathcal{F}}_2(\mathcal{H})$ each 2-connected claw-free graph without an induced copy of $F(0,0,3)$ is hamiltonian provided it has at least 10 vertices. Thus, instead of $\overline{\mathcal{F}}_k(\mathcal{A})$ one may want to look at the family $\mathcal{F}_k(\mathcal{A})$ which consists of all graphs F for which there exists a constant N_F such that each k-connected claw-free graph on at least N_F vertices without an induced copy of F has property $\mathcal{A}$. Note that, clearly, $\mathcal{F}_k(\mathcal{A}) \supseteq \overline{\mathcal{F}}_k(\mathcal{A})$. Moreover, in principle, at least for some properties $\mathcal{F}(\mathcal{A})$ might be an infinite proper subset of all finite graphs. In this article, we give an example of a property $\mathcal{A}$ for which this is indeed the case.

Let us observe first that $\mathcal{F}_k(\mathcal{H})$ is finite for $k=2,3$. Indeed, let us take any graph J which is obtained by attaching at least one pendant edge to each vertex of the Petersen graph. Let G be the line graph of J. It is easy to check (see [6]) that G is claw-free, 3-connected and non-Hamiltonian. Furthermore, although G can be arbitrarily large, the number of its induced subgraphs which do not contain K_4 is bounded by an absolute constant. Since it is easy to see that no graph from $\mathcal{F}(\mathcal{H})$ contains K_4 (see Theorem 1), $\mathcal{F}_3(\mathcal{H})$ must be finite.

Thus, instead of $\mathcal{F}_k(\mathcal{H})$, we consider the family $\mathcal{F}_k = \mathcal{F}_k(\mathcal{S})$, where $\mathcal{S}$ is the property that a graph G is *subpancyclic*, i.e., for every ℓ, $3 \le \ell \le \mathrm{circ}(G)$, G contains a cycle of length ℓ, where $\mathrm{circ}(G)$ denotes the length of the longest cycle in G.

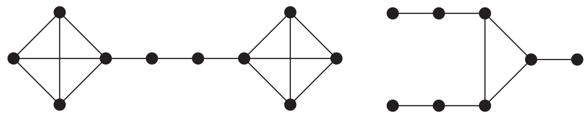

FIGURE 1. $Ł(4,3)$ and $F(1,2,2)$.

Note first that since a graph is subpancyclic if each of its blocks is subpancyclic, $\mathcal{F}_1 = \mathcal{F}_2$. The family $\mathcal{F}_2$ was studied by Faudree *et al.* [3] (see also Faudree and Gould [4]); we briefly recall their argument. Note that a cycle on $n \geq 4$ vertices is 2-connected but not subpancyclic, so the only possible candidates for members of $\mathcal{F}_2$ are paths. Now consider the graph G_n which consists of the complete graph on the set V_1 of $2n$ vertices, a perfect matching on the set V_2, where $V_1 \cap V_2 = \emptyset$ and $|V_2| = 2n$, and an additional perfect matching between these two sets. It is easy to see that G_n is hamiltonian, but it contains no cycles on $4n-1$ vertices. Moreover, G_n contains no induced path P_7 on seven vertices and so $P_k \notin \mathcal{F}_2$ for $k \geq 7$. Finally, Faudree *et al.* [3] proved that each 2-connected claw-free graph on at least ten vertices which contains no induced copy of P_6 is subpancyclic. Hence, $\mathcal{F}_2 = \{P_3, P_4, P_5, P_6\}$.

In order to study the family $\mathcal{F}_k$ for $k \geq 3$ we introduce two types of graphs (see also Figure 1). By $Ł(r,s)$ we denote the graph which consists of two vertex-disjoint complete graphs on r vertices which are connected by a path of length s (it is a generalization of a graph $Ł$ introduced in [5]). By $F(t_1, \dots, t_r)$, where $0 \leq t_1 \leq \cdots \leq t_r$, we mean the graph which consists of the complete graph K_r on r vertices with paths of lengths $t_1, \dots, t_r$ rooted at different vertices of K_r (as mentioned above, graphs $F(t_1, t_2, t_3)$ emerge naturally in studying hamiltonicity of claw-free graphs). We also remark that in other papers, the graphs $Ł(3,s)$ and $F(t_1, t_2, t_3)$ are sometimes called *double lasso* and *generalized net*, respectively.

Our first result narrows the set of possible candidates for elements of $\mathcal{F}_k$.

Theorem 1. *Let $k \geq 3$ and let $r = r(k) = \lceil k/2+1 \rceil$. Then there exists $m = m(k)$ such that every graph $G \in \mathcal{F}_k$ is a subgraph of either $Ł(r, 2s+1)$ for some $s \geq 0$, or $F(t_1, t_2, \dots, t_r)$ with $0 \leq t_1 \leq \cdots \leq t_{r-2}$ and $t_1 \leq m$.*

Furthermore, $m(3) = 2$, i.e., each graph from $\mathcal{F}_3$ is a subgraph of either $Ł(r, 2s+1)$ for some $s \geq 0$, or $F(t_1, t_2, t_3)$ with $0 \leq t_1 \leq 2$.

We conjecture that, in fact, the above theorem gives an almost complete characterization of $\mathcal{F}_k$ and the following two conjectures hold.

Conjecture A. *For every $k \geq 3$ there exists $\bar{m}(k)$ such that for each $0 \leq t_1 \leq t_2 \leq \cdots \leq t_r$, with $t_{r-2} \leq \bar{m}(k)$, there exists $\bar{N}$ such that each k-connected claw-free graph on at least $\bar{N}$ vertices without an induced copy of $F(t_1, \dots, t_r)$ is subpancyclic.*

Conjecture B. *For every $k \geq 3$ and $s \geq 0$ there exists $\hat{N}$ such that each k-connected claw-free graph on at least $\hat{N}$ vertices without an induced copy of $Ł(\lceil k/2+1 \rceil, 2s+1)$ is subpancyclic.*

In this paper, we study more closely the family $\mathcal{F}_3$. It has been proved by the authors of this article (see [5]) that if a claw-free graph G on at least 11 vertices is not pancyclic, then it contains induced copies of each of the graphs P_7, $Ł(3,1)$, $F(0,0,4)$, $F(0,1,3)$,

$F(1,1,2)$. Consequently, the above five graphs (and all their induced subgraphs) belong to $\mathcal{F}_3$. Here, we show that Conjecture A holds with $\bar{m}(3)=2$, i.e., all graphs $F(t_1,t_2,t_3)$ with $t_1 \le 2$ belong to $\mathcal{F}_3$.

Theorem 2. *The family $\mathcal{F}_3$ contains all graphs $F(t_1,t_2,t_3)$ with $t_1 \le 2$.*
In particular, for $k \ge 3$ the family $\mathcal{F}_k \supseteq \mathcal{F}_3$ is infinite.

2. NOTATION

In this paper, the term *graph* always stands for a simple graph. Graphs with multiple edges are called *multigraphs*. For all notation not defined here we refer the reader to [2]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$ (or sometimes just E), respectively. For a set of vertices $X \subseteq V(G)$, $N(X)$ stands for the *neighborhood of* X, i.e., the set of all vertices outside X adjacent to some vertices in X; sometimes we also use the *closed neighborhood of* X defined as $N[X]=N(X)\cup X$. By $\langle X\rangle$ we mean the subgraph of G induced by the set X. A vertex inside brackets denotes a vertex that may or may not be used in an induced subgraph.

For a directed path or cycle H and two vertices $x, y \in V(H)$, we write xHy for the $x-y$ path on H following the direction of H. By x^{--}, x^{-} and x^{+}, x^{++} we denote the predecessors and successors of x on H, respectively.

The *distance* between two vertices $x, y \in V(G)$ is the length of a shortest $x-y$ path, denoted by $d_G(x,y)$. If for any two vertices in a subgraph H of G, $d_H(x,y)=d_G(x,y)$, then we say H is *distance preserving*. The *diameter* of a connected graph, $\mathrm{diam}(G)$, is the maximum distance of two vertices in the graph. The *girth* of a graph is the length of a shortest cycle.

In Section 3, we give the proof to Theorem 1. In the following sections we first give a sketch of a fairly tedious proof of Theorem 2 (Section 4), and then proceed to prove it, first for graphs with small diameter (Section 5), and then for graphs with large diameter (Section 6).

3. PROOF OF THEOREM 1

In this section, we give a series of examples of k-connected claw-free graphs which are not subpancyclic. Clearly, each H from $\mathcal{F}_k$ must appear as an induced subgraph in all but a finite number of graphs of the above kind, so in such a way we shall gradually narrow the family of possible candidates for graphs in $\mathcal{F}_k$.

For a multigraph G we denote by $L(G)$ the line graph of G; the vertex set of $L(G)$ is the edge set of G, and two vertices in $L(G)$ are adjacent if the corresponding edges in G share at least one vertex in G. The set of all multigraphs F with $L(F)=G$ is $L^{-1}(G)$. The following fact list some useful properties of line graphs.

Fact 3. *Let G be a multigraph, let $L(G)$ be its line graph, and let F be a graph. Then*

(i) *$L(G)$ is k-connected if and only if G is essentially k-edge-connected, i.e., the removal of fewer than k edges in G leads to a multigraph with exactly one non-trivial component,*

(ii) *$L(G)$ is claw-free,*
(iii) *F is an induced subgraph of $L(G)$ if and only if G contains one of the multigraphs in $L^{-1}(F)$ as a (not necessarily induced) submultigraph.*

For a graph G, let sub(G) be the graph obtained from G by a subdivision of each edge of G into two edges.

We denote by $J(r,s)$ any r-regular essentially $(2r-2)$-edge-connected graph with girth at least s. It is well known that for any numbers $r,s \geq 3$ such a graph $J(r,s)$ exists. In fact, a positive fraction of all r-regular graphs ($r \geq 3$) have these properties, see for instance [1, Theorems II.19 and VII.32]. Observe that sub$(J(r,s))$ is essentially r-edge-connected for $r \geq 2$.

Now, let $k \geq 3$ and $r = \lceil k/2 \rceil + 1 \geq 3$. Then, $L(J(r,r+2))$ is a k-connected claw-free graph which is not subpancyclic (it contains no cycle of length $r+1$), so no graph from $\mathcal{F}_k$ contains a complete graph on more than r vertices. Furthermore, for every $s \geq (k+2)/2$, the graph $L(\text{sub}(J(k,s)))$ is a k-connected claw-free graph which contains cycles of length at least $2s \geq k+2$ but no cycles of length $k+1$, and so it is not subpancyclic. Consequently, each graph in $\mathcal{F}_k$ is an induced subgraph of $L(T)$ for some T, where T is a tree of maximum degree at most $r = \lceil k/2 \rceil + 1$ in which each two vertices of degree larger than two are connected by a path of even length.

To narrow down the list of candidates further, consider the tree T_t, consisting of a path $v_1 v_2 \ldots v_{t+5}$ on $t+5$ vertices, and two vertices u and w with $uv_3, v_{t+4}w \in E(T)$. Then $L(T_t)$ is the graph obtained from $Ł(3,t)$ by adding a pendant edge to one of the two triangles. Moreover, $L^{-1}(L(T_t)) = \{T_t, T_t'\}$, where T_t' is the multigraph obtained from T_t through identification of v_{t+5} and w (see Figure 2). Our aim is to show that no graph containing $L(T_t)$ as an induced subgraph for some $t \geq 1$ can be a member of $\mathcal{F}_k$.

To this end, start with the graph $J(k,k^2)$, and replace every edge by a path with $2t+2$ edges with all but the middle two edges of this path being multiplied by $k-1$ (see Figure 3 for an example), and call the resulting multigraph $J_t(k,k^2)$. Then, $L(J_t(k,k^2))$ is k-connected and it contains no induced subgraph isomorphic to $L(T_t)$, since $J_t(k,k^2)$ does not contain T_t or T_t' as a sub(multi)graph. All cycles in $L(J_t(k,k^2))$ which correspond to cycles in the original $J(k,k^2)$ have length at least $k^2(2t+2) > 2k^2 t$. All other cycles in $L(J_t(k,k^2))$ have length at most $k(t(k-1)+1) \leq k^2 t$, so $L(J_t(k,k^2))$ is not subpancyclic.

Thus, setting $r = \lceil k/2 \rceil + 1$, all graphs from the family $\mathcal{F}_k$ are either subgraphs of $F(t_1, t_2, \ldots, t_r)$ or subgraphs of $Ł(r,t)$ for some odd t.

Finally, we bound t_i for $i = 0, \ldots, r-2$. In order to do that, take n copies $G_1, \ldots, G_n$ of $J(k,k+1)$ and identify one vertex of G_i with one vertex of G_{i+1} for $i = 1, \ldots, n-1$

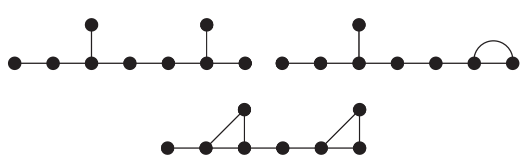

FIGURE 2. T_2, T_2' and $L(T_2)$.

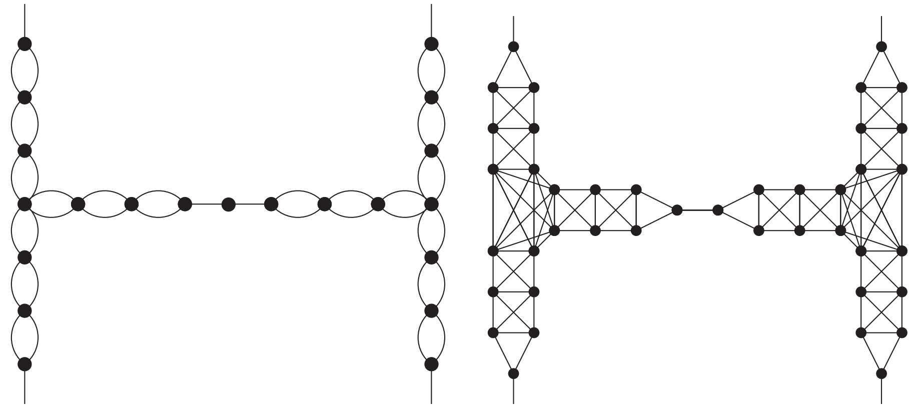

FIGURE 3. Parts of $J_3(3,s)$ and $L(J_3(3,s))$.

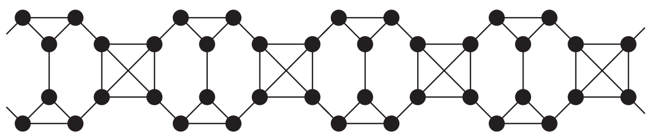

FIGURE 4. A part of $L(\mathrm{sub}(H_3^n))$.

in such a way that the graph H_k^n obtained in this way has maximum degree $2k$. Then the graph $L(\mathrm{sub}(H_k^n))$ is claw-free and k-connected, but not subpancyclic (it contains no cycles of length $2k+1$). If $J(k,k+1)$ has m vertices, then $L(\mathrm{sub}(H_k^n))$ contains no copies of $F(t_1,t_2,t_3)$ with $t_1 \geq 2m$. This completes the proof of the main part of Theorem 1.

We conclude the proof with a slight refinement of the graph H_3^n for the case $k=3$ to show that $m(3)=2$. Start with a cycle C_{2n} on $2n$ vertices and double every other edge to get the multigraph C'_{2n}. Let $H_3^n = L(C'_{2n})$. Then the graph $L(\mathrm{sub}(H_3^n))$ (see Figure 4) is a hamiltonian claw-free 3-connected graph, which contains no cycles of length five. Furthermore, $L(\mathrm{sub}(H_3^n))$ contains no induced copy of $F(3,3,3)$. Hence, the only possible candidates for the members of family $\mathcal{F}_3$ are paths, the graphs $Ł(3,2s+1)$, where $s \geq 0$, and the graphs $F(t_1,t_2,t_3)$, where $t_1 \leq 2$.

4. PROOF SKETCH FOR THEOREM 2

As the proof of Theorem 2 is rather tedious in parts, we want to give the reader a guideline with the main ideas and motivations before we get into the technical details.

We start with showing that all claw-free graphs with diameter much smaller than their order and minimum degree at least three are subpancyclic. The proof goes in two steps. In the first step, we use the fact that larger cycles must have shortcuts due to the small diameter, and thus we can find a slightly smaller cycle which we can then extend again to the desired length using the remaining vertices close to it. The second step is noting that such graphs have high maximum degree and therefore we can find cycles of small and moderate length in the closed neighborhood of a vertex.

In fact, we show a slightly stronger result in the first step. We show that for a given ℓ-cycle C, we can find an $(\ell-1)$-cycle through a given edge of C (Lemma 4). This stronger statement is used in Section 6 when we deal with graphs with large diameter.

If a graph G has large diameter, then it contains a long distance preserving path P. Suppose that G does not contain $F=F(r,k,k)$ as an induced subgraph for some fixed $r \ll k \ll \mathrm{diam}(G)$. Let v be a vertex in $V(G)\setminus N[P]$, and suppose there is an induced path R of length at least r from v to $N[P]$ with $N[P]\cap V(R)=\{w\}$, the final vertex of R. Then, w is adjacent to one of the first or last k vertices of P, as otherwise claw-freeness leads to a copy of F.

Therefore, G falls in one of two classes. Either, every vertex in $V(G)\setminus N[P]$ has paths to at most one end section of $N[P]$, in which case we say that G is *linear*, or there exists a vertex with paths into both end sections, in which case we call G *circular*.

In the linear case, G consists of a long mid-section with all vertices within distance r of P and two end sections with diameter at most $2k$. In the circular case, the structure is even simpler, as all vertices of G are within distance r of some long distance preserving cycle.

For the special case treated in Theorem 2, we have $r=2$. This gives us a lot of control over the mid-section in the linear case, and over all of G in the circular case. We take two disjoint long induced paths along a diameter path or a long distance preserving cycle, respectively, and show that there are so many edges between the two that we can find cycles of all lengths up to close to the order of the two paths.

If G is circular, we then present a way to lengthen the distance preserving cycle one by one all the way up to $|V(G)|$, so in this case, G is actually pancyclic. The argument in the linear case is somewhat more involved.

We first show that we can find cycles of all lengths up to the order of the mid-section. But this still leaves the possibility of a (maximal) missing cycle length ℓ for a larger $\ell<\mathrm{circ}(G)$. If an $(\ell+1)$-cycle C contains more than two-thirds of the vertices in the mid-section, then we can use our control there to find an ℓ-cycle. Otherwise, C must contain many vertices in one of the end sections. This is where Lemma 4 comes in again. After modifying C a bit on the cut set separating the mid-section from the end section, we can think of a part of C as one large cycle within the end section, one of its edges lying in the cut set. Now Lemma 4 lets us shorten this part of C by one without affecting the connection to the remainder of C, finishing the proof.

5. LARGE CLAW-FREE GRAPHS WITH SMALL DIAMETER

In this section, we show that all claw-free graphs with small diameter and minimum degree at least three are subpancyclic, provided they are large enough. The main tool for this is the following lemma which will be used again in Section 6.

Lemma 4. *Let G be a connected claw-free graph with minimum degree at least three and diameter d. Suppose there exists an $(\ell+1)$-cycle C in G through an edge $xy\in E(C)$ for some $\ell\geq(24d+22)^d$. Then G also contains a cycle of length ℓ through xy.*

Proof. Fix a direction on C such that $y = x^+$. If, for any $v \in V(C) \setminus \{x, y\}$, v^- and v^+ are adjacent, then the assertion follows. Hence, we may assume for the remainder of the proof that no such short chord exists. This assumption implies that there is an edge $v^- w$ or $v^+ w$ for every $w \in N(v) \setminus \{v^-, v^+\}$ as otherwise $\langle v, v^-, v^+, w \rangle$ would be a claw.

The strategy of the proof goes as follows. First, we find a cycle C' which is a little shorter than C but which shares a long segments with C, and then use the above observation to lengthen C' to an ℓ-cycle one vertex at a time. Let

$$W := \{v \in V(y^+ C x^-) : v \text{ is not an endvertex of a chord in } C\}.$$

Since G has minimum degree at least three, all vertices in W have neighbors outside of $V(C)$. We consider two cases.

Case 1. Suppose $W = \emptyset$.

Let $y_1 \in N(y) \cap V(C) \setminus \{y^+\}$ such that y has no neighbors in $y^{++} C y_1^-$ (possibly $y_1 = x$). Let uv be a chord of C such that

- $uCv \subseteq yCy_1$,
- there are no other chords with both endvertices on uCv,
- $|V(yCu)|$ is minimal.

If $x^- y \in E$ and $N(x) \cap V(u^+ C v^-) \setminus \{y^+\} \neq \emptyset$, let $x_1 \in N(x) \cap V(u^+ C v^-) \setminus \{y^+\}$ such that $|yCx_1|$ is minimal, and let $C' = y x x_1 C x^- y$. Otherwise, observe that $\langle \{x^-\} \cup (N(x) \cap V(u^+ C v^-) \setminus \{y^+\}) \rangle$ is complete to avoid a claw around x. Let $x_1 = v$ and $C' = x_1 C u x_1$ in this case. In either case, $|V(C')| \leq \ell$, and $N(y) \cap (V(C) \setminus V(C')) \subseteq \{y^+\}$.

We shall insert all but one vertex of $V(C) \setminus V(C')$ back into C' to create the desired ℓ-cycle. Let $w \in V(C) \setminus (V(C') \cup \{y^+\})$. If $w \in N(x)$, we know by the observation above that $wx^- \in E$, and w can be inserted between x^- and x. So assume that $w \notin N(x)$. Since $w \notin W$, there is a vertex $z \in V(x_1^+ C x^-)$, with $wz \in E$. As above we may assume that $z^- z^+ \notin E$. This implies that $wz^- \in E$ or $wz^+ \in E$, otherwise there is a claw at z. This enables us to insert w between z^- and z, or z and z^+, respectively.

Inserting vertices of $V(C) \setminus (V(C') \cup \{y^+\})$ one by one yields the ℓ-cycle. We will always use the first place $zz^+ \in E(x_1 C x)$ where the insertion is possible, (thus $wz^- \notin E$). The only problem we can encounter is a situation where we want to insert two different vertices $w, w' \in V(C) \setminus (V(C') \cup \{y^+\})$ between the same vertices $z, z^+ \in V(x_1 C x)$. But then $ww' \in E$ as otherwise $\langle z, z^-, w, w' \rangle$ is a claw. Thus, w and w' are neighbors on C due to the choice of u and v. In this case, we can extend C' through $zww'z^+$.

Case 2. Suppose $W \neq \emptyset$.

Since $W \neq \emptyset$, we have $V(C) \neq V(G)$. Thus, let $v \in V(G) \setminus V(C)$, and let T' be a spanning tree of G such that $d_{T'}(v, w) = d_G(v, w)$ for all $w \in V$, i.e., a breadth first search tree rooted at v. Let T be the minimal subtree of T' with $V(T) \supseteq V(C) \cup \{v\}$. Direct T away from v, and write z_- for the predecessor of a vertex $z \in V(T)$. By the diameter condition, we know that T has a vertex z with at least $12d + 12$ outneighbors in T. Let $v_1, v_2, \ldots, v_{12d+11} \in V(C) \setminus V(vTz)$ be $12d + 11$ vertices with $(V(C) \cup V(vTz)) \cap V(zTv_i) \subseteq \{z, v_i\}$. Out of these vertices we pick seven vertices $y_1, y_2, \ldots, y_7$ in this order on $y^{++} C x^{--}$, such that $|V(y_i C y_{i+1})| \geq 2d + 2$ for $1 \leq i \leq 6$. By the pigeon hole principle it is true that $|V(y_i^+Cy_{i+1}^-)\cap W|\le\frac{|W|}{6}$ for some $1\le i\le 6$. As it does not affect the remainder of the proof, we may assume that $i=1$.

If possible, choose $u,w\in V(y_1Cy_2)$, such that uw is a chord of C, and no other chords are inside uCw, and let $C'=wCuw$. If there is no such chord, let $u=y_1$, $w=y_2$ and $C'=wCuTw$. The only way that C' is not a cycle is that y_1Ty_2 uses z and $z\in V(C)$ (and thus $z\neq v$). In this case, let z_1,z_2 be the neighbors of z on y_1Ty_2. None of the edges z_-z_1, z_-z_2 can exist by the distance property of T. Hence, $z_1z_2\in E$ to avoid a claw at z. Now choose $C'=wCuTz_1z_2Tw$. In any case, $|V(C')|\le\ell$.

Let $X=\{z\in V(u^+Cw^-)\setminus W: N(z)\cap V(w^+Cu^-)\subseteq\{x,y\}\}$, let $X_1=X\cap N(x)$, $X_2=X\cap N(y)$. These are vertices which can create some problems later, and we show here that there are very few of them. Note that $\langle X_1\rangle$ is complete, otherwise there is a claw centered around x, using x^- and two independent vertices in X_1. Similarly, $\langle X_2\rangle$ is complete. This implies that $|X_1|,|X_2|\le 2$ as there are no chords inside u^+Cw^-. If $|X|>2$, there are vertices $x_1\in X_1$ and $x_2\in X_2$ with $x_1x_2\notin E$. To avoid claws around x and y, we must have $x^-y, xy^+\in E$ and thus $x^-y^+\in E$. In this case, we choose a cycle $C''=y^+Cx_2yxx_1Cx^-y^+$ or $C''=y^+Cx_1xyx_2Cx^-y^+$, such that $|V(C'')|\le\ell$. Otherwise, let $C''=C'$. In either case, $|X\cap(V(C)\setminus V(C''))|\le 2$.

Just as in Case 1, we can now insert vertices from $V(C)\setminus(V(C'')\cup X\cup W)$ into C'' to get a longest possible cycle C''' with

$$m-2-\left\lfloor\frac{|W|}{6}\right\rfloor\le|V(C''')|\le m-1.$$

Note that all these insertions happen at sections of C where C'' and C are identical. No vertex is inserted next to x or y as vertices of X are not inserted, and no vertex is inserted next to a vertex of W, since these vertices are not endvertices of chords. Note also that, since the minimum degree of G is at least three, each vertex in $W\cap V(C''')$ has at least one neighbor outside $V(C''')$. Furthermore, each such neighbor can be adjacent to at most four vertices from W without creating a claw. Therefore,

$$|N(W)\cap V(C''')|\ge\left\lceil\frac{1}{4}\times\frac{5}{6}|W|\right\rceil\ge\left\lfloor\frac{|W|}{6}\right\rfloor+1.$$

Since G is claw-free, all these vertices can be inserted into C''' one by one (if they are not already a part of C''' as vertices of uTw), and so G contains an ℓ-cycle, contrary to our assumption. ■

As a consequence of Lemma 4, we get the following theorem.

Theorem 5. *For every d there exists an $n=n(d)$ such that each connected claw-free graph G with at least n vertices, diameter at most d and minimum degree at least three is subpancyclic.*

Proof. Let G be a connected claw-free graph with n vertices, diameter at most d and minimum degree at least three. Lemma 4 implies that G contains a cycle of length ℓ, for each ℓ such that $\mathrm{circ}(G)\ge\ell\ge\ell_0=(24d+22)^d$. Note also that from the fact that G has diameter at most d it follows that it contains a vertex v of degree $D\ge(n-1)^{1/d}$.

Since G is claw-free, $\langle N(v)\rangle$ has independence number at most two, and it is an easy observation that $\langle N(v)\rangle$ contains a path of length $\lceil D/2\rceil$. Then $\langle N[v]\rangle$ contains a cycle of length ℓ for each ℓ, $3\le\ell\le D/2+1$, so it is enough to choose n large enough to have $(n-1)^{1/d}/2\ge\ell_0-2$. ■

6. GRAPHS WITH LARGE DIAMETER

For the proof of Theorem 2 it suffices to show that $\mathcal{F}_3$ contains $F(2,k,k)$ for all $k\ge 10$. So let $k\ge 10$ and $F=F(2,k,k)$.

We start with the following simple lemma.

Lemma 6. *For $u,v\in V(G)$, let P be an induced $u-v$ path in a claw-free graph G, and let $X\subseteq N(u^+Pv^-)$. Then G contains a $u-v$ path R with $V(R)=V(P)\cup X$ on which all vertices of $V(P)$ appear in the same order as on P.*

Proof. Without loss of generality we may assume that $V(G)=X\cup V(P)$. For $w\in V(P)\setminus\{u,v\}$, let $X_w=X\cap(N(w)\setminus N(w^-))$. Since G is claw-free, every $\langle X_w\cup\{w^+\}\rangle$ is complete.

Suppose, first that $\langle N(u)\rangle$ is 2-connected. Since G is claw-free, $\langle N(u)\rangle$ contains no independent sets of size three and thus it is hamiltonian. Consequently, we can construct a path Q from u to u^{++} through all vertices of $N[u]$. If $\langle N(u)\rangle$ is not 2-connected, then $\langle N(u)\rangle$ consists of two complete graphs sharing at most one vertex, where one of them contains u, the other u^{++}. Then, again, we can construct a path Q from u to u^{++} through all vertices of $N[u]$. Now let $R=uQu^{++}X_{u^{++}}u^{3+}\dots v^-X_{v^-}v$. ■

The next lemma tells us a lot about the structure of G.

Lemma 7. *Let G be $\{K_{1,3},F\}$-free, and let $P=v_0v_1\dots v_{4k+3}$ be an induced path in G. Let $x\in N(v_{2k+1})\cap N(v_{2k+2})$, $u\in N(x)\setminus N[P]$, and let $w\in N[u]\setminus N[P]$. Then $N[w]=N[u]$.*

Proof. First note that $wx\in E$ since otherwise $V(P)\cup\{x,u,w\}$ contains F. Suppose that $N(u)\ne N(w)$. Because of symmetry, we may assume that there exists $z\in N(u)\setminus N(w)$. If $z\in N(v_{k+1}Pv_{3k+2})$, then $\langle V(P)\cup\{z,u,w\}\rangle$ contains F, a contradiction. Hence $z\notin N(v_{k+1}Pv_{3k+2})$ and $xz\notin E$ to avoid a claw $\langle x;v_{2k+1},w,z\rangle$. But now $\langle V(v_{k+1}Pv_{3k+2})\cup\{x,u,z\}\rangle$ is a copy of F. This contradiction completes the proof of the lemma. ■

We call G *circular* if it contains a distance preserving cycle of length greater than $10k$ as a subgraph, and *linear* otherwise. We get the following two corollaries from Lemma 7.

Corollary 8. *If G is $\{K_{1,3},F\}$-free and contains an induced cycle C with $|V(C)|\ge 4k+5$, then $V(G)=N[N[C]]$. Furthermore, the diameter of G is at most $\frac{|V(C)|}{2}+3$.*

Corollary 9. *If G is $\{K_{1,3},F\}$-free and linear with diameter $N>10k$, and $P=v_0v_1\dots v_N$ is a diameter path of G, then every vertex $v\in V(G)\setminus N[N[P]]$ has distance at most $2k+1$ from one of v_{2k} and v_{N-2k}.*

We are ready to prove the existence of short and medium length cycles in the linear case now.

Lemma 10. *Suppose that G is 3-connected, $\{K_{1,3}, F\}$-free and linear, $N>10k$, and that $P=v_0v_1\dots v_N$ is a distance preserving path in G. Then G contains cycles of all lengths ℓ for $3\le\ell\le 2N-16k$.*

Proof. Let C be a minimal cycle through v_0 and v_N. Then v_0Cv_N and v_NCv_0 are induced paths, and we can apply Lemma 7 to them. Note also that all chords of C have ends in both $v_0^+Cv_N^-$ and $v_N^+Cv_0^-$. Let x_1y_1 and x_2y_2 be chords of C such that

- $|V(v_0Cx_1)|, |V(y_1Cv_0)|, |V(x_2Cv_N)|, |V(v_NCy_2)|>2k$, and
- $|V(y_1Cx_1)|$ and $|V(x_2Cy_2)|$ are minimal under this condition.

Such chords exist due to Corollary 8 and the fact that v_0 and v_N have distance N. Furthermore, by the same reasons we have $|V(x_1Cx_2y_2Cx_2x_1)|\ge 2N-16k$. Let

$$W=(V(x_1Cx_2)\setminus N(v_NCv_0))\cup(V(y_2Cx_2)\setminus N(v_0Cv_N)).$$

Then W is an independent set by Lemma 7 applied to v_0Cv_N and v_NCv_0, since any two consecutive vertices on each of the two paths have different closed neighborhoods.

Claim 1. *If x_3y_3 and x_4y_4 are chords in C, where $v_0, x_3, x_4, v_N, y_3, y_4$ appear on C in this order, then $\langle x_3, x_4, y_3, y_4\rangle$ is a complete graph.*

First, $x_4=x_3^+$ and $y_4=y_3^+$ as otherwise we could find a shorter cycle through v_0 and v_N. And then, $x_3y_4, x_4y_3\in E$ to avoid claws.

Claim 2. *If xy is a chord in C, and $x^-y, xy^+\notin E$, then $\langle x, x^+, y, y^-\rangle$ is a complete graph.*

We have $xy^-, x^+y\in E$ to avoid claws, the existence of the remaining edge follows from Claim 1.

Suppose that $\langle V(x_1Cx_2y_2Cy_1x_1)\rangle$ contains an $(\ell+1)$-cycle $C=x_3Cx_4y_4Cy_3x_3$ using exactly two chords of C, but no ℓ-cycle with this property for some ℓ with $3\le\ell\le 2N-15$. Then $\langle x_3^-, x_3, y_3, y_3^+\rangle$ and $\langle x_4, x_4^+, y_4^-, y_4\rangle$ are complete by Claim 2. If any of the edges $x_3^+y_3^-, x_3^+y_3^{--}, x_3^{++}y_3^-, x_4^-y_4^+, x_4^-y_4^{++}, x_4^{--}y_4^+$ are present, we can find an ℓ-cycle using only two chords of C. If this cycle contains vertices in $\{x_3^-, y_3^+, x_4^+, y_4^-\}$, we can subsequently skip these vertices one by one to create shorter cycles, ending up with an $(\ell-1)$-cycle or an $(\ell-2)$-cycle using only two chords of C with all vertices inside $V(x_1Cx_2y_2Cy_1x_1)$.

If none of these edges exist, $x_3^+, y_3^-, x_4^-, y_4^+\in W$ and $x_3^{++}y_3^{--}, x_4^{--}y_4^{++}\in E$ by Claim 1. Now $x_3^-Cx_4^{--}y_4^{++}Cy_3^+x_3^-$ is an $(\ell-1)$-cycle using only two chords of C (which can again be reduced to find a cycle with all vertices inside $V(x_1Cx_2y_2Cy_1x_1)$). If $\ell>5$, we can extend this cycle through a neighbor of x_3^+ to construct an ℓ-cycle.

Finally, if $\ell=5$, we have $x_3^+=x_4^-$ and $y_3^-=y_4^+$. Let $x\in(N(x_3^+)\cap N(v_NCv_0))\setminus\{x_3, x_4\}$. This vertex exists by Lemma 7 since G is 3-connected. If $x\in N(y_4Cy_3)$, then we can find a 5-cycle through x. Otherwise we can reduce $|V(C)|$, a contradiction. ■

Now we show Theorem 2 in the case of circular G.

Lemma 11. *Suppose that G is* 3*-connected*, $\{K_{1,3},F\}$*-free and G contains a distance preserving cycle of length* $N>20k$. *Then G is pancyclic.*

Proof. For cycle lengths ℓ with $3\le\ell<N$, we use a strategy very similar to the proof of Lemma 10. Let $C=v_0v_1\dots v_{N-1}v_0$ be a distance preserving cycle of length N in G. Treat all indices in the following modulo N.

Let $u\in V(G)\setminus V(C)$. Since C is distance preserving, we have $N(N(u))\cap V(C)\subseteq V(v_iCv_{i+4})$ for some $0\le i\le N-1$. This, together with Lemma 7, guarantees that $N[N[v_iCv_{i+3}]]$ is a cut set of $N[N[v_{i-k}Cv_{i+k}]]$ separating v_{i-k} and v_{i+k} for all i.

Consider $G_i=G\setminus N[N[v_iCv_{i+3}]]$. If G_i contains a cycle through v_{i-k} and v_{i+k}, we can use a shortest such cycle in place of C in the proof of Lemma 10, and show that G_i contains cycles of all lengths ℓ for $3\le\ell\le 2(N-2k)-16k$.

Otherwise, there is a j with $|i-j|>k$ such that v_j is a cut vertex of G_i separating v_{i-k} and v_{i+k}. Repeat the argument to show that G_j contains cycles of all lengths ℓ for $3\le\ell\le 2(N-2k)-16k$ unless there is a j' with $|j'-j|>k$ such that $v_{j'}$ is a cut vertex of G_j separating v_{j-k} and v_{j+k}. But then $\{v_j,v_{j'}\}$ is a cut set of G, contradicting that G is 3-connected.

For $\ell\ge N$, we start with C and extend it one vertex at a time. By Lemma 6, we can include any set $X\subseteq N(C)$ into the cycle.

For vertices $u\in N(N[C])$, let $Z_u=\{w\in N(N[C]):N[u]=N[w]\}$. If $|N(u)\cap N(v_i)\cap N(v_{i+1})|\ge 2$ for any i, we can extend the cycle through these two vertices in $N(C)$ and any number of vertices in Z_u. Otherwise, u has neighbors in both $N(v_i)\cap N(v_{i+1})$ and $N(v_i)\cap N(v_{i-1})$ for some i since G is 3-connected. Again, we can extend the cycle through these two neighbors in $N(C)$ and any number of vertices in Z_u. If there are two (or more) vertices u_1, u_2 with $N[u_1]\ne N[u_2]$ and $|N(u_1)\cap N(v_i)\cap N(v_{i+1})|=|N(u_2)\cap N(v_i)\cap N(v_{i+1})|=|N(u_1)\cap N(u_i)\cap N(v_{i-1})|=|N(u_2)\cap N(v_i)\cap N(v_{i-1})|=1$, we can go back and forth through any number of vertices in Z_{u_1} and Z_{u_2}. Note that Lemma 7 and claw-freeness of G guarantee that u_1 and u_2 have no common neighbors, so we can perform all these insertions independently. ■

Because of Lemmas 10 and 11 for the proof of Theorem 2 it remains to show the following lemma.

Lemma 12. *Suppose that G is* 3*-connected*, $\{K_{1,3},F\}$*-free and linear with diameter* $N>10k$. *For every* $\ell\ge 2N-16k$, *if G contains an* ℓ*-cycle*, *then G contains an* $(\ell-1)$*-cycle.*

Proof. In the following two claims, we show that a lot of the sets $N[v_i]\cap N[v_{i+1}]$ are cut sets of G.

Claim 3. *Suppose there exists an* $u\in N(N[P])$ *such that* $N(N(u))\cap V(P)\subseteq V(v_iPv_j)$, $v_i,v_j\in N(N(u))$ *and* $2k+1\le i\le j\le N-2k-1$. *Then* $N[v_i]\cap N[v_{i+1}]$ *and* $N[v_{j-1}]\cap N[v_j]$ *are cut sets of G separating* v_0 *and* v_N.

By symmetry, it is enough to prove the statement for the first set. For the sake of contradiction, assume that for $X=N[v_i]\cap N[v_{i+1}]$ there is a path from v_0 to v_N in $G-X$.

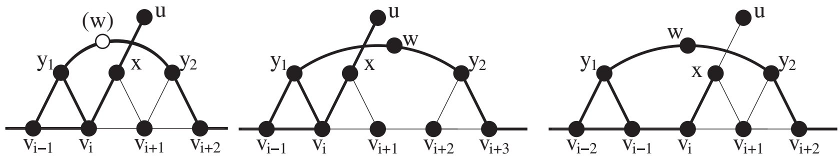

FIGURE 5. The three possible configurations in the proof of Claim 3.

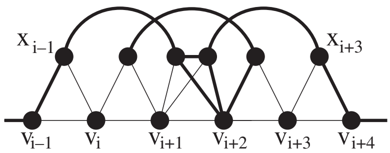

FIGURE 6. The configuration in the proof of Claim 4.

Let R be such a path which contains the minimal number of vertices outside of P. It is easy to see that R contains exactly two vertices y_1 and y_2 (in this order) in $N(P)$, and at most one vertex w outside of $N[P]$.

Let $x \in N(v_i) \cap N(u)$. The following edges cannot exist (see also Figure 5): uy_1 (since $v_{i-1} \notin N(N(u))$), uw (by Lemma 7 with $uy_1 \notin E$), uy_2 (claw centered at y_2), $v_i y_2$ ($y_2 \notin X$), wx (claw centered at x), $x_1 y_2$ (claw centered at y_2) and xy_1 (claw centered at y_1). Furthermore, one of $v_i y_1$ and $v_{i-2} y_1$ is not an edge, otherwise there is a claw centered at y_1.

But now it is easy to extend R to a copy of F through v_i, x and one of v_{i-1} and u, a contradiction.

Claim 4. *For $2k+5 \le i \le N-2k-8$, at least one of the sets $N[v_i] \cap N[v_{i+1}]$, $N[v_{i+1}] \cap N[v_{i+2}]$ and $N[v_{i+2}] \cap N[v_{i+3}]$ is a cut set of G separating v_0 and v_N.*

Suppose that none of the three sets cuts G. By Claim 3, no vertex in $N(N[P])$ has distance 2 to $v_i P v_{i+3}$. Thus, there exist three edges $x_{i-1}x_{i+1}$, $x_i x_{i+2}$ and $x'_{i+1}x_{i+3}$ with $x_j, x'_j \in N(v_j) \cap N(v_{j+1})$ for $i-1 \le j, j' \le i+3$ (see Figure 6). The edges $x_{i-1}x_i$ and $x_{i+2}x_{i+3}$ would lead to claws at x_{i-1} and x_{i+3}, respectively, and similarly, any edge of $x_i x_{i+1}$, $x_i x'_{i+1}$, $x_{i+1}x_{i+2}$ and $x'_{i+1}x_{i+2}$ would lead to a claw at x_{i+1} or x'_{i+1}, respectively, so none of the above edges exists. Furthermore, $x_{i+1} \ne x'_{i+1}$ to avoid a claw at x_{i+1}.

But this implies that

$$\langle v_{i+2}x_{i+2}x_i; x'_{i+1}x_{i+3}v_{i+4}Pv_{i+k+2}; x_{i+1}x_{i-1}v_{i-1}Pv_{i-k+1}\rangle$$

is a copy of F, a contradiction.

The last two claims guarantee that there are s, t with $2k+5 \le s \le 2k+7$ and $N-2k-7 \le t \le N-2k-5$ such that $X_L = N[v_s] \cap N[v_{s+1}]$ and $X_R = N[v_{t-1}] \cap N[v_t]$ are cut sets of G. Let L be the component of $G - X_L$ containing v_0, let R be the component of $G - X_R$ containing v_N and let $M = G - (L \cup R)$.

Observe that $\langle X_L \rangle$ is almost complete. The only edges which may be missing are pairs xy with $x \in N(v_{s-1}) \setminus \{v_s\}$ and $y \in N(v_{s+2}) \setminus \{v_{s+1}\}$ as all other missing edges would lead to claws. The symmetric statement is true for $\langle X_R \rangle$.

Claim 5. *The graph M is pancyclic.*

Let Q and S be two disjoint paths in M from X_L to X_R such that $|V(Q \cup S)|$ is minimal, let q_L, s_L, q_R, s_R be the end vertices of the paths. If $V(Q \cup S) \cap V(P) = \emptyset$, we replace Q by $v_{s+1} P v_{t-1}$, not changing $|V(Q \cup S)|$. Otherwise, let $q \in V(Q \cup S) \cap V(P)$ such that $V(Q \cup S) \cap V(v_{s+1} P q^-) = \emptyset$. By symmetry we may assume that $q \in V(Q)$. Then $|V(v_s q_L Q q)| \geq |V(v_s P q)|$, and we may replace $q_L Q q$ by $v_{s+1} P q$. Therefore, we may assume that $q_L = v_{s+1}$ and that $v_{t-1} \in \{q_R, s_R\}$.

Since v_s has no neighbors in $V(M) \setminus X_L$ and v_t has no neighbors in $V(M) \setminus X_R$ the paths $Q' = v_0 P v_{s+1} Q q_R (v_t) v_{t+1} P v_N$ and $S' = v_0 P v_{s-1} (v_s) s_L S s_R (v_t) v_{t+1} P v_N$ are induced. Furthermore, $v_s q_L Q q_R v_t$ is distance preserving in $M - S$, and $v_s s_L S s_R v_t$ is distance preserving in $M - Q$.

Let C be the cycle $v_s q_L Q q_R v_t s_R S s_L v_s$. Cycles of all lengths ℓ for $3 \leq \ell \leq |V(C)|$ can be found again in the same fashion as in the proof of Lemma 10. Observe also that all vertices in $N(C) \cap V(M)$ can be added to C one by one by Lemma 6. Note that this includes all vertices in $X_L \subset N[v_{s+1}]$ and $X_R \subset N[v_{t-1}]$, which can be inserted between x_s and x_{s+1} and between x_{t-1} and x_t, respectively.

All vertices in $V(M) \setminus N[C]$ must be in $N(N[Q']) \cap N(N[S'])$ by Lemma 7 applied to Q' and S', and therefore any such vertex has at least three neighbors in $N(Q') \cap N(S')$. None of these neighbors is in $V(L \cup R)$, so in the same fashion as in Lemma 11, we can add these vertices one by one to C.

Now let C be an ℓ-cycle with $\ell > |V(M)|$. For the sake of contradiction assume that G contains no $(\ell - 1)$-cycle. Consider $C \cap L$. If C contains a vertex outside L, this is a (possibly empty) collection of paths $P_0, P_1, \ldots$. Let $p_i, s_i \in V(C) \cap X_L$ be the predecessors and successors of the P_i on C.

Claim 6. *There is a path P_L beginning and ending in X_L with $V(P_L) \subseteq V(C) \cap N[L]$ which contains all P_i in any given order and direction.*

Let $P_i, P_j \subset C$ as above. Let y_i be the final vertex of P_i and x_j, the first vertex of P_j. If $s_i = p_j$, then P_i and P_j are consecutive on C and we can either skip s_i in C to get an $(\ell - 1)$-cycle, or $\langle s_i, v_{s+1}, x_j, y_i \rangle$ is a claw, which is a contradiction.

If $s_i p_j \notin E$, then $s_i \in N(v_{s-1})$ and $p_j \in N(v_{s+2})$ (or the other way around, but we may assume the former). To avoid a claw centered at p_j, we have $v_s x_j \in E$, as well as $v_{s-1} x_j \in E$ to avoid a claw at v_s. Now $v_{s-2} x_j \notin E$ to avoid a claw at x_j, concluding that $s_i x_j \in E$ to avoid a claw at v_{s-1}. Thus, we can connect P_i and P_j through s_i.

Similarly, as P_L, we can construct a (possibly empty) path P_R in $N[R]$. Next, we want to use paths similar to Q and S from above and join them with P_L and P_R into one big cycle C'. We want to be able to add all remaining vertices in M to C', so we have to take some care how we choose P_L, P_R, Q and S.

If $v_s = p_i$ (or $v_s = s_i$) for some $P_i \subset P_L$, let $x_L = s_i$ (or $x_L = p_i$, respectively). Otherwise, let $x_L = v_0$. Similarly define x_R in relation to v_t. Let Q and S be two disjoint paths in $M - \{x_L, x_R\}$ from $X_L - \{x_L\}$ to $X_R - \{x_R\}$ such that $|V(Q \cup S)|$ is minimal, $v_{s+1} \in V(Q)$ and $v_{t-1} \in V(Q \cup S)$. Such paths exist since G has no cut set consisting of x_L, x_R and a third vertex.

Let q_L, q_R, s_L, s_R be defined as above. By our choice of x_L, we can order the P_i on P_L such that P_L does not contain s_L or v_s as internal vertices. Similarly construct P_R. Now connect P_L with Q and S as follows. If s_L is an end vertex of P_L, then P_L and S are already connected. If s_L is adjacent to an end vertex of P_L which is not v_s, connect these two vertices. Otherwise, connect s_L with P_L through v_s. Finally, connect $q_L = v_{s+1}$ to the other end vertex of P_L (through v_s if v_s is not already used). Similarly connect P_R with Q and S to create the cycle C' with $V(C') = V(P_L \cup P_R \cup Q \cup S) \cup \{v_s, v_t\}$. If one of P_L and P_R is empty, use the vertex v_s or v_t instead of the path.

Case 1. Suppose that $|V(C')| < \ell$.

We can add most vertices in $V(M) \setminus V(C')$ one by one in the same fashion as in Claim 5. The only vertices which may pose problems are the ones which were inserted somewhere between q_L and s_L or between q_R and s_R.

For these vertices, note that every vertex in $X_L \setminus V(C')$ can be inserted between v_s and v_{s+1} after inserting v_s between v_{s+1} and P_L if possible. This is impossible only in the case that v_s and s_L are adjacent on C'. In this case, every vertex in $X_L \setminus V(C')$ can be inserted between v_s and s_L or between v_{s+1} and P_L as every vertex in X_L is adjacent to at least one of any two non-adjacent vertices in X_L.

Finally, for vertices $u \in N(X_L) \cap N(N[Q']) \cap N(N[S']) \cap V(M)$, note that every neighbor $x \in N(u) \cap X_L$ is connected to all vertices in X_L and has no neighbors in L (otherwise there would be a claw). So these vertices can be inserted in the same fashion as in Claim 5 without interfering with P_L, and similarly vertices in $N(X_R) \cap N(N[Q']) \cap N(N[S']) \cap V(M)$.

This way, we can extend C' to an $(\ell - 1)$-cycle, which is a contradiction.

Case 2. Suppose that $|V(C')| \geq \ell$.

Observe that $|V(M) \setminus V(C')| \geq N - 4k - 16$ as G is 3-connected and $|V(Q \cup S)|$ is minimal. Thus, either $|V(C') \cap N[L]| > \frac{N}{2} - 2k - 8$ or $|V(C') \cap N[R]| > \frac{N}{2} - 2k - 8$. By symmetry we may assume that $|V(C') \cap N[L]| > \frac{N}{2} - 2k - 8$. Note that $\langle N[L] \cup \{v_s\} \rangle$ has minimum degree at least three and diameter at most $4k$, so we can shorten the path $\langle V(C') \cap (N[L] \cup \{v_s\}) \rangle_{C'}$ one vertex at a time keeping the same end vertices by Lemma 4.

After going through this procedure possibly for several times, we eventually arrive at a cycle of length $\ell - 1$, which is a contradiction. ■

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