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tK_p-saturated graphs of minimum size

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ABSTRACT

A graph G is H-*saturated* if G does not contain H as a subgraph but for any nonadjacent vertices u and v, $G+uv$ contains H as a subgraph. The parameter $sat(H, n)$ is the minimum number of edges in an H-saturated graph of order n. In this paper, we determine $sat(H, n)$ for sufficiently large n when H is a union of cliques of the same order, an arbitrary union of two cliques and a generalized friendship graph.

1. Introduction

In this paper we consider only graphs without loops or multiple edges. We let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The *order* of G, usually denoted n, is $|V(G)|$ and the *size* of G is $|E(G)|$. For any vertex v in G, let $N(v)$ denote the set of vertices adjacent to v and $N[v] = N(v) \cup v$. The *degree* of a vertex v is $|N(v)|$ and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a vertex in G, respectively. We denote the complement of G by $\overline{G}$ and for any graph H let tH denote the graph composed of t vertex disjoint copies of H. For vertices $v_1, \ldots, v_t$ in $V(G)$, let $\langle v_1, \ldots, v_t\rangle$ denote the subgraph of G induced by these vertices. Furthermore, if $U \subset V(G)$, we will use $\langle U, v_1, v_2, \ldots, v_t\rangle$ to denote the subgraph of G induced by the vertices $v_1, \ldots, v_t$ and U. Given any two graphs G and H, their *join*, denoted $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G),\ h \in V(H)\}$.

Let G and H be graphs. We say that G is H-*saturated* if H is not a subgraph of G, but for any edge uv in $\overline{G}$, H is a subgraph of $G + uv$. For a fixed integer n, the problem of determining the maximum size of an H-saturated graph of order n is equivalent to determining the classical extremal function $ex(H, n)$. In this paper, we are interested in determining the *minimum* size of an H-saturated graph. Erdős, Hajnal and Moon introduced this notion in [5] and studied it for cliques. We let $sat(H, n)$ denote the minimum size of an H-saturated graph on n vertices. The value $sat(H, n)$ is called the *saturation number* for the graph H.

There are very few graphs for which $sat(H, n)$ is known exactly. In addition to cliques, some of the graphs for which $sat(H, n)$ is known include stars, paths and matchings [10], C_4 [11], and C_5 [3]. In [12] the value of $sat(K_{2,3}, n)$ is found asymptotically. See [1] for a survey of related results. Some progress has been made for arbitrary cycles and the current best known upper bound on $sat(C_t, n)$ can be found in [9]. The best upper bound on $sat(H, n)$ for an arbitrary graph H appears in [10], and it remains an interesting problem to determine a non-trivial lower bound on $sat(H, n)$.

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2. $sat(tK_p, n)$

In [5], Erdős, Hajnal and Moon determined that

$$sat(K_p, n) = (p-2)(n-1) - \binom{p-2}{2}$$

for all $p \geq 3$. The upper bound is obtained by considering the graph $K_{p-2} + \overline{K}_{n-p+2}$, which is K_p-saturated. In this section we extend this result by constructing a graph G that is tK_p-saturated for any $t \geq 1$ and $p \geq 3$. In addition to extending the result in [5] pertaining to $sat(K_p, n)$, our main result also extends a result from [10] which states that $sat(tK_2, n) = 3t - 3$ for $n \geq 3t - 3$.

Let $t \geq 1$, $p \geq 3$ and $n \geq pt + t - 3$ be fixed integers. Let $G_0 \cong (t-1)K_{p+1}$ and denote these copies of K_{p+1} by $H_1, \ldots, H_{t-1}$. The graph $G(n, p, t)$ is defined to be the join of $G_1 \cong K_{p-2}$ and $G_0 \cup \overline{K}_{n-pt-t+3}$. We first note that $G(n, p, t)$ contains no copy of tK_p. Indeed, any copy of K_p in $G(n, p, t)$ can only be composed of vertices from G_1 and exactly one H_i. Furthermore, no two disjoint copies of K_p in $G(n, p, t)$ can intersect any fixed H_i as together H_i and G_1 have only $2p - 1$ vertices. These two facts imply that if ℓK_p is contained in $G(n, p, t)$ then $\ell \leq t - 1$.

Let u and v be nonadjacent vertices in $G(n, p, t)$ and add uv to $G(n, p, t)$. Then u, v and the vertices of G_1 induce a copy of K_p in $G(n, p, t) + uv$. Since u and v cannot lie in the same H_i, it is possible to find a subgraph of $G(n, p, t)$ isomorphic to $(t-1)K_p$ that is disjoint from u, v and G_1, so that tK_p is a subgraph of $G(n, p, t) + uv$. This implies that $G(n, p, t)$ is tK_p-saturated. The main result of this section is as follows:

Theorem 2.1. *Let $t \geq 1$, $p \geq 3$ and $n \geq p(p+1)t - p^2 + 2p - 6$ be integers. Then*

$$sat(tK_p, n) = |E(G(n, p, t))| = (t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2).$$

Proof. Given p and t, let G be a tK_p-saturated graph of order $n \geq p(p+1)t - p^2 + 2p - 6$. We will assume that the size of G is strictly less than $|E(G(n, p, t))|$ and work to a contradiction.

By assumption, tK_p is not a subgraph of G, yet for any pair of nonadjacent vertices in $V(G)$, $G + uv$ must contain a subgraph F isomorphic to tK_p. This says that uv must lie in some copy of K_p in $G + uv$. As this must hold for all pairs of nonadjacent vertices in G, it follows that $\delta(G)$ is at least $p - 2$. When n is sufficiently large, we can make a stronger statement.

Claim 2.2. *If $n \geq p(p+1)t - p^2 + 2p - 6$ then $\delta(G) = p - 2$.*

Proof. Assume otherwise, so that every vertex in G has degree at least $\delta \geq p - 1$. Let v be a vertex of minimum degree δ, then each non-neighboring vertex u must therefore lie in a copy of K_p with v in $G + uv$. This implies that u is adjacent to at least $p - 2$ vertices in $N(v)$ and also implies that there is a copy of K_{p-2} contained in the subgraph induced by $N(v)$. Thus, the sum of the vertex degrees in $N(v)$ is at least $(n - \delta - 1)(p-2) + 2\binom{p-2}{2} + \delta$. This yields that

$$2|E(G)| \geq \delta(n-\delta) + (n-\delta-1)(p-2) + 2\binom{p-2}{2} + \delta.$$

Since $\delta \geq p - 1$, we have that

$$2|E(G)| \geq (n-p+1)(p-1) + (n-p)(p-2) + 2\binom{p-2}{2} + (p-1).$$

By assumption,

$$|E(G)| < |E(G(n, p, t))| = (t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2)$$

which implies that

$$(n-p+1)(p-1) + (n-p)(p-2) + 2\binom{p-2}{2} + (p-1)$$

is at most

$$2\left((t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2)\right).$$

Simplifying, we get that

$$n < p(p-2) + (t-1)p(p+1) - (p-2)(p-3)$$

or

$$n < p(p+1)t - p^2 + 2p - 6,$$

contradicting our assumption about the order of G. □

Let v be a vertex of degree $p-2$ in G and choose any vertex u that does not lie in $N(v)$. Such a vertex exists by our bound on n. Then $G + uv$ must contain tK_p such that u and v are both in the same copy of K_p. This immediately implies that the other $p-2$ vertices in this copy of K_p must be $N(v)$ and hence, as the degree of v is $p-2$, that $N(v)$ must induce a complete subgraph of G, which we will henceforth call S. Furthermore, since this holds for any choice of u, it must be that all of the vertices in S are adjacent to each vertex in $G - S$.

Since $G+uv$ contains tK_p in which one of the copies of K_p is $\langle S, u, v\rangle$, G must contain a subgraph isomorphic to $(t-1)K_p$ that does not intersect S. Let H be such a subgraph and let $H_1, \ldots, H_{t-1}$ denote the components of H. To further describe the structure of G, let R denote those vertices in G, in $S \cup \bar{V}(H)$, that are adjacent to at least one vertex in $V(H)$.

It is now our goal to show that there are at least $(t-1)p$ edges ux in G such that neither u nor x lies in S and ux is not in $E(H)$. If $t = 1$, there is nothing to prove, thus we need only consider $t \geq 2$. In this case, we would know that

$$|E(G)| \geq \binom{p-2}{2} + (p-2)(n-p+2) + (t-1)\binom{p}{2} + (t-1)p = |E(G(n,p,t))|,$$

hence equality must hold. We will accomplish this by uniquely associating each vertex h in H with an appropriate edge incident to h.

Assume that some vertex in H, say v_1 in H_1, is such that $N[v_1] = S \cup V(H_1)$. Select any other vertex x in H_1 and add the edge xv to G, where again we let v denote a vertex of degree $p-2$ in G. Then $G + xv$ contains a subgraph F isomorphic to tK_p in which $\langle S, x, v\rangle$ is one of the copies of K_p. Note that v_1 has degree $2p-3$ and hence cannot lie in F since $p-1$ of its neighbors are already used in the clique $\langle S, x, v\rangle$. Consequently, replacing $\langle S, x, v\rangle$ with $\langle S, v_1, x\rangle$ in F, yields a subgraph of G isomorphic to tK_p, contradicting the assumption that G is tK_p-saturated.

We can therefore assume that every vertex h in H has a neighbor u that lies in either R or H such that hu is not in $E(H)$. If each vertex in H has a neighbor in R, this would assure at least $(t-1)p$ additional edges in G, completing the proof. This must hold if $t-1 = 1$, so we may assume $t \geq 2$. We also assume that the subgraph H' given by $\langle V(H)\rangle - E(H)$ is nonempty.

The components of H' fall into three categories: those components containing a cycle, those components that are trees and contain a vertex which has a neighbor in R and those components that are trees such that no vertex in the component has an adjacency in R. Assume for a moment that there are no components of the third type. Let C be a component of the first type, so that there is some cycle in C. Choose any edge xy on this cycle and consider $C - xy$, which must be connected. Choose any spanning tree of $C - xy$ and root it at x. Define the map $f_C : V(C) \to E(C)$ such that $f_C(x)$ is xy and for each other vertex $w \neq x$ in C, $f_C(w)$ is the edge that precedes w in the rooted spanning tree. Note that f_C is an injection.

Next assume that C is a component of the second type, that is, C is a tree (possibly a trivial tree) and there are vertices x and r in C and R respectively such that xr is in $E(G)$. Root C at x and define the map $f_C : V(C) \to (E(C) \cup xr)$ such that $f(x)$ is xr and for each other vertex $w \neq x$ in C, $f(w)$ is the edge that precedes w in the rooting of C at x. Note again that f_C is injective.

If all of the components of H' fall into one of these two categories, then we will define the function $f : V(H) \to E(G)$ such that if w is in some component C of H', then $f(w) = f_C(w)$. For each component C, f_C is injective and $f_C(v)$ is an edge adjacent to v that either lies in C or has an endpoint in R, and these two properties imply that f must be injective, which would complete the proof.

It is therefore our goal to show that each component of H' that is a tree must contain a vertex which has a neighbor in R. Assume that T is such a component of H' and let u_1 be an end-vertex of T. Assume that u_1 lies in H_1 and let w denote the neighbor of u_1 in T, so that w lies in some H_i for $i \geq 2$. Let u_2 be any vertex in H_1 other than u_1 and assume that u_2w is not an edge in G. Choose any u_3 in H_1 distinct from u_1 and u_2 and add the edge u_3v to G, where v is any vertex of degree $p-2$ in G. Then $G + u_3v$ contains a subgraph F isomorphic to tK_p such that one of the copies of K_p is $\langle S, u_3, v\rangle$. Note that the neighborhood of u_1 is exactly S, w and the other vertices in H_1. This implies, since $\langle S, u_3, v\rangle$ is one of the cliques in F, that if u_1 was in F, it would have to be in a clique with w and $V(H_1) \setminus \{u_3\}$. This is impossible, as we have assumed that u_2w is not an edge in G, so u_1 is not in F. This implies that we could replace $\langle S, u_3, v\rangle$ in F with $\langle S, u_1, u_3\rangle$ which creates a subgraph of G isomorphic to tK_p, a contradiction.

Hence we may assume that w is adjacent to each vertex in H_1. Let $V(H_1) = \{u_1, \ldots, u_{p-1}, y\}$. If we choose u_1 to be an end-vertex of a longest path in T, we may assume that all but one of the neighbors of w in T are also end-vertices of T. Specifically, we will assume that $U = \{u_1, \ldots, u_{p-1}\}$ are end-vertices in T. By assumption, u_1 is not adjacent to any other vertex in the component of H containing w, so choose some vertex z in the same component of H as w and add the edge u_1z to G. This creates a subgraph F of $G + u_1z$ isomorphic to tK_p. Let C denote the component (clique) in F that contains u_1z and let $\mathcal{T}$ denote $F \setminus C$.

Note that $N(u_1) \cap N(z)$ is composed of S, w and possibly y (if yz is an edge in G). Also note that the common neighbors of the vertices in U are exactly w, S and y. We consider several cases.

Case 1: Suppose that $C = \langle S, u_1, z\rangle$.

Note that the vertices in U have exactly 2 common neighbors outside of C, namely y and w. Thus, if any vertices of U appear in $\mathcal{T}$, then they specifically appear in the clique $\langle y, w, u_2, \ldots, u_{p-1}\rangle$. If $\langle y, w, u_2, \ldots, u_{p-1}\rangle$ is a clique in $\mathcal{T}$, then we see that $H_1(= \langle U, y\rangle)$, $\langle S, w, z\rangle$ and the cliques in $\mathcal{T} \setminus \langle y, w, u_2, \ldots, u_{p-1}\rangle$ comprise a subgraph of G isomorphic to tK_p, contrary to our assumptions. Hence we may assume that $\langle y, w, u_2, \ldots, u_{p-1}\rangle$ is not one of the cliques in $\mathcal{T}$ and therefore that no vertex of U appears in $\mathcal{T}$. Then $\langle S, u_1, u_2\rangle$ together with $\mathcal{T}$ is a subgraph of G isomorphic to tK_p, a contradiction.

Case 2: Suppose that $C = \langle S', u_1, w, z\rangle$, where $S' = S \setminus \{s\}$.

Note that if $p = 3$, then $|S| = 1$ and $S' = \emptyset$. The vertices in U have exactly two common neighbors outside of C, namely y and s, so if any vertex of U appears in $\mathcal{T}$, then they specifically appear in the clique $\langle y, s, u_2, \ldots, u_{p-1}\rangle$. If $\langle y, s, u_2, \ldots, u_{p-1}\rangle$ is in $\mathcal{T}$, then H_1, $\langle S, w, z\rangle$ and the cliques in $\mathcal{T} \setminus \langle y, s, u_2, \ldots, u_{p-1}\rangle$ comprise a subgraph of G isomorphic to tK_p, contrary to our assumptions. Hence we may assume that $\langle y, s, u_2, \ldots, u_{p-1}\rangle$ is not one of the cliques in $\mathcal{T}$ and therefore that no vertex of U appears in $\mathcal{T}$. Then $\langle S', u_1, u_2, u_3\rangle$ together with $\mathcal{T}$ is a subgraph of G isomorphic to tK_p, a contradiction.

Case 3: Suppose that $C = \langle S'', u_1, w, y, z\rangle$, where $S'' = S \setminus \{s_1, s_2\}$.

Note that Case 3 does not exist if $p = 3$. Also note that the vertices in U have only s_1 and s_2 as common neighbors in $\bar{C}$, so once again if any vertex of U is in $\mathcal{T}$ then they specifically appear in the clique $\langle s_1, s_2, u_2, \ldots, u_{p-1}\rangle$. If $\langle s_1, s_2, u_2, \ldots, u_{p-1}\rangle$ is in $\mathcal{T}$, then H_1, $\langle S, w, z\rangle$ and the cliques in $\mathcal{T} \setminus \langle s_1, s_2, u_2, \ldots, u_{p-1}\rangle$ comprise a subgraph of G isomorphic to tK_p. If $\langle s_1, s_2, u_2, \ldots, u_{p-1}\rangle$ is not a clique in $\mathcal{T}$, then $H_1 \cup \mathcal{T}$ is a subgraph of G isomorphic to tK_p, a contradiction.

Case 4: Suppose that $C = \langle S', u_1, z, y\rangle$, where $S' = S \setminus \{s\}$.

Note that the vertices in U have only w and s as common neighbors in $\bar{C}$, so as above if any vertex of U is in $\mathcal{T}$, then they specifically appear in the clique $\langle s, w, u_2, \ldots, u_{p-1}\rangle$. If $\langle s, w, u_2, \ldots, u_{p-1}\rangle$ is in $\mathcal{T}$, then H_1, $\langle S, w, z\rangle$ and the cliques in $\mathcal{T} \setminus \langle s, w, u_2, \ldots, u_{p-1}\rangle$ comprise a subgraph of G isomorphic to tK_p. If $\langle s, w, u_2, \ldots, u_{p-1}\rangle$ is not a clique in F, then $H_1 \cup \mathcal{T}$ is a subgraph of G isomorphic to tK_p, a contradiction.

As noted above, $N(u_1) \cap N(z)$ is composed of S, v and possibly y (if yz is an edge in G) so these four cases suffice to exhaust the possible compositions of C.

Consequently, it follows that each component of H' which is a tree must contain a vertex which has a neighbor in R. By our previous discussion, we can therefore associate each vertex in H with a unique edge in $\bar{H}$ that is not incident to any vertex in S. This assures that there are at least $(t-1)p$ edges in G aside from those in H and those adjacent to at least one vertex in S, completing the proof. □

One of the difficulties in determining $sat(H, n)$ is that frequently the extremal graphs are not unique. In [5], it was shown that $G(n, p, 1) = K_{p-2} + \overline{K}_{n-p+2}$ was the unique K_p-saturated graph of minimum size. As a consequence of the main result of the next section we will also show that $G(n, p, 2)$ is the unique $2K_p$-saturated graph of order n with minimum size. In this vein, we show the following.

Theorem 2.3. *If $p \geq 3$ and $n \geq 3p(p+1) - p^2 + 2p - 6$, then $G(n, p, 3)$ is the unique $3K_p$-saturated graph of order n with minimum size.*

Proof. Let G be a $3K_p$-saturated graph of minimum size amongst all such graphs of order $|G| = n \geq 3p(p+1) - p^2 + 2p - 6$. Many of the structural observations about G made in the proof of Theorem 2.1 still hold. In particular, there must be a set S of $p-2$ vertices in G each having degree $n-1$. Additionally, G has a subgraph H which is disjoint from S and isomorphic to $2K_p$. Let H_1 and H_2 be the components of H and note that since G is $3K_p$-saturated of minimum size, there are exactly $2p$ edges in G that lie in $\bar{H}$ and are not incident to any vertex in S.

As in the proof of Theorem 2.1 we may also assume that each vertex h in H has a neighbor u such that u is not in S and hu is not an edge of H. Let R again denote those vertices in $V(\bar{H}) \cup S$ that have a neighbor in H. We first wish to show that $|R| \geq 2$. Assume that $|R| \leq 1$ and that there are nonadjacent vertices h_1 and h_2 in H_1 and H_2, respectively. Then $G + h_1h_2$ must contain $3K_p$, but the only vertices of degree at least $p-1$ in $G + h_1h_2$ lie in H, S and possibly R. This accounts for at most $|S| + |H| + |R| \leq p - 2 + 2p + 1 = 3p - 1$ vertices of degree at least $p-1$, implying that $3K_p$ cannot be a subgraph of $G + h_1h_2$. Thus, if $|R| \leq 1$ each vertex h_1 and h_2 in H_1 and H_2 respectively, must be adjacent. This implies that there are at least p^2 edges in G that lie in $\bar{H}$ and are not incident to any vertex in S. Since $p^2 > 2p$ for $p \geq 3$, this is a contradiction.

Next we note that each vertex in R must be adjacent to at least p vertices in H. Assume that there is some r in R that is adjacent to strictly less than p vertices in H. Let x be any neighbor of r in H and let v be a vertex of degree $p-2$ in G. Then $G + xv$ contains a subgraph F isomorphic to $3K_p$ in which $\langle S, x, v\rangle$ is one of the copies of K_p. The fact that there are exactly $2p$ edges in G not induced by R, or H, nor incident with S, it follows that r cannot lie in F. This implies that $\langle S, r, x\rangle$ is a copy of K_p in G that is disjoint from $F \setminus \langle S, x, v\rangle$ so that G must contain $3K_p$, a contradiction.

Since $|R| > 1$ and each vertex in R is adjacent to at least p vertices in H, we must have that $R = \{r_1, r_2\}$. Let h be some neighbor of r_1 in H, specifically assume that h is in H_1. Let v be a vertex of degree $p-2$ in G and add the edge hv to G. Then $G + hv$ contains some subgraph F isomorphic to $3K_p$, and $\langle S, h, v\rangle$ is one of the copies of K_p in F. If r_1 does not lie in F, then we could simply replace $\langle S, h, v\rangle$ in F with $\langle S, h, r_1\rangle$, implying that there is a copy of $3K_p$ in G. Thus r_1 must be in F and $N_F(r_1)$, the neighborhood of r_1 in F, must be a clique of order $p-1$. Furthermore, this clique must be disjoint from S since $\langle S, h, v\rangle$ is in F and hence must lie entirely in one component of H. If $N_F(r_1)$ was contained in H_2, then recall that r_1 is adjacent to exactly p vertices in H and repeat this argument by adding the edge h_2v to G, where h_2 is any vertex in $N_F(r_1) \cap H_2$. Then r_1

would have to be adjacent to a clique of order $p-1$ that included h, but excluded h_2 which is impossible because this would imply that r_1 would be adjacent to more than p vertices in H.

Hence we may assume that $N(r_1)$ and $N(r_2)$ both induce components of H. If these components are distinct then G is isomorphic to $G(n,3,p)$, so assume without loss of generality that $N(r_1)=N(r_2)=H_2\cup S$. In this case, choose any vertex h_1 in H_1 and any vertex v of degree $p-2$ in G, and add the edge h_1v to G. Then $\langle S,h_1,v\rangle$ is a K_p in some subgraph F of $G+h_1v$ isomorphic to $3K_p$. The assumption that $N(r_1)=N(r_2)=H_2\cup S$ in G along with the fact that $\langle S,h_1,v\rangle$ is a K_p in F implies that no vertex $h\neq h_1$ lies in F. This implies that we can replace $\langle S,h_1,v\rangle$ in F with $\langle S,h_1,h\rangle$ demonstrating that $3K_p$ is a subgraph of G, a contradiction. Thus it must be that, without loss of generality, $N(r_1)=H_1$ and $N(r_2)=H_2$, so G is isomorphic to $G(n,p,3)$. □

2.1. *Generalized friendship graphs*

Let F_k be the graph comprised of k triangles intersecting in a common point, often called the *friendship graph*. Extending this notion, let $F_{t,p,\ell}$ denote the graph comprised of t copies of K_p intersecting in a common K_ℓ. The graph $F_{t,p,\ell}$ generalizes the notion of a friendship graph. Both of these graphs have been of interest in the extremal literature. The extremal function $ex(F_k,n)$ was determined in [4] and was subsequently extended in [2] to determine $ex(F_{t,p,\ell})$ when $\ell=1$.

We will use techniques nearly identical to those in the proof of Theorem 2.1 to determine $sat(F_{t,p,\ell},n)$. We begin by constructing a graph $FG(t,p,\ell)$ that is $F_{t,p,\ell}$-saturated. For $p\geq 3$, $t\geq 2$ and $p-2\geq\ell\geq 1$, let $FG(t,p,\ell)$ denote the graph formed by taking the join of $G_1=K_{p-2}$ and $(t-1)K_{p-\ell+1}\cup\overline{K}_{n-(p-2)-(t-1)(p-\ell+1)}$. We wish to verify that $FG(t,p,\ell)$ is $F_{t,p,\ell}$-saturated.

If $FG(t,p,\ell)$ contained a copy of $F_{t,p,\ell}$, then the common K_ℓ would have to lie in G_1. However, there is no subgraph of $FG(t,p,\ell)$ isomorphic to $tK_{p-\ell}$ that is disjoint from any ℓ-element subset of $V(G_1)$. If u and v are nonadjacent vertices in $FG(t,p,\ell)$, then in $FG(t,p,\ell)+uv$ there is a copy of $F_{t,p,\ell}$ constructed from G_1, u, v and any $(t-1)$ copies of $K_{p-\ell}$ that are disjoint from G_1, u and v.

Theorem 2.4. *Let $p\geq 3$, $t\geq 2$ and $p-2\geq\ell\geq 1$ be integers. Then, for sufficiently large n,*

$$sat(F_{t,p,\ell},n)=|E(FG(t,p,\ell))|=(p-2)(n-p+2)+\binom{p-2}{2}+(t-1)\binom{p-\ell+1}{2}.$$

As mentioned above, the proof of this theorem will closely mirror that of Theorem 2.1. As such, we will give only a sketch of the proof and leave the details to the reader.

Proof (*Sketch*). Let G be an $F_{t,p,\ell}$-saturated graph, and assume that $|E(G)|<|E(FG(t,p,\ell))|$. Assume that u and v are nonadjacent vertices in G. Then $G+uv$ has a subgraph F isomorphic to $F_{t,p,\ell}$ that contains the edge uv. This implies that u and v each must have degree at least $\delta(F_{t,p,\ell})=p-1$ in $G+uv$ and hence that $\delta(G)\geq p-2$. By an argument similar to Claim 2.2, for n sufficiently large we may assume $\delta(G)=p-2$. Let v be a vertex of degree $p-2$ in G. For any other vertex w in $G\setminus N[v]$, $G+vw$ contains a subgraph $F\cong F_{t,p,\ell}$ such that vw lies in some K_p. Then w and v each have a copy of K_{p-2} in their neighborhoods, and since v has degree $p-2$ in G, we know that $\langle N(v)\rangle\cong K_{p-2}$. Let $S=N(v)$.

The preceding argument holds for all choices of w, and as such, each vertex in S must be adjacent to every vertex in $V(G)\setminus S$. Additionally, since $\langle S,v,w\rangle$ must be the clique containing wv in $G+wv$, we may assume that the common K_ℓ in the subgraph of $G+uw$ isomorphic to $F_{t,p,\ell}$ lies in S. This implies that in $G\setminus S$ there are $(t-1)$ disjoint copies of $K_{p-\ell}$, denoted by $H_1,\ldots,H_{t-1}$.

Let $H=\cup_{1\leq i\leq t-1}H_i$. As in the proof of Theorem 2.1, we wish to show that there are at least $(t-1)(p-\ell)$ edges in G that are neither in H nor adjacent to a vertex in S. This would imply that G has at least $|E(FG(t,p,\ell))|$ edges. It is not difficult to show that each vertex x in H has a neighbor v_x such that v_x is not in S and xv_x is not in $E(H)$. If, for each vertex x in H, there is some choice for v_x that lies in $\bar{H}$, we are done. Hence we will consider the subgraph $H_1=\langle V(H)\rangle-E(H)$. Using arguments similar to those above, it is not difficult to show that each component C of H_1 either contains a cycle or is a tree with a vertex v that is adjacent to some vertex in $V(G)\setminus(S\cup H)$. As above, this completes the proof. □

3. Determining $sat(K_p\cup K_q,n)$

In this section, we will consider the problem of determining the saturation number of a union of cliques that are not all of the same order. Specifically, for $3\leq p\leq q$ we will determine $sat(K_p\cup K_q,n)$. Let $H(n,p,q)$ denote the graph formed by taking the join of K_{p-2} and $K_{q+1}\cup\overline{K}_{n-p-q+1}$ and note that $H(n,p,q)$ is structurally similar to each of the extremal graphs in the preceding section. This graph has only $p+q-1$ vertices of degree at least $p-1$, and as such cannot contain a copy of $K_p\cup K_q$. It is not difficult to see that for any nonadjacent vertices u and v in $H(n,p,q)$, the addition of the edge uv creates a copy of $K_p\cup K_q$ in $H(n,p,q)+uv$. The following is the main result of this section.

Theorem 3.1. *Let $2 \leq p \leq q$ and $n \geq q(q+1)+3(p-2)$ be integers. Then*

$$sat(K_p \cup K_q, n) = |E(H(n,p,q))| = (p-2)(n-p+2) + \binom{p-2}{2} + \binom{q+1}{2}.$$

Furthermore, $H(n,p,q)$ is the unique $(K_p \cup K_q)$-saturated graph of minimum size when $n \geq q(q+1)+3(p-2)$.

Proof. Given $q \geq p \geq 2$, let G be a $K_p \cup K_q$-saturated graph of order $n \geq q(q+1)+3(p-2)$. We will assume that $|E(G)| \leq |E(H(n,p,q))|$ and work to show that equality must hold. Choose any nonadjacent u and v in G. Since G is $K_p \cup K_q$-saturated, we know that in $G+uv$ there is a clique of order at least p that contains uv. This implies that u and v have degree at least $p-1$ in $G+uv$, and hence that $\delta(G) \geq p-2$. In fact, via an argument that is nearly identical to Claim 2.2 of Theorem 2.1, our choice of $n \geq q(q+1)+3(p-2)$ allows us to assume that $\delta(G) = p-2$.

Let v be a vertex of degree $p-2$ in G and let w be any other vertex in G that is not adjacent to v. Then $G+vw$ contains a subgraph F that is isomorphic to $K_p \cup K_q$ such that vw is in F. Since the degree of v is $p-1$ in $G+vw$ the edge vw must lie in a clique of order p. Therefore, if $p \geq 3$, G must contain a clique S of order $p-2$ with every vertex of S adjacent to both v and w. In particular, $N(v) = S$ and since this must hold for all choices of w it follows that each vertex in S must therefore be adjacent to each vertex in $G \setminus E(S)$. If $p=2$, v was an isolated vertex and w may or may not have been isolated. To complete the proof of this theorem, it will suffice to show that there are at least $\binom{q+1}{2}$ edges in $G \setminus E(S)$.

Also note that since $G+vw$ contains $K_p \cup K_q$ and vw must be in some copy of K_p, we can also assume that G has a subgraph H that is isomorphic to K_q such that H contains no vertices from S. Choose some vertex x in H and again let v have degree $p-2$ in G. Then $G+vx$ contains a copy of $K_p \cup K_q$ in which $\langle S, v, x\rangle$ must be the K_p and some subgraph H_x of G, distinct from H (but possibly intersecting), must be the K_q. For $p \geq 3$, if $|V(H) \cap V(H_x)| = t < q-1$, then $G \setminus E(S)$ has at least

$$\binom{q}{2} + \binom{q-t}{2} + t(q-t) \geq \binom{q+1}{2}$$

edges, implying that $|E(G)| \geq |E(H(n,p,q))|$. If $q=p=2$, then $t \neq 0$ or else $2K_2$ already exists. But then again,

$$\binom{2}{2} + \binom{1}{2} + 1(2-1) \geq \binom{2+1}{2},$$

again implying $|E(G)| \geq |E(H(n,2,2))|$.

Therefore, we may assume that for each x in H there is some vertex v_x that lies in neither S nor H such that v_x and $q-1$ vertices of H form a K_q in G. If for distinct x_1 and x_2 in $V(H)$, $v_{x_1} \neq v_{x_2}$ then there are at least $\binom{q}{2} + 2(q-1) > \binom{q+1}{2}$ edges in $G \setminus E(S)$, contradicting our assumption that G has at most as many edges as $H(n,p,q)$. Hence, there is some vertex y such that $v_x = y$ for each x in $V(H)$. This implies that $H \cup y$ induces a K_{q+1} contained in $G \setminus E(S)$, thus, G has at least as many edges as $H(n,p,q)$, which implies that the K_{q+1} induced by $V(H) \cup y$ must be the entirety of edges of $G \setminus E(S)$. Thus, G must be isomorphic to $H(n,p,q)$. □

For integers $3 \leq p_1 \leq p_2 \leq \cdots \leq p_t$, it is interesting to consider the problem of determining $sat(K_{p_1} \cup \cdots \cup K_{p_t}, n)$. In fact, one may consider adapting the structure of the extremal graphs used thus far in this paper in the following way. Let $\Sigma p_i = m$ and consider the graph G formed by taking the join of K_{p_1-2} and $K_{p_2+1} \cup \cdots \cup K_{p_t+1} \cup \overline{K}_{n-m-t+3}$. Clearly, if u and v are nonadjacent vertices in G, then $G+uv$ contains a copy of $K_{p_1} \cup \cdots \cup K_{p_t}$. However, for appropriate choices of the p_i, G may also contain a copy of this subgraph. Indeed, for any integers $3 \leq \ell \leq p$, choose $p_1 = \ell$, $p_2 = p$ and $p_3 = p+1$. In this case, the graph G would be $K_{\ell-2}$ joined to $K_{p+1} \cup K_{p+2} \cup \overline{K}_{n-\ell-2p+1}$. The copies of $K_{\ell-2}$ and K_{p+2} form a $K_{\ell+p}$ which contains $K_\ell \cup K_p$. This, together with the K_{p+1} already in G comprises a subgraph of G isomorphic to $K_\ell \cup K_p \cup K_{p+1}$. This precludes G from being $(K_\ell \cup K_p \cup K_{p+1})$-saturated.

4. Conclusion

With an eye towards further extending the results from [10], it would be of interest to continue investigating the saturation number of a union of cliques of different sizes, particularly in light of the observation made above about the case $K_\ell \cup K_p \cup K_{p+1}$. For the sake of completeness, the issue of the uniqueness (or non-uniqueness) of $G(n,t,p)$ for $t > 3$ and n large enough would also be of interest.

A non-negative integer sequence π is said to be *graphic* if it is the degree sequence of some graph G and we then say that G is a *realization* of π. For an arbitrary graph H, define $\sigma(H,n)$ (see for example [8]) to be the minimum even integer m such that any n-term graphic sequence π with sum at least m has some realization that contains H as a subgraph. In [8], it is conjectured that $2sat(H,n) < \sigma(H,n)$. Comparing Theorems 2.1 and 3.1 to the results in [6] and Theorem 2.4 to the results in [1,7] affirms this conjecture for tK_p, $K_p \cup K_q$ and $F_{t,p,\ell}$.

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