Distance between two k-sets and Path-Systems Extendibility

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Abstract

Given a simple graph G on n vertices, let $\sigma_2(G)$ be the minimum sum of the degrees of any two non adjacent vertices. The graph G is said to be connected if any two distinct vertices may be joined by a path. It is easy to see that if $\sigma_2(G) \geq n-1$ then G is not only connected, but we can choose the connecting path to be of size at most two. Ore [4] proved that if $\sigma_2(G) \geq n+1$ we may always choose this path to cover all the vertices of G. In this paper we extend these results to systems of vertex disjoint paths connecting two vertex k-sets of G.

1 Preliminaries

In this paper, $G = (V, E)$ will denote a simple loopless graph with $|G| = |V(G)| = n$. The order and the size of a graph are respectfully the number of vertices and the number of edges in this graph. Definitions and notation that are not found here may be found in [2].

Let $u, v \in V(G)$. If $u \neq v$, a $[u, v]$-path is a subgraph P of G constituted of a sequence of distinct vertices

$$u = z_1, z_2, \ldots, z_{p-1}, z_p = v$$

along with edges between z_i and z_{i+1} (for all $1 \leq i \leq p-1$). We will consider a vertex u to be a $[u, u]$-path of order one; in this case we say the path is singular. The size of a path is the number of it's edges.

A graph G is said to be *connected* if for any two distinct vertices $u, v \in V(G)$, there is a $[u, v]$-path in G. In extremal graph theory, one is interested in determining how large (or small) a given graph parameter has to be to imply a given graph property. Consider the following parameter: Given a non-complete graph G, let

$$\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}.$$

We have the following simple Fact:

Fact 1 *If G is a graph of order n with $\sigma_2(G) \geq n - 1$, then G is connected.*

Indeed, take any two non-adjacent vertices u and v of G; since $d(u) + d(v) \geq n - 1 > |G - u - v|$, there must be a vertex $w \in N(u) \cap N(v)$, hence the connectivity of G.

The distance $\mathrm{dist}(u, v)$ between two vertices u and v of a graph G is defined to be the minimum size of a $[u, v]$-path. The diameter $\mathrm{diam}(G)$ of G is the maximum possible distance between two vertices of G. If G is disconnected (not connected) we let $\mathrm{diam}(G) = \infty$. The argument of the previous paragraph shows us the following:

Fact 2 *If $\sigma_2(G) \geq n - 1$, then $diam(G) \leq 2$.*

Note that the condition $\sigma_2(G) \geq n - 2$ does not even ensure connectivity, as exemplified by a graph having two complete components. This shows that the lower bound $\sigma_2(G) \geq n - 1$ is best possible.

The graph G is said to be Hamilton-connected if for any pair (u, v) of vertices of G, there exists a Hamilton path between u and v (that is, a $[u, v]$-path covering all the vertices of G). Ore [4] proved:

Theorem 1 *If $\sigma_2(G) \geq n + 1$ then G is Hamilton-connected.*

The lower bound on $\sigma_2(G)$ is the best possible, as exemplified by a balanced complete bipartite graph (a graph composed of two sets X and Y of $\frac{n}{2}$ vertices each, no edges inside X or Y, but all edges between X and Y).

In this paper we wish to generalize these concepts of extremal size paths between two vertices found in Fact 2 and Theorem 1 to the idea of extremal size path systems between two k-sets of vertices.

A graph G is said to be k-connected if one must remove at least k vertices to either disconnect the graph, or leave only one vertex. In other words, G

is k-connected if for any set $S \in V(G)$, if $G - S$ has only one vertex, or more than one component, then $|S| \geq k$. The connectivity $\kappa(G)$ of a graph G is the maximum k such that G is k-connected.

Definition 1 *A path-system $\mathcal{P}$ of G is a family of vertex-disjoint paths $P_1, \ldots, P_k$ of G. Let $\mathcal{S}_k(G)$ be the family of all path-systems of G of order k.*

As a consequence of Menger's famous theorem of 1927 [3], we have the following

Theorem 2 *A graph G is k-connected if and only if for any pair (A, B) of disjoint k-sets of $V(G)$, there exists a path-system $\mathcal{P} = \{P_1, \ldots, P_k\}$ such that for all $i \in [k]$, P_i is a $[a_i, b_i]$-path, $\{a_1, \ldots, a_k\} = A$, and $\{b_1, \ldots, b_k\} = B$.*

In light of the equivalence pointed out by Theorem 2, graph theorists were brought to the following strong connectivity condition:

Definition 2 *A graph G is said to be k-linked if for every $2k$ distinct vertices $a_1, \ldots, a_k, b_1, \ldots, b_k$, G has a path-system $\mathcal{P} = P_1, \ldots, P_k$ such that, for all i, P_i is an $[a_i, b_i]$-path.*

We will generalize the idea of distance by using the concepts of connectivity described in Theorem 2 and Definition 2 since these both extend the idea of a path between two vertices to the idea of a system of vertex-disjoint paths between two disjoint k-sets of vertices. Let us mention right away the discrepancies and problems we will encounter in this generalization.

First, note that each path of the path-systems of Definition 2 has specified end-vertices whereas those of Theorem 2 have only the global requirement of connecting each vertex of A to a vertex of B injectively. Both are natural generalizations since they overlap when $|A| = |B| = 1$ on the concept of a single path joining two specified vertices. We will thus generalize the idea of distance using both these concepts.

Second, where there is only one way of naturally measuring a path to bring forth the idea of distance between the two vertices it joins, namely by the size of this path (the number of edges of this path), it is not clear how we should measure the distance between two k-sets of vertices using a path-system between them. We could think of simply taking the total size of this path-system, but taking the size of the smallest path, or the size of the largest one, provide alternate metrics for distance. Again, these three different choices are identical in the case of one single path.

Finally, note that the path-systems found in Theorem 2 and Definition 2 join k-sets which are disjoint. Yet, we wish to define the distance between any two k-sets, so some attention will have to be brought to the case where the k-sets overlap.

Using Theorem 2, one may easily see that k-connectivity may be defined in terms of path-systems between any pair of vertex k-sets of G:

Corollary 3 *A graph G is k-connected if and only if for any two k-sets A and B of vertices of G, there are $k - |A \cap B|$ disjoint paths in $G - (A \cap B)$ injectively joining every vertex of $A - B$ to a vertex of $B - A$*

We make the following definition in order to adapt Definition 2 to the situation where we want to link two k-sets that overlap.

Definition 3 *A graph G is said to be (k,t)-linked $(0 \leq t \leq k-1)$ if for any set T of t vertices of G, $G - T$ is $(k-t)$-linked.*

Note that this definition unifies the notions of k-connected and k-linked in the sense that $(k, k-1)$-linked is equivalent to k-connected and $(k, 0)$-linked is equivalent to k-linked.

A graph G is said to be k-Hamilton-connected if removing any $k-1$ vertices of G leaves a Hamilton-connected graph. Bondy and Chvátal [1] extended Theorem 1 as follows:

Theorem 4 *If $\sigma(G) \geq n + k$ then G is k-Hamilton-connected.*

Saying that a graph is k-Hamiltonian-connected is equivalent to saying that for any vertex k-sets A and B of G such that $|A \cap B| = k-1$, there is a system of paths as described in Corollary 3, covering all the vertices of $G - (A \cap B)$. We will extend this Theorem further, allowing arbitrary orders for $|A \cap B|$.

2 Distance between k-sets and Extendibility of Path Systems

Let $V_k(G)$ be the family of all k-sets of vertices of a graph G. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two elements of $V_k(G)$, $T = A \cap B$, and $|T| = t$.

An (A,B)-system $\mathcal{P}$ is the union of T and a set of $k-t$ vertex-disjoint paths $P_1,\ldots,P_{k-t}$ joining the vertices of $A-T$ to the vertices of $B-T$ (that is, having one of it's end-vertices in $V(A-T)$ and the other in $V(B-T)$). Let $\mathcal{S}(A,B)$ denote the family of all (A,B)-systems of G. Note that by Corollary 3, G is k-connected if, and only if, for all $(A,B)\in V_k(G)^2$, G has an (A,B)-system.

Let $\Pi(A,B)$ be the family of bijective maps

$$\pi : A \longrightarrow B$$

such that for all $a\in A$ and $b\in B$,

$$\text{if } \ a=b \text{ , then } \ \pi(a)=b. \tag{1}$$

Let $\pi\in\Pi(A,B)$. An (A,B,π)-linkage is an (A,B)-system $\mathcal{P}=\{P_1,\ldots,P_k\}$ where for all $i\in[k]$, P_i is an $[a_i,\pi(a_i)]$-path. Note that the condition (1) shows that $\mathcal{P}$ is the union of $|A\cap B|$ and an $(A-T,B-T)$-system whose end-vertices are imposed by π.

Let $\mathcal{L}(A,B,\pi)$ denote the family of all (A,B,π)-linkages of G. We see that

$$\mathcal{S}(A,B)=\cup_{\pi\in\Pi(A,B)}\mathcal{L}(A,B,\pi).$$

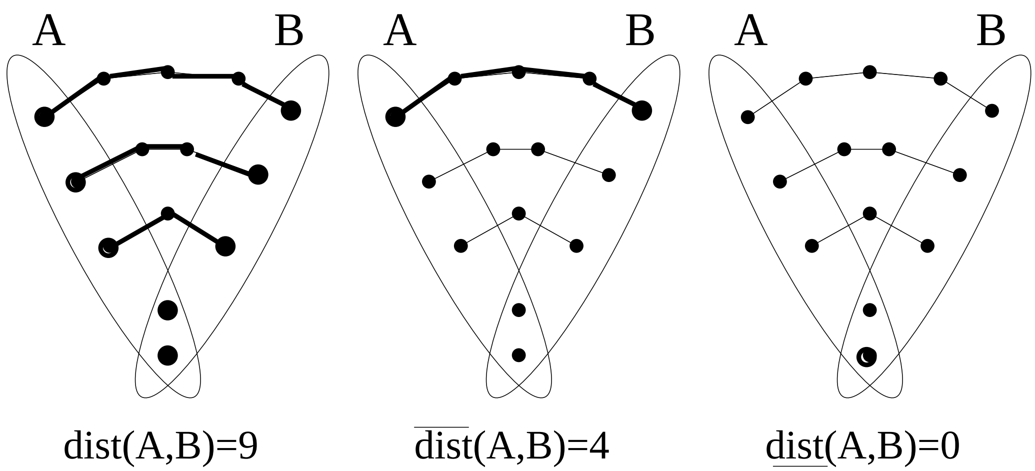
Note that G is (k,t)-linked if, and only if, for all $(A,B)\in V_k(G)^2$ such that $|T|=|A\cap B|=t$, and all $\pi\in\Pi(A,B)$, $\mathcal{L}(A,B,\pi)\neq\emptyset$.

For a given $\pi\in\Pi(A,B)$ we make the following definitions:

$$\begin{aligned}
\mathrm{dist}_\pi(A,B) &= \min_{\mathcal{P}\in\mathcal{L}(A,B,\pi)} |E(\mathcal{P})|,\\
\overline{\mathrm{dist}}_\pi(A,B) &= \min_{\mathcal{P}\in\mathcal{L}(A,B,\pi)} \max_{P\in\mathcal{P}} |E(P)|, \text{ and}\\
\underline{\mathrm{dist}}_\pi(A,B) &= \min_{\mathcal{P}\in\mathcal{L}(A,B,\pi)} \min_{P\in\mathcal{P}} |E(P)|.
\end{aligned}$$

If $\mathcal{L}(A,B,\pi)=\emptyset$, we let $\mathrm{dist}_\pi(A,B)=\overline{\mathrm{dist}}_\pi(A,B)=\underline{\mathrm{dist}}_\pi(A,B)=\infty$.

Figure 1: Different types of distances



Using this, we define the following distance and diameter measures:

$$\begin{aligned}
\mathrm{dist}(A,B) &= \min_{\pi\in\Pi(A,B)} \mathrm{dist}_\pi(A,B),\\
\overline{\mathrm{dist}}(A,B) &= \min_{\pi\in\Pi(A,B)} \overline{\mathrm{dist}}_\pi(A,B),\\
\underline{\mathrm{dist}}(A,B) &= \min_{\pi\in\Pi(A,B)} \underline{\mathrm{dist}}_\pi(A,B),\\
\mathrm{diam}_k(G) &= \max_{(A,B)\in V_k(G)^2} \mathrm{dist}(A,B),\\
\overline{\mathrm{diam}}_k(G) &= \max_{(A,B)\in V_k(G)^2} \overline{\mathrm{dist}}(A,B), \text{ and}\\
\underline{\mathrm{diam}}_k(G) &= \max_{(A,B)\in V_k(G)^2} \underline{\mathrm{dist}}(A,B).
\end{aligned}$$

We see that $\mathcal{S}(A,B)$ is empty if, and only if,

$$\mathrm{dist}(A,B) = \overline{\mathrm{dist}}(A,B) = \underline{\mathrm{dist}}(A,B) = \infty,$$

so saying that G is k-connected is equivalent to saying that any of the k-diameters are finite.

The linked-distances between two k-sets and the corresponding (k,t)-

linked diameters are defined as follows:

$$
\begin{aligned}
\mathrm{ldist}(A,B) &= \max_{\pi\in\Pi(A,B)} \mathrm{dist}_\pi(A,B),\\
\overline{\mathrm{ldist}}(A,B) &= \max_{\pi\in\Pi(A,B)} \overline{\mathrm{dist}}_\pi(A,B),\\
\underline{\mathrm{ldist}}(A,B) &= \max_{\pi\in\Pi(A,B)} \underline{\mathrm{dist}}_\pi(A,B),\\
\mathrm{ldiam}_{k,t}(G) &= \max_{\substack{(A,B)\in V_k(G)^2\\ |A\cap B|=t}} \mathrm{ldist}(A,B),\\
\overline{\mathrm{ldiam}}_{k,t}(G) &= \max_{\substack{(A,B)\in V_k(G)^2\\ |A\cap B|=t}} \overline{\mathrm{ldist}}(A,B), \text{ and}\\
\underline{\mathrm{ldiam}}_{k,t}(G) &= \max_{\substack{(A,B)\in V_k(G)^2\\ |A\cap B|=t}} \underline{\mathrm{ldist}}(A,B).
\end{aligned}
$$

We see that $\mathcal{L}(A,B,\pi)=\emptyset$ for some $\pi\in\Pi(A,B)$ if, and only if, $\mathrm{ldist}(A,B)=\overline{\mathrm{ldist}}(A,B)=\underline{\mathrm{ldist}}(A,B)=\infty$, so saying that any one of these (k,t)-diameters are finite is equivalent to saying that G is (k,t)-linked.

Finally we make the following definitions which extend in two different directions the notion of k-Hamilton-connectedness due to Bondy and Chvátal.

We say that G is Hamilton k-connected if for any $(A,B)\in V_k(G)^2$, there is a path-system $\mathcal{P}$ in $\mathcal{S}(A,B)$ such that $\mathcal{P}$ covers all the vertices of G. We say that G is Hamilton (k,t)-linked if for any $(A,B)\in V_k(G)^2$ with $|A\cap B|=t$ and any $\pi\in\Pi(A,B)$, there is a $\mathcal{P}$ in $\mathcal{L}(A,B,\pi)$ such that $\mathcal{P}$ covers all the vertices of G. Note that k-Hamilton-connected is equivalent to Hamilton-$(k,k-1)$-linked (which is also equivalent to Hamilton k-connected restricted to k-sets that intersect on a $(k-1)$-set).

3 Results

One may easily see that

$$\sigma_2(G)\geq n+k-2 \ \text{ implies } \ \kappa(G)\geq k. \tag{2}$$

Indeed, this is the contrapositive of

$$\kappa(G)\leq k-1 \ \text{ implies } \ \sigma_2(G)\leq n+k-3. \tag{3}$$

This can be seen to be true since if C is a cut set of order $k-1$ and A and B were two components of $G-C$, then taking two vertices $x \in A$ and $y \in B$, we see that $xy \in E(G)$ yet

$$d(x) + d(y) \leq (|A| - 1) + |C| + (|B| - 1) + |C| \leq n + k - 3.$$

By considering the case where A and B are the only components of G, and both $(A \cup C)$ and $(B \cup C)$ induce complete graphs, we see that $\sigma_2(G) = n + k - 3$, yet $\kappa(G) = k - 1$, so the bound on $\sigma_2(G)$ is the best possible.

Proposition 1 *Let G be a graph on n vertices and $k \in [n]$ be such that $\sigma_2(G) \geq n + k - 2$. Then for any k-sets A and B of vertices of G, we have*

$$dist(A, B) \leq 2(k - |A \cap B|).$$

This implies that $\operatorname{diam}(G) \leq 2k$, and we will show that if $n \geq 3k$, there are graphs G with $\sigma_2(G) \geq n + k - 2$ and $\operatorname{diam}(G) = 2k$. This diameter is essentially the lowest possible in the sense that, in order to reduce it further, one must have a graph that is nearly complete. But the actual lowest possible k-diameter of a graph is k, so for completeness, we include the bounds on $\sigma_2(G)$ implying lower diameters than $2k$. The following Theorem shows the effect of $\sigma_2(G)$ on the k-diameters of G.

Theorem 5 *Let G be a graph of order $n \geq 2k$ and $l \in [k]$. The following table relates the value of $\sigma_2(G)$ to the lowest upper bound on the k-diameter of G.*

	$diam_{\mathbf{k}}(\mathbf{G}) \leq$	$\overline{diam}_{\mathbf{k}}(\mathbf{G}) \leq$	$\underline{diam}_{\mathbf{k}}(\mathbf{G}) \leq$
$\sigma_{\mathbf{2}} \leq n + k - 3$	∞	∞	∞
$n + k - 2 \leq \sigma_{\mathbf{2}} \leq 2n - 2k - 2$	$2k$	*2*	*2*
$\sigma_{\mathbf{2}} = 2n - 2k - 2 + l$	$2k - l$	*2*	*1*
$2n - k - 2 \leq \sigma_{\mathbf{2}}$	k	*1*	*1*

We see that the minimum bound on $\sigma_2(G)$ ensuring $\operatorname{diam}_k(G) < \infty$ (k-connectivity) is $n+k-2$, and when this happens, we have automatically the small diameter of $2k$. Then, until $2n - 2k - 1$ we cannot lower the diameter further. At $2n-k-2$ we attain the smallest possible diameter $\operatorname{diam}_k(G) = k$ (equivalently, $\underline{\operatorname{diam}}_k = \overline{\operatorname{diam}}_k = 1$).

Note that $\sigma_2(G)$ cannot be larger than $2n-4$, and that those graphs G for which $\sigma_2(G)=2n-4$ have the property that a vertex cannot have more than one non-adjacency. Hence these graphs are isomorphic to $K_n - M_m$ where M_m is a set of m independent edges of K_n for some $m \in [\lfloor n/2 \rfloor]$. The following Theorem shows the effect of $\sigma_2(G)$ on the linked-diameters of G. Since even $\sigma_2(G)=2n-4$ is not sufficient to force $\mathrm{ldiam}_k(G) < 2k$, we include the linked-diameters of the $K_n - M_m$ graphs.

Theorem 6 *Let G be a graph of order $n \geq 4k$ and $0 \leq t \leq k-l \leq k-1$. Let M_{k-t-l} be a set $k-t-l$ independent edges of a complete graph K_n. The following table relates the value of $\sigma_2(G)$ to the lowest upper bound on the linked-diameters of G.*

	$ldiam_{\mathbf{k,t}}(\mathbf{G}) \leq$	$\overline{ldiam}_{\mathbf{k,t}}(\mathbf{G}) \leq$	$\underline{ldiam}_{\mathbf{k,t}}(\mathbf{G}) \leq$
$\sigma_{\mathbf{2}} \leq n+2k-t-4$	∞	∞	∞
$\sigma_{\mathbf{2}} = n+2k-t+l-4$	$3(k-t)-l$	*3*	*2*
$n+3k-t-4 \leq \sigma_{\mathbf{2}}$	$2(k-t)$	*2*	*2*
$G = K_n - M_{k-t-l}$	$2(k-t)-l$	*2*	*1*
$G = K_n$	$k-t$	*1*	*1*

Note that, for $t=k-1$ and $l=1$, Theorem 6 implies (2).

In order to extend Theorem 4 we prove the following Theorem:

Theorem 7 *If $n \geq 4k$ and $\sigma_2(G) \geq n+k$ then any (A,B)-system $\mathcal{P}$ can be extended to an (A,B)-system $\mathcal{P}'$ covering all the vertices of G such that the paths of $\mathcal{P}$ and $\mathcal{P}'$ have the same end-vertices. On the other hand, if $\sigma_2(G) \geq n+k-1$ there are sets $A,B \in V_k(G)^2$ for which there is no (A,B)-system covering all the vertices of G.*

The last statement of this Theorem shows that lower bound on σ_2 is not only the best possible to allow path systems to be extended while conserving the end-vertices of each individual path, but is also the best possible if we allow these paths to swap end-vertices. Using this Theorem, we see that Theorems 5 and 6 have the following corollaries:

Corollary 8 *If $n \geq 3k$ and $\sigma_2(G) \geq n+k$ then G is Hamilton-k-connected.*

Corollary 9 *If $n \geq 4k$, $0 \leq t \leq k-3$, and $\sigma_2(G) \geq n+2k-t-3$, then G is Hamilton-(k,t)-linked.*

And the lower bounds on σ_2 are best possible here again.

4 Proofs

Proof of Proposition 1: Suppose G satisfies the conditions of Proposition 1 and take any $(A,B) \in V_k(G)^2$ where $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$. Let $T = A \cap B$, $t = |T|$ and k_1 be the maximum number of independent edges in $E(A-T, B-T)$. Without loss of generality we may assume that $T = \{a_1, \ldots, a_t\} = \{b_1, \ldots, b_t\}$ and that the k_1 independent edges are $a_{t+1}b_{t+1}, \ldots, a_{t+k_1}b_{t+k_1}$. Let $A_1 = \{a_{t+1}, \ldots, a_{t+k_1}\}$, $B_1 = \{b_{t+1}, \ldots, b_{t+k_1}\}$, $A_2 = A - T - A_1$, and $B_2 = B - T - B_1$. Let $k_2 = |A_2| = |B_2| = k - k_1 - t$.

If $k_2 = 0$, we are done, so assume $k_2 \neq 0$. Now for all $t + k_1 + 1 \leq i \leq k$,

$$d(a_i, A-T) + d(b_i, B-T) \leq |A| - 1 + |B| - 1 = 2(k-1), \text{and} \tag{4}$$

$$d(a_i, B-T) + d(b_i, A-T) \leq k_1, \tag{5}$$

since the maximality of k_1 shows that $E(A_2, B_2) = \emptyset$ and if $d(a_i, B_1) + d(b_i, A_1) > k_1$, there was a j with $t+1 \leq j \leq k_1+t$ such that $a_ib_j, b_ia_j \in E(G)$, thus replacing a_jb_j with these two edges, we would contradict the maximality of k_1.

Since $a_ib_i \notin E(G)$, we have $d(a_i) + d(b_i) \geq n + k - 2$, so (4) and (5) show that

$$\begin{aligned} d(a_i, G-A-B) + d(b_i, G-A-B) &\geq n + k - 2 - 2k + 2 - k_1 \\ &= (n - 2k + t) + (k - t - k_1) \\ &= |G - A - B| + k_2. \end{aligned} \tag{6}$$

This shows that $|N(a_i, G-A-B) \cap N(b_i, G-A-B)| \geq k_2$, ensuring that there are k_2 distinct vertices $z_1, \ldots, z_{k_2}$ of $G - A - B$ such that z_j is adjacent to both a_i and b_i for $1 \leq j \leq k_2$. Using these z_j vertices, one may easily construct the required (A,B)-system.

The (A,B)-system constructed verifies $\overline{\text{dist}}(A,B) \leq 2$, thus $\text{dist}(A,B) \leq 2(k-t) = 2(k - |A \cap B|)$. $\square_{Proposition\ 1}$

Proof of Theorem 5: We use the same definitions as the above proof.

Since the pair (A,B) constructed above was arbitrary, and Proposition 1 verified $\overline{\text{dist}}(A,B) \leq 2$, $\overline{\text{diam}}(G) \leq 2$ and $\text{diam}_k(G) \leq 2k$.

If $\sigma_2(G) \geq 2n-2k-2+l$, where $l \in [k]$, then for all $t+k_1+1 \leq i \leq k$,

$$d(a_t, B) + d(b_t, A) \geq \\ (2n-2k-2+l) - \big(d(a_i, G-A-B) + d(b_i, G-A-B) + d(a_i, A) + d(b_i, B)\big), \tag{7}$$

thus, since

$$d(a_i, G-A-B) + d(b_i, G-A-B) \leq 2|G-A-B| = 2n-4k,$$

using (4), we get

$$\begin{array}{rcl} d(a_t, B) + d(b_t, A) & \geq & (2n-2k-2+l) - (2n-4k+2(k-1)) \\ & = & l. \end{array}$$

By (5) then, we get $k_1 \leq l$, which shows that $\underline{\mathrm{diam}}(G) = 1$ and $\mathrm{diam}(G) \leq 2k-l$.

We have already seen, in Section 3 that $\sigma_2(G) \leq n+k-2$ is the lowest bound implying k-connectivity, or equivalently a finite diameter. To see that $\sigma_2(G) = 2n-2k-2+l$ is the smallest value of $\sigma_2(G)$ implying $\mathrm{diam}_k(G) \leq 2k-l$, consider the complete graph K_n, and two disjoint k-sets A and B and a subset $B_{k-l+1} \subset B$ of order $k-l+1$. Then the graph $G = K_n - E_{K_n}(A, B_{k-l+1})$ verifies $\sigma_2(G) = 2n-2k-3+l$, yet $\mathrm{dist}(A,B) = 2k-l+1$. By letting $l=1$ we also see that $\sigma_2(G) = 2n-2k-2$ is not enough to yield $\underline{\mathrm{diam}}_k(G) = 1$ and $\mathrm{diam}_k(G) < 2k$. $\Box_{Theorem\ 5}$

Proof of Theorem 6: First we take care of the case $t=0$ and $l=1$:

Claim 1 *If G is a graph on $n \geq 4k$ vertices and $\sigma_2(G) \geq n+2k-3$ then $ldiam_{k,0}(G) \leq 3$.*

Let G be a graph such that

$$\sigma_2(G) \geq n+2k-3 \tag{8}$$

where k is an integer such that $n \geq 4k$. Let A and B be two disjoint k-sets of $V(G)$, $S = A \cup B$, and $\pi \in \Pi(A,B)$. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be such that for every $i \in [k]$, $\pi(a_i) = b_i$. Let $\mathcal{P} = \{P_1, \ldots, P_{k'}\}$ be a family of paths linking k' vertices of A to the corresponding vertices of

B, where all paths have order no more than 4. Without loss of generality, we may assume that for some non-negative integers k_1, k_2, k_3, and k_4 such that $k = k_1 + k_2 + k_3 + k_4$ and $k' = k_1 + k_2 + k_3$, we have $|P_1| = \ldots = |P_{k_1}| = 2$, $|P_{k_1+1}| = \ldots = |P_{k_1+k_2}| = 3$, and $|P_{k_1+k_2+1}| = \ldots = |P_{k_1+k_2+k_3}| = 4$. We choose $\mathcal{P}$ so that

$$k' = |\mathcal{P}| \text{ is maximal,} \tag{9}$$

and under this condition,

$$\Sigma_{i=1}^{k'} |P_i| \text{ is minimal,} \tag{10}$$

and under this condition,

$$\max_{i>k'} \min\{d(a_i, G - S - V(\mathcal{P})), d(b_i, G - S - V(\mathcal{P}))\} \text{ is maximal.} \tag{11}$$

Let $R = V(G) - S$. By (10), for every i ($k_1 + 1 \leq i \leq k$), $a_i b_i \notin E(G)$ so that

$$d(a_i, S), d(b_i, S) \leq |S| - 2 = 2k - 2, \text{ thus} \tag{12}$$
$$d(a_i, S) + d(b_i, S) \leq 4k - 4. \tag{13}$$

Note that if $k' = k$ then we have our result, so assume $k' < k$ (i.e. $k_4 \geq 1$) and let $S_1 = S \cap \cup_{i=1}^{k_1} V(P_i)$, $S_2 = S \cap \cup_{i=k_1+1}^{k_1+k_2} V(P_i)$, $S_3 = S \cap \cup_{i=k_1+k_2+1}^{k_1+k_2+k_3} V(P_i)$, and $S_4 = S - S_1 - S_2 - S_3$. Let $R_2 = R \cap \cup_{i=k_1+1}^{k_1+k_2} V(P_i)$, $R_3 = R \cap \cup_{i=k_1+k_2+1}^{k_1+k_2+k_3} V(P_i)$, and $R_4 = R - R_2 - R_3$.

Let $u = a_j$ and $v = b_j$ where $k' + 1 \leq j \leq k$ is such that

$$\min\{d(u, R_4), d(v, R_4)\} = \max_{i>k'} \min\{d(a_i, R_4), d(b_i, R_4)\}.$$

Let $\alpha = d(u, R_4)$ and $\beta = d(v, R_4)$ and assume, without loss of generality, that $\alpha \leq \beta$. Note that

$$d(\{u, v\}, R_3) \leq 2k_3 \tag{14}$$

since otherwise there would be an $k_1 + k_2 + 1 \leq i \leq k_1 + k_2 + k_3$ such that $d(\{u, v\}, P_i \cap R) \geq 3$, implying that one of the two vertices w of $P_i - a_i - b_i$ is adjacent to both u and v. Yet then the path P_i of order 4 may be replaced with the path uwv of order 3, contradicting the minimality (10).

Case 1: Assume $\alpha \geq 1$. Then let x and y be any vertices of $N(u, R_4)$ and $N(v, R_4)$ respectively.

We prove a few upper bounds on the number of edges between vertices u, v, x and y, and different parts of the graph. First of all,

$$d(\{x, y\}, S_2 \cup R_2) + d(\{u, v\}, R_2) \leq 6k_2. \tag{15}$$

Indeed, if this isn't the case, then for some $k_1 + 1 \leq i \leq k_1 + k_2$, we must have $d(\{x, y\}, P_i) + d(\{u, v\}, P_i \cap R) \geq 7$. Note that

$$|\{x, y\}| \cdot |P_i| + |\{u, v\}| \cdot |P_i \cap R| = 8,$$

so there is at most one missing edge. Let $P_i = a_i w b_i$. If edge uw is missing then

$$\mathcal{P}' = (\mathcal{P} - P_i) \cup vwxu \cup a_i y b_i$$

contradicts the maximality (9). One may verify that every other case of a missing edge leads to a similar situation where one may find two disjoint paths; a $[u, v]$-path of order 3 and an $[a_i, b_i]$-path of order 3, contradicting (9).

Further,

$$d(\{x, y\}, S_3 \cup S_4) \leq 2(k_3 + k_4) \tag{16}$$

as if this were not true, there would be an $k_1 + k_2 + 1 \leq i \leq k$ such that $d(\{x, y\}, \{a_i, b_i\}) \geq 3$, ensuring the existence of the path $a_i x b_i$ (or $a_i y b_i$) of order 3. If $k_1 + k_2 + 1 \leq i \leq k_1 + k_2 + k_3$, this contradicts (10), and if $k_1 + k_2 + k_3 + 1 \leq i \leq k$, this contradicts (9).

Since $|S_1| = 2k_1$ and $|R_3| = 2k_3$ we have

$$d(\{x, y\}, S_1 \cup R_3) \leq 4(k_1 + k_3). \tag{17}$$

Finally, if $d(x, N(v) \cap R_4) \neq 0$ or $d(y, N(u) \cap R_4) \neq 0$, then (9) would be contradicted, so

$$\begin{aligned} d(x, R_4) &\leq |G - x| - |S| - |R_2| - |R_3| - |N(v, R_4)| \\ &= n - 1 - 2k - k_2 - 2k_3 - \beta \end{aligned} \tag{18}$$

and

$$\begin{aligned} d(y, R_4) &\leq |G - y| - |S| - |R_2| - |R_3| - |N(u, R_4)| \\ &= n - 1 - 2k - k_2 - 2k_3 - \alpha. \end{aligned} \tag{19}$$

One may verify that

$$
\begin{array}{ll}
\multicolumn{2}{l}{d(x)+d(y)+d(u)+d(v) \leq} \\
\quad d(\{u,v\},S) & \\
\quad +\, d(\{u,v\},R_3) & +d(\{u,v\},R_4) \\
\quad +\, d(\{x,y\},S_2\cup R_2) & +d(\{u,v\},R_2) \\
\quad +\, d(\{x,y\},S_3\cup S_4) & +d(\{x,y\},S_1\cup R_3) \\
\quad +\, d(x,R_4) & +d(y,R_4)
\end{array}
$$

Using (13), (14), (15), (16), (17), (18) and (19), we see that

$$
\begin{array}{ll}
\multicolumn{2}{l}{d(x)+d(y)+d(u)+d(v) \leq} \\
\quad 4k-4 & \\
\quad +\, 2k_3 & +\alpha+\beta \\
\quad +\, 6k_2 & \\
\quad +\, 2(k_3+k_4) & +\,4(k_1+k_3) \\
\quad +\, n-1-2k-k_2-2k_3-\beta & +n-1-2k-k_2-2k_3-\beta
\end{array}
$$

Simplifying this expression, and using the fact that $k_1+k_2+k_3+k_4=k$, we see that

$$d(x)+d(y)+d(u)+d(v) \leq 2n-6+4k-2k_4.$$

Since $uy, vx \notin E(G)$, our degree sum condition (8) shows on the other hand that

$$d(x)+d(y)+d(u)+d(v) \geq 2n+4k-6.$$

This shows that we must have $k_4=0$, a contradiction.

Case 2: Assume $\alpha=0$. First we show that $\beta \geq 3$. Indeed, $uv \notin E(G)$, so using (14) we get

$$
\begin{array}{rcl}
d(u,R_4)+d(v,R_4) & \geq & n+2k-3-d(\{u,v\},S) \\
 & - & |N(\{u,v\},R_2)|-d(\{u,v\},R_3) \\
 & \geq & n+2k-3-(4k-4)-2k_2-2k_3 \\
 & = & n-2k+1-2(k_2+k_3),
\end{array}
$$

and since $k_4 \geq 1$, $k_2+k_3 \leq k-1$, hence using the fact that $n \geq 4k$ and $d(u,R_4)=0$, we have

$$\beta = d(v,R_4) \geq n-4k+3 \geq 3.$$

Let y be a vertex of $N(v, R_4)$. Note that

$$d(u, R_2) + d(y, S_2) \leq 2k_2 \tag{20}$$

since otherwise, for some $k_1 + 1 \leq i \leq k_1 + k_2$ we would have $d(u, P_i \cap R) + d(y, P_i \in S) \geq 3$, implying that $ya_i, yb_i, uw \in E(G)$ where w is the middle vertex of P_i. But then replacing P_i with the path $a_i y b_i$, we obtain a system of paths satisfying conditions (9) and (10), but contradicting (11) since u is adjacent to w and v is still adjacent to at least 2 vertices of $G - S - V(\mathcal{P})$. Further,

$$d(u, R_3) + d(y, R_3 \cup S_3) \geq 4k_3. \tag{21}$$

Indeed, if this were not the case, for some $k_1 + k_2 + 1 \leq i \leq k_1 + k_2 + k_3$, we would have $\mathrm{d}(u, P_i \cap R) + d(y, P_i) \geq 5$. Since we cannot have both $ya_i \in E(G)$ and $yb_i \in E(G)$ (or (10) would be contradicted), this shows that letting $P_i = a_i w z b_i$, we have $yw, yz, uw, uz \in E(G)$, and without loss of generality, $yb_i \in E(G)$. Replacing P_i by the path $a_i w y b_i$ one may verify that we again contradict (11). Finally,

$$d(y, S_4) \leq k_4 \tag{22}$$

or there would be a $k_1 + k_2 + k_3 + 1 \leq i \leq k$ with $xa_i, xb_i \in E(G)$, hence a path $a_i x b_i$ contradicting (9).

Now

$$\begin{aligned} d(u) + d(y) = d(u, S) \quad &+ \quad d(u, R_2) + d(y, S_2) \ + d(u, R_4) \\ &+ \quad d(u, R_3) + d(y, S_3 \cup R_3) \\ &+ \quad d(y, R_2 \cup R_4) \ + d(y, S_1) + d(y, S_4) \end{aligned}$$

Using (12), (20), (21), (19), we find that

$$\begin{aligned} d(u) + d(v) &\leq (2k - 2) + 0 + 2k_2 + 4k_3 + (n - 1 - 2k - 2k_3) + 2k_1 + k_4 \\ &= n - 3 + 2(k_1 + k_2 + k_3 + k_4) - k_4 \\ &= n + 2k - 3 - k_4 \\ &< n + 2k - 3, \end{aligned}$$

since $k_1 + k_2 + k_3 + k_4 = k$ and $k_4 \geq 1$. Yet since $uy \notin E(G)$ this contradicts (8).

Hence $k_4 = 0$, so G is $(k,0)$-linked and since we required all paths of $\mathcal{P}$ to be of order at most four, we see that, in fact, $\overline{\text{ldiam}}_{k,0}(G) \leq 3$. $\square_{Claim\ 1}$

The following claim takes care of the case $t = 0$ and $1 \leq l \leq k$:

Claim 2 *Let G be a graph of order n, k and l be positive integers such that $n \geq 4k$ and $l \in [k]$. If $\sigma_2(G) \geq n + 2k + l - 4$ then $ldiam_k(G) \leq 3k - l$.*

Proof: Let G be a graph satisfying the conditions of the Claim. Let S, R, A, B, $\mathcal{P}$, k_1, k_2, k_3 and k_4 be defined as in the proof of Claim 1. The said Claim shows that $k_4 = 0$, so that $k = k_1 + k_2 + k_3$. If $k_1 + k_2 \geq l$, then

$$\begin{aligned} |\mathcal{P}| &= 2k_1 + 3k_2 + 4k_3 \\ &= 4(k_1 + k_2 + k_3) - (k_1 + k_2) - k_2 \\ &\leq 4k - l, \end{aligned}$$

which implies $\text{ldiam}_k(G) = |\mathcal{P}| - k \leq 3k - l$, which is to be proven. Hence we assume

$$k_2 + k_3 \leq l - 1. \tag{23}$$

Now for every $k_1 + k_2 + 1 \leq i \leq k$ we have $a_i b_i \notin E(G)$, so

$$\begin{aligned} d(a_i, b_i, \mathcal{P} - S) &\geq \sigma_2(G) - 2(2k - 2) \\ &\geq n - 2k + l \\ &= |G - S| + l, \end{aligned}$$

implying that there are at least l vertices in G which are adjacent to both a_i and b_i. The minimality of $|\mathcal{P}|$ implies that none of these vertices may be in P_i or in $G - \mathcal{P}$ since otherwise a (a_i, b_i)-path of order four could be replaced by a path of order three. Also, by (23), at least one of these vertices must be in $P_j - \{a_j, b_j\}$, where $k_1 + k_2 + 1 \leq j \leq k$ and $j \neq i$.

Let D be a digraph of order k_3 obtained by taking $P_{k_1+k_2+1}, \ldots, P_k$ to correspond to the vertices, and where there is an edge from P_i to P_j $(i \neq j)$ if and only if there is a vertex w in $P_j - \{a_j, b_j\}$ such that $a_i w, b_i w \in E(G)$. One may easily verify that if D had a directed cycle then one could replace every path P_i of order 4 corresponding to the vertices of this directed cycle with an (a_i, b_i)-path of order 3, hence contradicting the minimality of $|\mathcal{P}|$. Yet the previous paragraph implies that every vertex of D has at least one edge coming out of it, and this can be seen to imply the existence of a directed cycle in D (note that we allow this cycle to be of order two).

Indeed, take the last vertex z of a directed path Z of D of maximal order. Since z must be adjacent to a vertex z' of D, and that z' cannot be in $D-Z$, or the maximality of Z would be contradicted, we see that z' must be in Z, creating a directed cycle in D, and hence completing the proof of our Claim. $\square_{Claim\ 2}$

Now suppose $1 \leq t \leq k-l \leq k-1$, $\sigma_2(G) \geq n+2k-t+l-4$ and that $A, B \in V_k(G)$ intersect on t vertices. Let $T = A \cap B$. It is easy to see that

$$\begin{aligned} \sigma_2(G-T) &\geq \sigma_2(G) - 2|T| \\ &= n+2k-t+l-4-2t = (n-t)) + 2(k-t) + l - 4 \\ &= |G-T| + 2(k-t) + l - 4. \end{aligned}$$

Hence, by Claim 2, $\mathrm{ldist}_{k-t,0}(A-T, B-T) \leq 3(k-t)-l$, $\overline{\mathrm{ldist}}_{k-t,0}(A-T, B-T) \leq 3$, and $\underline{\mathrm{ldiam}}_{k-t,0}(A-T, B-T) \leq 2$, so $\mathrm{ldist}_{k-t,0}(A, B) \leq 3(k-t)-l$, and since A and B were arbitrary,

$$\begin{aligned} \mathrm{ldiam}_{k,t}(G) &\leq 3(k-t)-l, \\ \overline{\mathrm{ldiam}}_{k,t}(G) &\leq 3, \text{ and} \\ \underline{\mathrm{ldiam}}_{k,t}(G) &\leq 2. \end{aligned}$$

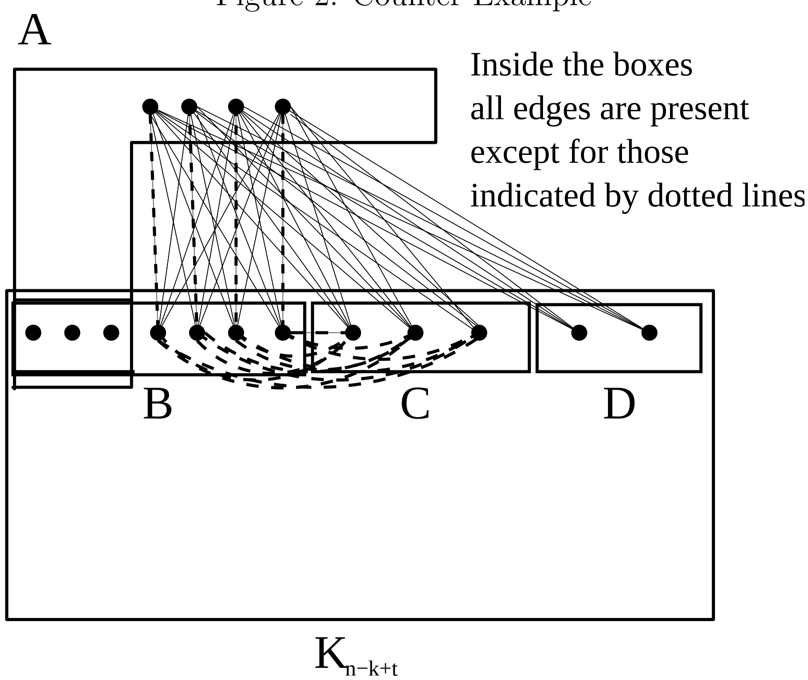
If $l = k-t$, then all the linking paths have order at most three, so $\overline{\mathrm{ldiam}}_{k,t}(G) \leq 2$.

One may easily verify that the graph $K_n - M_{k-t}$, where M_{k-t} is a set of t independent edges of K_n, verifies $\sigma_2(K_n - M_{k-t}) = 2n-4$, yet $\mathrm{ldiam}_{k,t}(K_n - M_{k-t}) = 2(k-t)$ and $\overline{\mathrm{ldiam}}_{k,t}(K_n - M_t) = \underline{\mathrm{ldiam}}_{k,t}(K_n - M_t) = 2$. On the other hand, for any two disjoint t-sets $A', B'V_t(K_n - M_{k-t-l})$ there must be at least l independent edges in $E_{K_n - M_{k-t-l}}(A', B')$. This shows that

$$\begin{aligned} \mathrm{ldiam}_{k,t}(K_n - M_{k-t-l}) &\leq 2(k-t)-l, \\ \overline{\mathrm{ldiam}}_{k,t}(K_n - M_{k-t-l}) &\leq 2, \text{ and} \\ \underline{\mathrm{ldiam}}_{k,t}(K_n - M_{k-t-l}) &= 1. \end{aligned}$$

To see that if $n \geq 4k-3t+l-4$, the lower bound $\sigma_2(G) \geq n+2k-t+l-4$ is the smallest possible ensuring that the (k,t)-linked-diameters are finite (for

Figure 2: Counter Example



$l = 1$) and that $\mathrm{ldiam}_{k,t}(G) \leq 3(k-t)-l$, $\overline{\mathrm{ldiam}}_{k,t}(G) \leq 3$ and $\underline{\mathrm{ldiam}}_{k,t}(G) \leq 2$ consider the following construction. Take the complete graph K_{n-t}, and let B, C and L be disjoint subgraphs of K_{n-k+t} or orders k, $k-t-1$ and $l-1$ respectively. The vertices of B will be labeled $b_1, \ldots, b_k$ and B' will be the set $\{b_1, \ldots, b_{k-t}\}$. Let $A = \{a_1, \ldots, a_k\}$ be a complete graph on k vertices where $A' = \{a_1, \ldots, a_{k-t}\}$ is disjoint from K_{n-k+t} and for all $t+1 \leq i \leq k$, $a_i = b_i$. Consider the graph

$$G = (A' + K_{n-k+t}) \quad -E(B', C) - E(A', K_{n-k+t} - B' - C - L) \\ -\{a_1 b_1, \ldots, a_{k-t} b_{k-t}\}$$

of order n. A sketch of this graph may be found in figure 2.

Let u and v be any two non-adjacent vertices of G. If $u \in V(B')$ and

$v \in V(C)$, then

$$\begin{aligned} d(u)+d(v) &= d(u, G-A'-C)+d(u,A')+d(v,G-A'-B')+d(v,A') \\ &= ((n-k+t-1)-(k-t-1))+(k-t-1) \\ &\quad +((n-k+t-1)-(k-t-1))+k-t \\ &= 2n-2(k-t)-1 \\ &\geq n+2k-t+l-5 \end{aligned}$$

since $n \geq 4k-3t+l-4$. If $u \in V(A')$ and $v \in V(G)$ then first note that whether $v \in V(B')$ or $v \in V(G-B-C-L)$, we have $d(v) = n-k+t-1$, and so

$$\begin{aligned} d(u)+d(v) &= d(u,A')+d(u,B')+d(u,C\cup L)+d(v) \\ &= (t-1)+(k-1)+(t-1+l-1)+n-t-1 \\ &= n+2k-t+l-5. \end{aligned}$$

These are up to symmetry, the only possibilities for u and v, with $uv \notin E(G)$, thus $\sigma_2(G) = n+2k-t+l-5$.

Now suppose that $\pi \in \Pi(A,B)$ is defined to be such that for all i, $\pi(a_i) = b_i$. Since we removed the edges $a_1b_1, \ldots, a_{k-t}b_{k-t}$, the edges of $E(A',B')$ cannot be used in a (A,B,π)-linkage $\mathcal{P}$. The only edges left from A' to the rest of the graph G are those of $E_G(A', C\cup L)$, so the paths linking A' to B' must go through C or L.

Hence, if $l=1$, since $|C| = k-t-1$, we see that there is no (A,B,π)-linkage in G, hence

$$\mathrm{ldiam}_{k,t}(G) = \overline{\mathrm{ldiam}}_{k,t}(G) = \underline{\mathrm{ldiam}}_{k,t}(G) = \infty.$$

Since $E_G(C,B') = \emptyset$, paths going through C will have order at least four, the only paths having order three being those going through L. Since $|C| = t$ and $|L| = l-1$,

$$\Sigma_{P\in\mathcal{P}}|P| \geq 3(l-1)+4(k-t-l+1) = 4(k-t)-l+1,$$

implying that $\mathrm{ldiam}_{k,t}(G) \geq 3(k-t)-l+1$.

This concludes the proof of Theorem 6 $\square_{Theorem\ 6}$

Proof of Theorem 7: First, we show that the lemma is true for $t=0$, i.e. $A\cap B = \emptyset$. Let G satisfy the conditions of the Lemma and let

$A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two disjoint k-sets of vertices of $V(G)$ such that there is an (A, B)-system $\mathcal{P} = \{P_1, \ldots, P_k\}$ in G where for $1 \leq i \leq k$, P_i is an $[a_i, b_i]$-path. Take $\mathcal{P}$ to be of maximal order and let $q = |Q|$, $p = |P|$, and for every $1 \leq i \leq k$, let $p_i = |P_i|$. Let $P = (\mathcal{P})_G$ and $Q = G - P$.

Note that for any $u \in Q$ and $P_i \in \mathcal{P}$, if $z \in N(u, P_i)$, then $z^+ \notin N(u, P_i)$ or replacing P_i with $[a_i, z]_{P_i} \cup zu \cup uz^+ \cup [z^+, b_i]_{P_i}$ we would contradict the maximality of $\mathcal{P}$. This implies that $d(u, P_i) \leq \left\lfloor \frac{p_i+1}{2} \right\rfloor$, thus

$$d(u, P) \leq \left\lfloor \frac{p+k}{2} \right\rfloor, \tag{24}$$

which in turn yields that for any two vertices u and v of Q,

$$d(w, Q) + d(w', Q) \geq n + k - (p + k) = q,$$

showing that Q is connected.

The fact that $\sigma_2(G) \geq n+(k+2)-2$ shows by Theorem 5 that G is $(k+2)$-connected. Thus $|N(Q, P)| \geq k + 2$, so by the pigeon-hole principal, some member of $\mathcal{P}$, without loss of generality P_1, satisfies $|N(Q, P_1)| \geq 2$. Let x and y be such that $\{x, y\} \in N(Q, P_1)$, y appears after x in P_1, and $|[x, y]_{P_1}|$ is minimal. Let $u, v \in V(Q)$ be such that $ux, vy \in E(G)$ and $R = [x^+, y^-]_{P_1}$. We cannot have $y = x^+$ or the maximality of $\mathcal{P}$ would be contradicted, so $R \neq \emptyset$. Let $r = |R|$, $P_1' = [a_1, x]_{P_1}$ and $P_1'' = [y, b_1]_{P_1}$.

By the minimality of $|[x, y]_{P_1}|$, $d(S, R) = 0$, so inequality (24), when applied to $\mathcal{P}'$, shows that for all $w \in V(Q)$,

$$d(w, \mathcal{P}) = d(w, \mathcal{P}') \leq \frac{p - r + k + 1}{2}. \tag{25}$$

Since for all $w \in V(Q)$ and $z \in V(R)$, $wz \notin E(G)$, our degree condition yields

$$\begin{aligned} d(z, \mathcal{P}') &\geq \sigma_2(G) - d(w, Q) - d(w, \mathcal{P}') - d(z, R) \\ &\geq (n + k) - (q - 1) - \frac{p - r + k + 1}{2} - (r - 1) \\ &= \frac{p - r + k + 3}{2}. \end{aligned} \tag{26}$$

If $|R| = 1$, since

$$d(x^+, \mathcal{P}') \geq \frac{p - r + k + 3}{2} > \frac{p - r + k + 1}{2},$$

there is a path $P \in \mathcal{P}'$ and a vertex $z \in V(P)$ such that $x^+z, x^+z^+ \in E(G)$, so we can insert x^+ into P, and obtain a larger (A, B)-linkage than $\mathcal{P}$.

If $|R| = 2$ then $x^+ \neq y^-$, and (26) shows that

$$d(x^+, \mathcal{P}') + d(y^-, \mathcal{P}') \quad \geq \quad p - r + k + 3.$$

This implies that for some $Z \in \mathcal{P}'$,

$$d(x^+, Z) + d(y^-, Z) \geq |Z| + 1.$$

Note that we can choose Z to be of order at least 2 since for $2 \leq i \leq k$, $|P_i| \geq 2$, and if both P_1' and P_1'' have order 1, we still have

$$d(x^+, P - P_1) + d(y^-, P - P_1) \geq |P - P_1| + 1.$$

This shows that for some vertex $z \in Z$ such that $zx^+, z^+y^- \in E(G)$ or $z^+x^+, zy^- \in E(G)$. Let us assume we are in the later case, since the other case is similar. If $Z = P_i$ for some $2 \leq i \leq k$, replacing the path P_1 of $\mathcal{P}$ with

$$[a_1, x]_{P_1} \cup xu \cup S \cup vy \cup [y, b_1],$$

and P_i with

$$[a_i, z]_{P_i} \cup zy^- \cup R \cup x^+z^+ \cup [z^+, b_i]_{P_i},$$

we contradict the maximality of $\mathcal{P}$. If $Z = P_1'$, we can replace the path P_1 of $\mathcal{P}$ with

$$[a_1, z]_{P_1} \cup R \cup x^+y^+ \cup [y^+, x]_{P_1} \cup xu \cup S \cup vy \cup [y, b_1]_{P_1},$$

we again have a contradiction. The case $Z = P_1''$ is similar to the previous one. Hence the Lemma is true for $t = 0$.

If $t = |T| = |A \cap B| \neq 0$ now, notice that for all non-adjacent vertices u and v of $G - T$,

$$d(u, G - T) + d(v, G - T) \geq n + k - 2|T| = (n - t) + k - t$$

This shows that $\sigma_2(G - T) \geq |G - T| + k - t$, so that any $(A - T, B - T)$-system of $G - T$ may be extended to an $(A - T, B - T)$-system covering all the vertices of $G - T$ while conserving the endpoints of the paths of the original system.

To see that the condition $\sigma_2(G) \geq n + k - 1$ isn't even enough to extend some (A, B)-systems to a Hamilton (A, B)-system, consider the graph $G =$

$X + Y$ where $n \geq 4k$, X is an empty graph (no edges) on $\frac{n-k+1}{2}$ vertices and Y is a complete graph on $\frac{n+k-1}{2}$ vertices. Note that $G_{n,k}$ can be seen to be obtained by taking the complete graph K_n and removing all vertices of a subgraph X on $\frac{n-k+1}{2}$ vertices of $V(K_n)$.

One will verify that $\sigma_2(G) = n+k-1$ yet if A and B are k-sets of vertices of X intersecting on t vertices, there can be no (A, B)-system in G covering all the vertices of the graph. $\square_{Theorem\ 7}$

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