Minimum Degree and Dominating Paths

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Joint work with Ralph Faudree, Mike Jacobson and Doug West

Dedicated to Ralph J. Faudree

October 17, 2015

Figure : Ralph Faudree - Outstanding Mathematician and Great Friend

Figure : Ralph Faudree - Powerful Administrator

Figure : From experience he did not believe all he heard!

Figure : Ralph and Pat on the 3-gorges river cruise.

Figure : The Gang of 7: Back: Burr, Jacobson, Rousseau, Schelp
Front: RG, Uncle Paul, Ralph: March 1984

Figure : Dinner in Budapest with Miki Simonovits.

Background

Definition

A set $S \subset V(G)$ is a connected dominating set provided $G[S]$ is connected and each vertex of $V(G) - S$ has a neighbor in S.

Fact: large minimum degree implies small connected dominating set.

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Theorem

$(Caro - West - Yuster, 2000)$
For large fixed k, every n-vertex graph G with $\delta(G) \geq k$ has a connected dominating set with size at most

$$\frac{(1+o(1))ln\ k}{k}n$$

Question

What do we known about $G[S]$*?*

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Spanning trees are connected dominating sets, with the leaves as the dominated vertices.

But what can we say about the structure of $G[S]$, even if it is a tree of some sort?

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One of our goals will be to try and say something about one special case of a connected dominating set that is a tree - namely a path.

More background - Maybe the set is not so small

Theorem

$(Alon, Alon - Wormald, 1990, 2010)$

Some k-regular graphs have no dominating set of size less than

$$\frac{1+ln\ (k+1)}{k+1}n$$

A probabilistic argument for this.

Control $G[S]$ - dominating paths

Definition

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Theorem

(*Dirac*, 1952)
Every n-vertex graph with $\delta(G) \geq (n-1)/2$ has a spanning path, hence a dominating path with $n-2$ vertices.

Definition

Vertices u and v are λ-distant provided $dist(u, v) \geq \lambda$.

Theorem

(*Broersma*, 1988)
Let G be a k-connected graph ($k \geq 1$) and let $\lambda \geq 2$. If the degree sum of any $k + 2$ mutually ($2\lambda - 1$) distant vertices is at least $n - 2k - 1 - (\lambda - 2)k(k + 2)$, then G has a path where every vertex is at distance less then λ of this path.

Corollary

Let G be a k-connected graph. If the degree sum of any $k + 2$ mutually 3-distant vertices is at least

$$n - 2k - 1$$

then G has a dominating path.

Dominating cycles were usually long cycles

Theorem

(*Yoshimoto*, 2008)
If

$$deg\ e_1 + deg\ e_2 > |V(G)| - 4$$

for any two remote edges e_1, e_2, *then all longest cycles in* G *are dominating and this bound is best possible.*

More long cycles

Theorem

(*Bondy*, 1980)
If G is 2-connected of order n and

$$\sigma_3(G) \geq n + 2,$$

then each longest cycle of G is dominating.

Theorem

(*Yamashita*, 2008)
If G is 3-connected of order n and

$$\sigma_4(G) \geq n + \kappa(G) + 3,$$

then G contains a longest cycle which is dominating.

Our driving question??????

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What minimum degree guarantees a "small" dominating path?

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Question

How small is small?????

Our Results

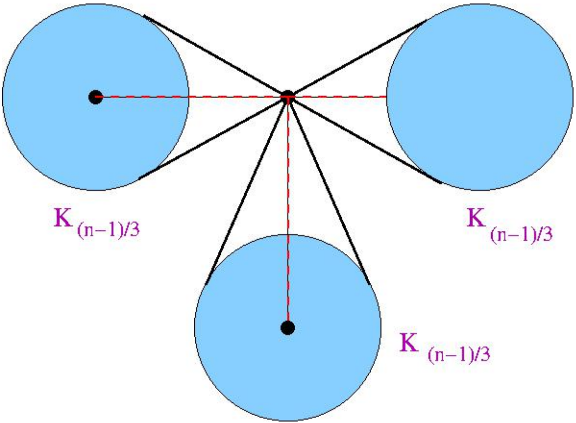
Theorem

Every n-vertex connected graph G with

$$\delta(G) \geq n/3 - 1$$

contains a dominating path, and the inequality is sharp.

Sharpness Example



More connectivity - less degree

Theorem

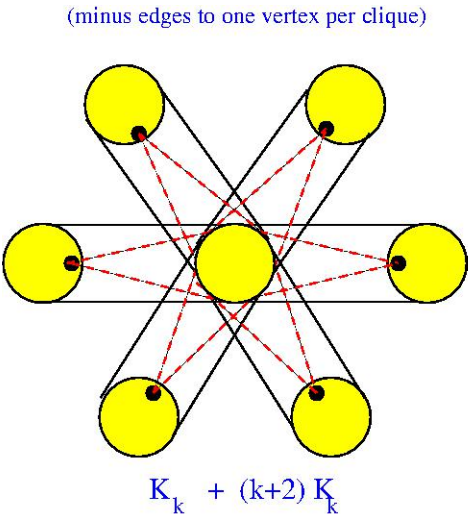
If G is an n-vertex 2-connected graph with

$$\delta(G) \geq (n+1)/4,$$

then G contains a dominating path.

(This is almost sharp in the sense there is an example with $\delta(G) = (n-6)/4$ that fails.)

Sharpness Example



More control of the path length

Theorem

If $\delta(G) \geq n/3$, then G has a dominating k vertex path for every k from the least value to at least

$$\min \ \{n, 2\delta(G) + 1\}$$

and this is sharp.

Better yet!

Theorem

If $\delta(G) \geq cn$ with $c > 1/3$, then G has a dominating path with length logarithmic in n (base depends only on c).

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If $\delta(G) \geq cn$ with $c > 1/3$, then G has a dominating path with length logarithmic in n (base depends only on c).

Remark: The Alon - Wormald result implies min deg k guaranteeing an s-vertex dominating path requires $s > \frac{ln\ k}{k} n$. But when s is constant a direct argument gives more.

Theorem

Fix $s \in N$ and $c \in R$ with $c < 1$. For n sufficiently large, with

$$\delta(G) \geq n - 1 - cn^{1-1/s}$$

the graph contains an s-vertex dominating path.

Theorem

Given $s \in N$ and $c > 1$, for n suff. large, some n-vertex graph with $\delta \geq n - c(s \ln\, n)^{1/2}\; n^{1-1/s}$ has no dominating set of size at most s.

Theorem

Given $s \in N$ and $c > 1$, for n suff. large, some n-vertex graph with $\delta \geq n - c(sln\ n)^{1/2}\ n^{1-1/s}$ has no dominating set of size at most s.

Corollary

In particular, no s-vertex dominating path.

Controling the path length

Theorem

Let G be a connected n-vertex graph with

$$\delta(G) \geq an + log_{a/(1-a)} n$$

where $a > 1/2$. For n suff. large, G has an s-vertex dominating path whenever

$$log_{a/(1-a)} n \leq s \leq n$$

starting from any vertex.

Improvement

Theorem

For $1/3 < a < 1$, there is a constant $c = c(a)$ such that if n is suff. large and $\delta(G) \geq an$, then G contains a dominating path with at most $clog_{1/(1-a)}n$ vertices.

Theorem

Fix $s \in N$ and $c \in R$ with $c < 1$. For suff. large n,

$$\delta(G) \geq n - 1 - cn^{1-1/s}$$

ensures an s=vertex dominating path.

Near Sharpness.

Theorem

Given $s \in N$ and $c > 1$, for suff. large n, some n-vertex graph with $\delta(G) \geq n - c(sln\ n)^{1/s} n^{1-1/s}$ has no dominating set of size at most s.

Balanced Caterpillars

Definition

spanning caterpillar = spanning tree consisting of a single path (*spine*) plus leaves.

Definition

balanced = if the vertices of the spine all have the same number of neighbors; nearly balanced = the numbers differ by at most 1.

Balance

Theorem

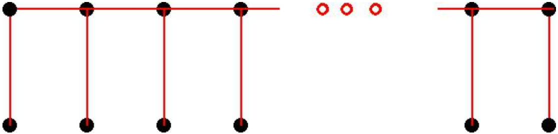
Fix $p \in N$. For n suff. large, with $(p+1)$ dividing n and

$$\delta(G) \geq (1 - \frac{p}{(p+1)^2})n$$

then G contains a balanced spanning caterpillar with $\frac{n}{p+1}$ spine vertices.

Special Case:

Case: $p = 1$ says $\delta \geq 3n/4$ and n even. This implies



Theorem

Fix a positive integer s and a real constant c less than 1. Let G be an n-vertex graph such that $\delta(G) \geq n - cn^{1-1/s}$. If n is suff. large, then G contains a nearly balanced spanning caterpillar with k spine vertices for each k such that

$$s \leq k \leq .5 \frac{log\ \ n}{log\ \ log\ \ n}.$$

Open Problems

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at most $n - \Omega(n^{1-1/s}$ and at least $n - O(sln\ n)^{1/s}n^{1-1/s}$.