Extending Vertex and Edge Pancyclic Graphs

Megan Cream[*] Ronald J. Gould[†] Kazuhide Hirohata[‡]

December 5, 2017

Abstract

A graph G of order $n \geq 3$ is *pancyclic* if G contains a cycle of each possible length from 3 to n, *vertex* (*edge*) *pancyclic* if each vertex (edge) is contained on a cycle of each possible length from 3 to n. A *chord* is an edge between two vertices of a cycle that is not an edge on the cycle. A *chorded cycle* is a cycle containing at least one chord. We define a graph G of order $n \geq 4$ to be *chorded pancyclic* if G contains a chorded cycle of each possible length from 4 to n. In this article, we consider extentions of the property of being chorded pancyclic to chorded vertex pancyclic and chorded edge pancyclic.

1 Introduction

The study of the cycle structure in graphs has a long and well developed history. Many aspects of a cycle have been considered, for example cycle length or elements contained on the cycle are two such aspects. Given a graph G of order $n \geq 3$, we say that G is *Hamiltonian* if G contains a cycle that spans $V(G)$, the vertex set of G. We say that G is pancyclic if G contains a cycle of each possible length from 3 to n. Besides these length considerations, we may ask for more. The graph G is called *vertex pancyclic* (*edge pancyclic*) if each vertex (edge) is contained on a cycle of each possible length from 3 to n.

More recently another cycle property has been studied (see for example, [7], [9], [4]). We say that a cycle in G is *chorded* if G contains an edge between two vertices of the cycle that is not an edge on the cycle. Further, we say that G is *chorded pancyclic* if G contains a chorded cycle for each possible length from 4 to n. The graph G is *chorded vertex pancyclic* if each vertex is contained in a chorded cycle of each possible length from 4 to n.

In the early 1970's, J. A. Bondy ([1], [2]) stated his well-known meta conjecture that almost any condition that implies a graph is Hamiltonian will imply the graph is pancyclic, possibly with a well defined class of exceptional graphs. Bondy supported his meta-conjecture with several results (see [1], [2]). Bondy's meta-conjecture was extended in [4] to almost any condition that implies a graph is Hamiltonian will imply it is chorded pancyclic, possibly with some class of well defined exceptional graphs and some small order exceptional graphs. Again, this extension has been supported by several results (see [3], [4]). The purpose of this paper is to further consider these ideas and to

[*]Department of Mathematics, Spelman College, Atlanta, GA. Email: mcream@spelman.edu

[†]Department of Mathematics and Computer Science, Emory University, Atlanta, GA. Email: rg@mathcs.emory.edu

[‡]Department of Electronic and Computer Engineering, Ibaraki National College of Technology, Ibaraki, Japan. Email: hirohata@ece.ibaraki-ct.ac.jp

examine the relationship between various degree conditions and chorded vertex (edge) pancyclic graphs.

We only consider finite simple graphs in this paper. Let G be a graph of order n. A cycle of length k is called a *k-cycle*. For an integer $r \geq 3$, a vertex of G is called *r-pancyclic* if it is contained in a k-cycle for every $r \leq k \leq n$, and G is also called *vertex r-pancyclic* if every vertex is r-pancyclic. If $V(G) = \{x\}$, then we denote G by x. We denote the set $N_G(u) = \{x \in V(G) \mid ux \in E(G)\}$ and $\deg_G(u) = |N_G(u)|$. Further, $N_G[x] = N_G(x) \cup \{x\}$. Let H be a subgraph of G, and let $S \subseteq V(G)$. For $u \in V(G) - V(H)$, we denote $N_H(u) = N_G(u) \cap V(H)$ and $\deg_H(u) = |N_H(u)|$. For $u \in V(G) - S$, $N_S(u) = N_G(u) \cap S$. The subgraph of G induced by S is denoted by $\langle S \rangle$, $G - S = \langle V(G) - S \rangle$, and $G - H = \langle V(G) - V(H) \rangle$. If $S = \{u\}$, then we write $G - u$ for $G - S$. For two disjoint subgraphs (resp. subsets) A and B in G (resp. $V(G)$), $E(A, B)$ is the set of edges between A and B. For two disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes the union of G_1 and G_2, and $G_1 + G_2$ denotes the join of G_1 and G_2. For t $(t \geq 2)$ disjoint graphs $G_1, G_2, \ldots, G_t$, we denote by $G_1 + G_2 + \cdots + G_t$ the graph satisfying $G_i + G_{i+1}$ for every $1 \leq i \leq t-1$ and $E(G_i, G_j) = \emptyset$ for every $1 \leq i < j \leq t$ $(j \neq i+1)$. We denote the complement of G by $\overline{G}$, and $R_1 \subseteq G \subseteq R_2$ means that G is isomorphic to a subgraph of R_2 containing R_1. Let

$$\sigma_2(G) = \min\{d_G(u) + d_G(v) \mid u, v \in V(G), uv \notin E(G)\},$$

and $\sigma_2(G) = \infty$ when G is a complete graph. For terms not defined here see [8].

Ore's classic theorem will be usefull later.

Theorem 1 (Ore [11])**.** *If G is a graph of order $n \geq 3$ with $\sigma_2(G) \geq n$, then G is hamiltonian.*

2 Vertex Pancyclic Extensions

In this section we examine extensions of vertex pancyclic properties to chorded vertex pancyclic properties. We begin with a result from Hendry [10].

Theorem 2 (Hendry [10])**.** *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n+1}{2}$, then G is vertex pancyclic.*

We now extend Hendry's Theorem to chorded graphs.

Theorem 3. *If G is a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+1}{2}$, then G is chorded vertex pancyclic.*

Proof. Let G be a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+1}{2}$. By Theorem 2, any vertex is on a cycle of every possible length, so suppose $x \in V(G)$ is on a cycle C of length $k \geq 5$ such that C is chordless. Now $|V(G) - V(C)| = n - k \leq n - 5$. Further, for any vertex $w \in V(C)$, $|N_{G-C}(w)| \geq \frac{n+1}{2} - 2 = \frac{n-3}{2}$. Thus, the intersection of the neighbors of any two vertices of C, off of the cycle C, contains at least two vertices. Let $C = x_1, x_2, x_3, \ldots, x_k, x_1$ with $x = x_1$. Note that there exists $w_1 \in N_{G-C}(x_1) \cap N_{G-C}(x_2)$ and there exists $w_2 \in N_{G-C}(x_2) \cap N_{G-C}(x_5)$ such that $w_1 \neq w_2$ as x_2 and x_5 are nonadjacent and share no neighbors on C. Then $x_1, w_1, x_2, w_2, x_5, \ldots, x_1$ is a k-cycle with chord x_1x_2. Again, by Theorem 2, we know $x = x_1$ also lies on a triangle, say $C' = x_1, x_2, x_3, x_1$. Since $|V(G) - V(C')| = n - 3$, at least one intersection of the form $N_{G-C'}(x_a) \cap N_{G-C'}(x_b)$, for $a, b \in \{1, 2, 3\}$ with $a \neq b$, must be nonempty. Say $w \in N_{G-C'}(x_a) \cap N_{G-C'}(x_b)$, then the graph induced by the vertex set $\{w, x_1, x_2, x_3\}$ contains a chorded 4-cycle containing $x = x_1$. Therefore, G is chorded vertex pancyclic. □

The above minimum degree condition is actually strong enough to imply a bit more as we see in the next theorem.

Theorem 4. *Let G be a graph of order $n \geq 10$ with $\delta(G) \geq \frac{n+1}{2}$. Then for every $k \geq 5$ and any vertex $x \in V(G)$, there is a doubly chorded cycle in G containing x, that is G is doubly chorded 5-pancyclic.*

Proof. By Theorem 3, we know G is chorded vertex pancyclic. Let $C_k : x_1 = x, x_2, \ldots, x_k, x_1$, with $k \geq 7$ be such a cycle in G containing the specified vertex x. If C_k has two or more chords, then we have the desired cycle. Thus, suppose that it has only one chord. Now by the minimum degree condition, any two consecutive vertices of C_k have degrees to $G - C_k$ of at least $\frac{n-5}{2}$ and $\frac{n-3}{2}$ as at most one of these vertices is the end vertex of the chord. As $|V(G) - V(C_k)| = n - k \leq n - 7$, we see that any two consecutive vertices of C_k have at least three common neighbors off of C_k. Let $w_1 \in N_{G-C_k}(x_k) \cap N_{G-C_k}(x_1)$ and let $w_2 \in N_{G-C_k}(x_1) \cap N_{G-C_k}(x_2)$ with $w_1 \neq w_2$. Now, if x_2x_6 is not the chord for C_k, then there exists $w_3 \in N_{G-C_k}(x_2) \cap N_{G-C_k}(x_6)$, with w_3 not equal to either w_1 or w_2. Now

$$x_k, w_1, x_1, w_2, x_2, w_3, x_6, x_7, \ldots, x_k$$

is a k-cycle containing $x = x_1$ with chords x_kx_1 and x_1x_2.

If however, x_2x_6 is the chord of C_k, then we note there exists a $w_3 \neq w_1$ such that $w_3 \in N_{G-C_k}(x_5) \cap N_{G-C_k}(x_7)$. Now

$$x_k, w_1, x_1, x_2, x_6, x_5, w_3, x_7, \ldots, x_k$$

is a k-cycle containing $x = x_1$ with chords x_kx_1 and x_6x_7. (Note that $x_k = x_7$ is possible.) Thus, a doubly chorded cycle containing x exists when $k \geq 7$.

Next we claim that every vertex x is on a chorded 5-cycle. First suppose that $deg\ (x) = n - 1$. Then by the degree condition, it is easily seen that a P_4 exists in $N(x)$, hence x is on a doubly chorded 5-cycle. Now suppose $deg\ x < n - 1$ and let y be any vertex not adjacent to x. By the minimum degree condition, x and y have at least three common neighbors. Let M be the set of common neighbors of x and y. Note that the vertices of M must be independent or a doubly chorded 5-cycle containing x (and y) is easily found. Further, let $A = N(x) - M$ and $B = N(y) - M$. Note that each vertex in A has at least two adjacencies to M for otherwise for such $a \in A$

$$deg\ a + deg\ y \leq (|A| - 1) + 1 + 1 + |M| + |B| \leq n - 1 < n + 1$$

a contradiciton. Thus, if am_1 and am_2 are two such adjacencies of a in M then,

$$x, a, m_1, y, m_2, x$$

is a 5-cycle containing x with chords xm_1 and am_2. Thus, every vertex of G is on a doubly chorded 5-cycle as claimed.

Finally, we need to show that every vertex is on a doubly chorded 6-cycle. By Theorem 3 we know every vertex is on a chorded 6-cycle. We may then assume $C_6 : x = x_1, x_2, \ldots, x_6, x_1$ is a chorded 6-cycle containing x. If C_6 is doubly chorded we have the desired cycle. Therefore, we can assume C_6 has exactly one chord. By the degree condition, the degrees of any two consecutive vertices on C_6 to $V(G) - V(C)$ must be at least $\frac{n-5}{2}$ and $\frac{n-3}{2}$ as only one of them can be incident to the chord. Thus, any such pair have at least two common neighbors off C_6. If x_1x_3 is not the chord, then they have at least three common neighbors off C_6. Now there exists $w_1 \in N_{G-C_6}(x_1) \cap$

$N_{G-C_6}(x_2)$ and a $w_2 \in N_{G-C_6}(x_2) \cap N_{G-C_6}(x_3)$ with $w_1 \neq w_2$ and $w_3 \in N_{G-C_6}(x_1) \cap N_{G-C_6}(x_3)$ with $w_3 \neq w_1$ or w_2. Hence,

$$x_1, w_1, x_2, w_2, x_3, w_3, x_1$$

is the desired doubly chorded 6-cycle containing $x_1 = x$ with chords x_1x_2 and x_2x_3.

Now suppose that x_1x_3 is the chord of C_6. Then we repeat the same argument on the consecutive pairs x_6, x_1, and x_1, x_2 and the pair x_6, x_2 to get the 6-cycle

$$x_6, w_1, x_1, w_2, x_2, w_3, x_6$$

with chords x_6x_1 and x_1x_2. Hence, we see that G is doubly chorded vertex 5-pancyclic. $\square$

Example 1. *To see that the graph of the previous theorem need not be doubly chorded 4-pancyclic, let $n \cong 3(mod4)$ and consider the graph $K_{\frac{n+1}{2},\frac{n-1}{2}}$ along with a perfect matching in the larger partite set. This graph is $\frac{n+1}{2}$ regular. However, no vertex lies in a K_4, hence there are no doubly chorded 4-cycles in this graph.*

A result of Randerath et al. [12] is the following:

Theorem 5 (Randerath et al. [12])**.** *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$. Then G is vertex 4-pancyclic.*

We next determine what this σ_2 condition implies for chorded vertex pancyclic graphs.

Theorem 6. *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$. Then G is chorded vertex 5-pancyclic.*

Proof. Consider a graph G of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$. By Theorem 5, G is vertex 4-pancyclic, so every vertex in $V(G)$ is contained in a cycle of every length from 4 to n.

Claim 1: Every vertex in G is contained on a chorded k-cycle for every k where $6 \leq k \leq n$.

Let $x_1 \in V(G)$ and suppose $C = x_1, x_2, x_3, \ldots, x_k, x_1$ is a k-cycle in G containing x_1. If C does not contain a chord, then x_1 and x_3 are nonadjacent and thus $\deg_G(x_1) + \deg_G(x_2) \geq n+1$. Therefore, x_1 and x_3 must share at least three common neighbors not on C. Let w_1 be one such neighbor. Similarly, x_2 and x_6 must also have a common neighbor, say $w_2 \notin V(C)$ with $w_1 \neq w_2$. Then $x_1, w_1, x_3, x_2, w_2, x_6, \ldots, x_1$ is a k-cycle containing x_1 with the chord x_1x_2. Thus, the Claim holds.

Next suppose there is a vertex $x \in V(G)$ such that x is not on any chorded 5-cycle in G. But x does lie on 5-cycles by Theorem 5. Say $C : x, c, y, b, a, x$ is one such 5-cycle. Then x and y are nonadjacent in G and so $deg\ x + deg\ y \geq n+1$. We partiton $V(G)$ as follows: Let C be the set of common neighbors of x and y. Let $A = N(x) - C$ and let $B = N(y) - C$. By the degree sum condition, $|C| \geq 3$. Note that there can be no edge between two vertices in C or a chorded 5-cycle containing x is easily constructed. Also, there can be no edges from A to C or again a chorded 5-cycle containing x exists. Similarly, there are no edges from B to C.

If $a_1 \in A$ has no adjacencies to B, then $deg\ a_1 + degy \leq |A| - 1 + 1 + |B| + |C| \leq n-2$, a contradiction. But that means that every vertex in A has adjacencies in B and similarly every vertex in B has adjacencies in A. But then $deg\ c_1 + deg\ c_2 = 4$, again a contradiction. Thus, x_1 must be on a chorded 5-cycle and hence G is chorded vertex 5-pancyclic. $\square$

Example 2. *In Theorem 6, the degree sum condition is sharp. The balanced complete bipartite graph,* $K_{\frac{n}{2},\frac{n}{2}}$ *has* $\sigma_2 = n$ *but it does not contain any odd cycles. Thus it is not pancyclic, so clearly it is not chorded vertex pancyclic either.*

Example 3. *Theorem 6 is also sharp in terms of 5-pancyclicity. Consider the graph* $G = (K_n - E(K_c))$*, where the integer* $c \geq 3$*, with an additional vertex* v *adjacent to each vertex of the* K_c *whose edges were removed. Then* v *clearly lies on no chorded 4-cycle. Further, the* $\sigma_2(G)$ *condition is realized by* v *and any vertex in* $V(G) - N[v]$ *as* $c+n-2 \geq n+1$ *as long as* $c \geq 3$*. Thus,* G *need not be chorded vertex pancyclic under this degree sum condition.*

Example 3 shows that $\sigma_2(G) \geq n + c$ for any constant c will fail to imply the graph is vertex pancyclic when n is sufficiently large. In [12], a sharp minimum degree sum condition implying the existence of vertex pancyclic graphs was determined.

Theorem 7 (Randerath et al. [12])**.** *Let* G *be a graph of order* $n \geq 3$ *such that* $\sigma_2(G) \geq \lceil \frac{4n}{3} \rceil - 1$*. Then* G *is vertex pancyclic.*

Our next result extends Theorem 7 to chorded vertex pancyclic graphs.

Theorem 8. *Let* G *be a graph of order* $n \geq 8$ *such that* $\sigma_2(G) \geq \lceil \frac{4n}{3} \rceil - 1$*. Then* G *is chorded vertex pancyclic.*

Proof. First note that by Theorem 7, G is vertex pancyclic.

Claim 1: For every $x \in V(G)$ there is a chorded 4-cycle in G containing x.

Consider any vertex $x \in V(G)$, and take $y \in V(G)$ such that $xy \notin E(G)$. Such a y exists or else, by the degree sum condition, $N(x)$ certainly contains a path on three vertices, and hence a chorded 4-cycle containing x exists. Now partition the remaining vertices of G as follows:

$$M = N_G(x) \cap N_G(y),$$

$$X = N_G(x) - M,$$

$$Y = N_G(y) - M,$$

$$D = V(G) - (\{x, y\} \cup M \cup X \cup Y).$$

Since $\sigma_2(G) \geq \lceil \frac{4n}{3} \rceil - 1$, we see that $|M| \geq \frac{n}{3} + 1$. Note that M must be an independent set, otherwise x is contained in a chorded 4-cycle. Consider $a, b \in M$. Since each vertex a (and b) can be adjacent to every other vertex in G except for the other vertices in M, hence $\deg_G(a) + \deg_G(b) \leq 2\left(\frac{2n}{3} - 1\right) = \frac{4n}{3} - 2$. This contradicts the degree sum condition of G since $ab \notin G$. Therefore M cannot be an independent set. If $ab \in E(G)$, then x, a, y, b, x is a 4-cycle containing x with chord ab.

Next we show that if x_1 is on a chorded 4-cycle, then it is on a chorded 5-cycle. So let $C : x_1, x_2, x_3, x_4, x_1$ be a 4-cycle containing x_1 with chord x_1x_3. Then by the degree conditon we know G is 2-connected. Hence there exists a vertex w not on C that is adjacent to a vertex of C. Say wx_2 is an edge of G. Now if w is also adjacent to either x_1 or x_3, then a chorded 5-cycle containing x_1 exists. Otherwise, w and x_3 are nonadjacent and hence have a common neighbor w_1

not on C. But then $x_1, x_2, w, w_1, x_3, x_1$ is a 5-cycle with chord x_2x_3 that contains x_1. Note that a similar argument holds if w is adjacent to x_3 or the chord of C is x_2x_3.

Next suppose that the vertex x_1 is contained on a cycle C_t, $t \geq 6$. Say $C_t : x_1, x_2, \ldots, x_t, x_1$. If C_t is chorded we are done, so suppose that it is not chorded. Then x_1 and x_3 are nonadjacent and by the degree condtion, they have at least $n/3 + 1$ common neighbors. Let $w_1 \notin V(C)$ be such a common neighbor. Similarly, x_2 and x_6 have a common neighbor $w_2 \neq w_1$ off of C. But then $x_1, w_1, x_3, x_2, x_6, \ldots, x_1$ is a t-cycle containing x_1 with chord x_1x_2.

Thus, G is chorded vertex pancyclic. □

Definition 1. *A graph G of order n is called (h,k)-**pancyclic** if every set of h vertices in G is contained in a cycle of every length from k to n.*

Theorem 9 (Faudree, Gould, Jacobson, 2009 [5])**.** *Let $2 \leq k$, $2k \leq m$ and $m < n$ be integers and let G be a graph of order n. If $\delta(G) \geq \lfloor \frac{n+2}{2} \rfloor$, then G is (k,m)-pancyclic.*

Theorem 10. *Let $k \geq 2$ and $n > 2k$, and let G be a graph of order n with $\delta(G) \geq \lfloor \frac{n+2}{2} \rfloor$. Then G is chorded $(k, 2k+1)$-pancyclic.*

Example 4. *In order to see that $2k+1$ is sharp when $k = 2$ consider a graph G of even order $n \geq 8$, formed by creating a matching in each partite set of the complete balanced bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Let x and y be nonadjacent from the same partite set. Then x and y are not together on any chorded 4-cycles in G. However, $\delta(G) = \frac{n}{2} + 1 = \lfloor \frac{n+2}{2} \rfloor$ as n is even.*

Proof. (of Theorem 10) Let G be a graph of order $n \geq 2k+1$ such that $\delta(G) \geq \lfloor \frac{n+2}{2} \rfloor$. Consider the set $X = \{x_1, x_2, \ldots, x_k\}$ of k vertices in G. By Theorem 9, the vertices in X lie on cycles of every length from $2k$ to n. Let $C : v_1 = x_1, v_2, \ldots, v_r$, with $r \geq 2k+1$ be a cycle containing X and assume C is not chorded, or else we have the desired cycle. Since $|V(C)| \geq 2k+1$, at least one of the k intervals $[x_i, x_{i+1})$ for $i = 1, 2, \ldots, k-1$ and $[x_k, x_1)$ must have at at least two vertices from $V(C) - X$. Without loss of generality say $[x_1, x_2)$ contains two such vertices v_i and v_{i+1}.

Note that $|V(G) - V(C)| \leq n - 5$. Also note that $2\delta(G) \geq n+1$. Now by the minimum degree conditon, two vertices on C have at least two common neighbors off C as $n + 1 - 4 = n - 3$. Thus, there exists a vertex $w_1 \notin V(C)$ such that $w_1 \in N_{G-C}(v_{i-2}) \cap N_{G-C}(v_{i-1})$. Also, there exists a vertex $w_2 \neq w_1$ with $w_2 \in N_{G-C}(v_{i-1}) \cap N_{G-C}(v_{i+2})$. Now, $v_{i-2}, w_1, v_{i-1}, w_2, v_{i+2}, \ldots, v_{i-2}$ is an r-cycle containing X with chord $v_{i-2}v_{i-1}$. □

In [12] a σ_2 bound was given for G to be vertex pancyclic.

Theorem 11 (Randerath et al. [12])**.** *Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n$. Then G is vertex 4-pancyclic unless n is even and $G = K_{n/2,n/2}$.*

In [6] a stronger σ_2 bound was given for G to be (k,m)-pancyclic when $k \geq 2$.

Theorem 12 ([6])**.** *Let $k \geq 2$ and $n > 2k$. Then if G is a graph of order n with $\sigma_2(G) \geq 2\lfloor \frac{n+2}{2} \rfloor + 1$, then G is $(k, 2k)$-pancyclic.*

We next extend Theorem 12.

Theorem 13. *Let $k \geq 2$ and $n > 2k$. If G is a graph of order n with $\sigma_2(G) \geq 2\lfloor \frac{n+2}{2} \rfloor + 1$, then*
(1) If $k \geq 4$, then G is chorded $(k, 2k)$-pancyclic,
(2) If $k = 2$ or 3, then G is chorded $(k, 2k+1)$-pancyclic.

Proof. We note that by the σ_2 conditon, $\sigma_2(G) \geq n+2$ if n is odd and $\sigma_2(g) \geq n+3$ if n is even. Let $X = \{x_1, x_2, \ldots, x_k\}$ be a set of k vertices in G. By Theorem 12, the k vertices of X are on cycles of length $2k$ to n. Assume the one of these cycles $C_r : u_1, u_2, \ldots, u_r, u_1$, is not chorded. Then the vertices of X partiton C_r into k intervals $[x_i, x_{i+1})$ for $i = 1, \ldots k-1$ and $[x_k, x_1)$. Note that if $k \geq 3$, the cycle C_r has length at least 6. Now if one of these intervals contains at least two vertices of $V(C) - X$, say that u_i, u_{i+1} are these two vertices. As C_r is not chorded, u_{i-3} and u_{i-1} are not adjacent. Now, $|V(G) - V(C_k)| = n - r \leq n - 6$. Thus, by the σ_2 condition, these vertices have at least three common neighbors off C_r. Let w_1 be such a vertex. Further, u_{i-2} and u_{i+2} are nonadjacent and also have at least three common neighbors off C_k. Let w_2 be such a vertex. Now,

$$u_{i-3}, w_1, u_{i_1}, u_{i-2}, w_2, u_{i+2}, \ldots, u_{i-3}$$

is an r-cycle containing X with chord $u_{i-3}u_{i-2}$ as a chord.

Now assume $k = 2$. We will show that any pair of vertices are on a chorded cycle of length 5. Suppose x and y are a nonadjacent pair of vertices. By the σ_2 condition, x and y have at least three common neighbors. Let B be the set of common neighbors of x and y. Let $A = N(x) - B$ and let $Y = N(y) - B$. It is easy to see that the vertices of B must be independent or a 5-cycle containing x and y is easily found. Further, there is an edge from $b \in B$ to $v \in A$, and if $b_1 \in B$ and $b_1 \neq b$, then x, v, b, y, b_1, x is a 5-cycle containing x and y with chord xb. A similar argument shows any edge from B to Y produces a chorded 5-cycle containing x and y. But if there are no edges from B to X or B to Y, then the σ_2 condition is violated by any pair of vertices in B. Thus, there is a chorded 5-cycle containing x and y.

Now assume $xy \in E(G)$. Let z be some vertex not adjacent to x and define B as the common neighbors of x and z, and $A = N(x) - B$, and $Z = N(z) - B$. Now an argument similar to the one in the previous case shows that any edge from A to B places x and y on a chorded 5-cycle. But such edges must exist or else if $y \in A$ then

$$deg\ y + deg\ z \leq |X| + |B| + |Z| < n,$$

a contradiction. If $y, y_1 \in B$, then

$$degy + deg\ y_1 < n,$$

again a contradiction.

Thus, in either case any pair of vertices is on a chorded 5-cycle, hence, when $k = 2$, the graph G is chorded $(2, 5)$-pancyclic and thus, for all $k \geq 2$, the graph G is chorded $(k, 2k+1)$-pancyclic.

To see that we cannot reduce to chorded 4-cycles when $k = 2$, consider the following graph H: Let n be odd. Take a pair of nonadjacent vertices x and y with exactly five independent vertices that are common neighbors of x and y. Call this set of common neighbors M. Let n be odd. Let $R = V(G) - \{x, y\} - M$ with $|R| = n - 7$. Let x be adjacent to $(n-7)/2$ vertices of R and let y be adjacent to the remaining $(n-7)/2$ vertices of R. Let the vertices of M each be adjacent to all of R. Now the nonadjacent pair x, y are on no chorded 4-cycles, as all such 4-cycles use two vertices of M. Further,

$$\sigma_2(H) = deg\ x + deg\ y = |R| + 10 = n + 3$$

. Thus, H fails to be chorded $(k, 2k)$-pancyclic.

Next suppose that $k \geq 4$. We know that the vertices of X lie on $2k$-cycles by Theorem 12. From the argument above we also know that on any such $2k$-cycle C, the vertices must alternate between a vertex of X and a vertex in $V(G) - X$. Let M_i be the common neighbors of the pair

x_i, x_{i+1} with M_k the common neighbors of x_k, x_1. Note the following:
(1) Any common neighbor of x_i, x_{i+1} can replace the common neighbor on the $2k$-cycle C, creating a new $2k$-cycle containing X which must also be chordless.
(2) Any vertex in M_i is nonadjacent to any vertex in M_j, $i \neq j$ or a chorded cycle would exit.
(3) Without loss of generality, let $|M_2| \leq |M_i|$, for $i \neq 2$.
(4) Let $R = V(G) - V(C)$. Then $|R| = n - k - \sum_{i=1}^{k} |M_i|$.

Then, $deg\ x_2 + deg\ x_3 \leq |M_1| + 2|M_2| + |M_3| + |R|$ which implies that

$$n + 3 \leq n + |M_2| - \sum_{i=4}^{k} |M_i| - k < n$$

as $|M_2| \leq |M_4|$. Thus, there must be a chorded $2k$-cycle containing X.

Next let $k = 3$. We define the graph H_1 as follows: For M_i the common neighbors of x_i, x_{i+1} we assume that $|M_i| = \frac{n+15}{6}$, for each $i = 1, 2, 3$. Let $< M_i >$ be a clique for each i. Let $R = G - H_1$ where $|R| = n - 3(\frac{n+15}{6}) - 3$. Let each vertex from X have exactly $\frac{1}{3}|R|$ distinct adjacencies in R as there are no common adjacencies of such pairs in R. Now $\sigma_2(H_1)$ is determined by any two of the x_i. Hence,

$$deg\ x_1 + deg\ x_2 = 4(\frac{n+15}{6}) + \frac{2}{3}(n - 3(\frac{n+15}{6}) - 3) = n + 3.$$

Then H_1 has no chorded 6-cycles containing X. Hence, $k = 3$ fails to have chorded $(3, 6)$-cycles. $\square$

Next we consider what happens when we reduce the bound on σ_2.

Theorem 14. *Let G be a graph of order $n \geq 4$, and let x be any specified vertex of G. If $\sigma_2(G) \geq n$, then one of the following statements holds.*
(i) *G is chorded vertex pancyclic.*
(ii) *$\overline{K}_{n/2} + \overline{K}_{n/2} \subseteq G \subseteq \overline{K}_{n/2} + (K_1 \cup F)$ (n is even), where F is a spanning subgraph of $K_{n/2-1}$, satisfying the following conditions:*

- *if $E(F) = \emptyset$, then $x = v$ for any $v \in V(G)$,*
- *if $E(F) \neq \emptyset$, then $x \in V(K_1)$, or $x = v$ such that $\deg_F(v) = 0$ for $v \in V(F)$.*

(iii) *G is a spanning subgraph of $H = B + x + \overline{K}_a + (K_1 + K_c + K_d)$, ($|V(H)| = n, a \geq 2, c \geq 1, 0 \leq d \leq a - 2$) with all the edges of $B + (K_c \cup K_d)$, where B is a graph of order $b \geq 0$ with $|E(B)| \leq 1$ satisfying the following conditions:*

- *$h = h_1 + h_2 \leq 1$, where $h_1 = |E(B)|$ and $h_2 = |E(\overline{K}_a, B)|$,*
- *if $z_1 z_2 \in E(B)$ for $z_1, z_2 \in V(B)$, then $N_{K_c \cup K_d}(z_1) \cap N_{K_c \cup K_d}(z_2) = \emptyset$,*
- *if $mz \in E(G)$ for $m \in V(\overline{K}_a)$ and $z \in V(B)$, then $N_{K_c \cup K_d}(m) \cap N_{K_c \cup K_d}(z) = \emptyset$.*

Proof. Let G be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n$. Suppose that G is not a graph satisfying (ii) and (iii) in Theorem 14. If G is a complete graph, then the theorem holds. Thus G is not a complete graph. Note that G is Hamiltonian by Ore's theorem (Theorem 1). Let C^* be a Hamiltonian cycle in G, say $C^* = v_1 v_2 \ldots v_n v_1$. Let x be any specified vertex in G. If $n = 4$, then either $G = K_{2,2}$, or G is a chorded 4-cycle containing x and G is chorded vertex pancyclic. Thus we may assume that $n \geq 5$.

Suppose that $n = 5$. By the $\sigma_2(G)$ condition, C^* has at least two chords, and then C^* is a chorded 5-cycle containing x. Thus we need only to prove that G contains a chorded 4-cycle containing x. If the two chords are adjacent, then G contains a chorded 4-cycle containing x, no matter where x is on the cycle. Thus, we may assume that C^* has crossing chords which are independent. If x is an end vertex of one of these chords, then G contains a chorded 4-cycle containing x. Otherwise, G is a graph satisfying (iii), ($a = 2, b = 0, c = 1, d = 0$), a contradiction.

Suppose that $n \geq 6$. By Theorem 11, G is either vertex 4-pancyclic, or n is even and $G = K_{n/2,n/2}$. Suppose that $xu \in E(G)$ for all $u \in V(G-x)$. We now consider Hamiltonian cycle C^* as above. Without loss of generality, we may assume that $x = v_1$. By our assumption, we have $v_1v_i \in E(G)$ for all $2 \leq i \leq n$. Then $v_1v_2 \ldots v_iv_1$ for all $4 \leq i \leq n$ is a chorded i-cycle containing x. Thus G is chorded vertex pancyclic.

Therefore, there exists some $y \in V(G-x)$ with $xy \notin E(G)$. Partition $V(G) - \{x, y\}$ as follows:

$$\begin{aligned}
M &= N_G(x) \cap N_G(y),\\
X &= N_G(x) - M,\\
Y &= N_G(y) - M,\\
D &= V(G) - (\{x, y\} \cup M \cup X \cup Y).
\end{aligned}$$

Note that $\sigma_2(G)$ condition implies $|M| \geq 2$. Let $|M| = 2 + t$, where $t \geq 0$.

Claim 1. *$|D| \leq t$.*

Proof. Suppose that $|D| \geq t + 1$. Since $xy \notin E(G)$, by $\sigma_2(G)$ condition, we have

$$\begin{aligned}
n \leq \sigma_2(G) \leq \deg_G(x) + \deg_G(y) &\leq |V(G - \{x, y\})| - |D| + |M|\\
&\leq (n-2) - (t+1) + (2+t) = n - 1, \text{ a contradiction.}
\end{aligned}$$

□

Claim 2. *There exists a chorded n-cycle in G containing x.*

Proof. Since $n \geq 6$ and G contains a Hamiltonian cycle C^*, it is easy to see that C^* is a chorded n-cycle containing x by $\sigma_2(G)$ condition. □

Claim 3. *There exists a chorded 4-cycle in G containing x.*

Proof. Suppose not. Since $|M| \geq 2$, let $m_1, m_2 \in M$. If $m_1m_2 \in E(G)$, then $m_1ym_2xm_1$ is a 4-cycle with chord m_1m_2 containing x, a contradiction. Thus $m_1m_2 \notin E(G)$. This implies that M is an independent set. Suppose that $X = Y = \emptyset$. By $\sigma_2(G)$ condition, since

$$|M| + |\{x, y\}| + |D| = n \leq \deg_G(m_1) + \deg_G(m_2) \leq 2(|\{x, y\}| + |D|), \tag{1}$$

we have $|M| \leq 2 + |D|$. Since $|M| = 2 + t$, $|D| \geq t$. By Claim 1, $|D| = t$. Thus $\deg_G(m_i) = 2 + |D|$ for all $i \in \{1, 2\}$ by the inequality (1). This implies that $N_G(m) = \{x, y\} \cup D$ for any $m \in M$. Thus G is a graph satisfying the statement (ii), a contradiction. Therefore, $X \cup Y \neq \emptyset$. If $X = \emptyset$, then $Y \neq \emptyset$, and G is a graph satisfying the statement (iii) ($b = 0$), a contradiction. Thus $X \neq \emptyset$.

Subclaim 1. *For any $m \in M$, $\deg_X(m) \leq 1$.*

Proof. Suppose that $\deg_X(m) \geq 2$ for some $m \in M$. Let $z_1, z_2 \in N_X(m)$. Then xz_1mz_2x is a 4-cycle with chord xm containing x, a contradiction. Thus the subclaim holds. □

Subclaim 2. *For any $z \in X$,* $\deg_{X\cup M}(z) \le 1$.

Proof. Suppose that $\deg_{X\cup M}(z_0) \ge 2$ for some $z_0 \in X$. First, suppose that $\deg_X(z_0) \ge 2$. Let $z_1, z_2 \in N_X(z_0)$. Then $xz_1z_0z_2x$ is a 4-cycle with chord xz_0 containing x, a contradiction. Next, suppose that $\deg_M(z_0) \ge 2$. Let $m_1, m_2 \in N_M(z_0)$. Then $xm_1z_0m_2x$ is a 4-cycle with chord xz_0 containing x, a contradiction. Finally, suppose that $\deg_X(z_0) = 1$ and $\deg_M(z_0) = 1$. Let $z_1 \in N_X(z_0)$ and $m \in N_M(z_0)$. Then xmz_0z_1x is a 4-cycle with chord xz_0 containing x, a contradiction. Thus the subclaim holds. □

Let $R = M \cup \{x, y\} \cup X$. Under the condition $X \neq \emptyset$, we claim that $Y \neq \emptyset$. Suppose not. Let $z \in X$. Since $zx \in E(G)$, $\deg_R(z) \le 2$ by Subclaim 2. Since $yz \notin E(G)$, by $\sigma_2(G)$ condition, we have

$$|M| + |\{x, y\}| + |X| + |D| = n \le \deg_G(y) + \deg_G(z) \le |M| + (\deg_R(z) + |D|) \le |M| + 2 + |D|,$$

and then $|X| \le 0$, a contradiction. Thus $Y \neq \emptyset$.

Subclaim 3. $|E(\langle M \cup X\rangle)| \le 1$.

Proof. Suppose that $|E(\langle M \cup X\rangle)| \ge 2$. Note that $|E(\langle M\rangle)| = 0$, since M is an independent set. We consider three cases.

Case 1. $|E(\langle X\rangle)| \ge 2$.

By Subclaim 2, $E(\langle X\rangle)$ is an independent edge set. In this case, note that $|X| \ge 4$. Let $\{z_1z_2, z_3z_4\} \subseteq E(\langle X\rangle)$. Since $z_1z_3 \notin E(G)$, $\deg_G(z_1) + \deg_G(z_3) \ge n$. By Subclaim 2, $\deg_R(z_i) = 2$ for all $i \in \{1, 3\}$. Since $\deg_G(z_i) = \deg_R(z_i) + \deg_{Y\cup D}(z_i) = 2 + \deg_{Y\cup D}(z_i)$ for all $i \in \{1, 3\}$, $\deg_{Y\cup D}(z_1) + \deg_{Y\cup D}(z_3) \ge n - 4$. Suppose that $\deg_{Y\cup D}(z_1) < (n - 4)/2$. Then $\deg_{Y\cup D}(z_3) \ge (n - 4)/2$. Since $z_1z_4 \notin E(G)$, $\deg_G(z_1) + \deg_G(z_4) \ge n$. Since $\deg_{Y\cup D}(z_1) < (n - 4)/2$ by our assumption, $\deg_{Y\cup D}(z_4) \ge (n - 4)/2$ by the same arguments above. If $N_{Y\cup D}(z_3) \cap N_{Y\cup D}(z_4) = \emptyset$, then $|Y \cup D| \ge 2(n - 4)/2 = n - 4$. On the other hand, since $|M| \ge 2$ and $|X| \ge 4$, $|Y \cup D| = |V(G) - (M \cup \{x, y\} \cup X)| \le n - 8$, a contradiction. Thus $N_{Y\cup D}(z_3) \cap N_{Y\cup D}(z_4) \neq \emptyset$. Let $w \in N_{Y\cup D}(z_3) \cap N_{Y\cup D}(z_4)$. Then xz_3wz_4x is a 4-cycle with chord z_3z_4 containing x, a contradiction. Thus $\deg_{Y\cup D}(z_1) \ge (n - 4)/2$. By the same arguments above, we have $\deg_{Y\cup D}(z_2) \ge (n - 4)/2$. Then since $N_{Y\cup D}(z_1) \cap N_{Y\cup D}(z_2) \neq \emptyset$, there exists a chorded 4-cycle containing x, a contradiction.

Case 2. $|E(M, X)| \ge 2$.

Then $E(M, X)$ is an independent edge set by Subclaims 1 and 2. Let $m_1, m_2 \in M$ and $z_1, z_2 \in X$, and let $\{m_1z_1, m_2z_2\} \subseteq E(M, X)$. Since $z_1z_2 \notin E(G)$ by Subclaim 2, $\deg_G(z_1) + \deg_G(z_2) \ge n$. Since $\deg_G(z_i) = \deg_R(z_i) + \deg_{Y\cup D}(z_i) = 2 + \deg_{Y\cup D}(z_i)$ for all $i \in \{1, 2\}$, $\deg_{Y\cup D}(z_1) + \deg_{Y\cup D}(z_2) \ge n - 4$. Suppose that $\deg_{Y\cup D}(z_2) < (n - 4)/2$. Then $\deg_{Y\cup D}(z_1) \ge (n - 4)/2$. Since $m_1z_2 \notin E(G)$ by Subclaim 1, $\deg_G(m_1) + \deg_G(z_2) \ge n$. Then since $\deg_G(m_1) = \deg_R(m_1) + \deg_{Y\cup D}(m_1) = 3 + \deg_{Y\cup D}(m_1)$ and $\deg_G(z_2) = \deg_R(z_2) + \deg_{Y\cup D}(z_2) = 2 + \deg_{Y\cup D}(z_2)$, $\deg_{Y\cup D}(m_1) + \deg_{Y\cup D}(z_2) \ge n-5$. Since $\deg_{Y\cup D}(z_2) < (n-4)/2$ by our assumption, $\deg_{Y\cup D}(m_1) \ge (n - 5)/2$. If $N_{Y\cup D}(z_1) \cap N_{Y\cup D}(m_1) = \emptyset$, then $|Y \cup D| \ge (n - 4)/2 + (n - 5)/2 = n - 9/2$. On the other hand, since $|M| \ge 2$ and $|X| \ge 2$, $|Y \cup D| = |V(G) - (M \cup \{x, y\} \cup X)| \le n - 6$, a contradiction. Thus $N_{Y\cup D}(z_1) \cap N_{Y\cup D}(m_1) \neq \emptyset$. Let $w \in N_{Y\cup D}(z_1) \cap N_{Y\cup D}(m_1)$. Then xm_1wz_1x is a 4-cycle with chord m_1z_1 containing x, a contradiction. Thus $\deg_{Y\cup D}(z_2) \ge (n - 4)/2$. By the same arguments above, since $\deg_G(m_1) + \deg_G(m_2) \ge n$ and $\deg_G(m_2) + \deg_G(z_1) \ge n$, we have

$\deg_{Y\cup D}(m_2) \geq (n-6)/2$. Then since $N_{Y\cup D}(z_2) \cap N_{Y\cup D}(m_2) \neq \emptyset$, there exists a chorded 4-cycle containing x, a contradiction.

Case 3. $|E(\langle X\rangle)| = 1$ and $|E(M, X)| = 1$.

Let $m \in M$, and let $z_1, z_2, z_3 \in X$. Note that $\{mz_1, z_2z_3\}$ is an independent edge set by Subclaim 2. Since $z_1z_2 \notin E(G)$ by Subclaim 2, $\deg_G(z_1) + \deg_G(z_2) \geq n$. Since $\deg_G(z_i) = \deg_R(z_i) + \deg_{Y\cup D}(z_i) = 2 + \deg_{Y\cup D}(z_i)$ for all $i \in \{1,2\}$, $\deg_{Y\cup D}(z_1) + \deg_{Y\cup D}(z_2) \geq n-4$. Suppose that $\deg_{Y\cup D}(z_1) < (n-4)/2$. Then $\deg_{Y\cup D}(z_2) \geq (n-4)/2$. Since $z_1z_3 \notin E(G)$ by Subclaim 2, $\deg_G(z_1) + \deg_G(z_3) \geq n$. Since $\deg_{Y\cup D}(z_1) < (n-4)/2$ by our assumption, $\deg_{Y\cup D}(z_3) \geq (n-4)/2$. If $N_{Y\cup D}(z_2) \cap N_{Y\cup D}(z_3) = \emptyset$, then $|Y \cup D| \geq 2(n-4)/2 = n-4$. On the other hand, since $|M| \geq 2$ and $|X| \geq 3$, $|Y \cup D| = |V(G) - (M \cup \{x, y\} \cup X)| \leq n-7$, a contradiction. Thus $N_{Y\cup D}(z_2) \cap N_{Y\cup D}(z_3) \neq \emptyset$. Let $w \in N_{Y\cup D}(z_2) \cap N_{Y\cup D}(z_3)$. Then xz_2wz_3x is a 4-cycle with chord z_2z_3 containing x, a contradiction. Thus $\deg_{Y\cup D}(z_1) \geq (n-4)/2$. By the same arguments above, since $\deg_G(m) + \deg_G(z_i) \geq n$ for all $i \in \{2,3\}$, we have $\deg_{Y\cup D}(m) \geq (n-5)/2$. Then since $N_{Y\cup D}(z_1) \cap N_{Y\cup D}(m) \neq \emptyset$, there exists a chorded 4-cycle containing x, a contradiction.

Therefore, the subclaim holds. □

Let $h_1 = |E(\langle X\rangle)|$ and $h_2 = |E(M, X)|$, and let $h = h_1 + h_2$. By Subclaim 3, $h \leq 1$.

By Claim 1 and Subclaim 3, G is a graph satisfying the statement (iii), a contradiction. This completes the proof of Claim 3. □

Claim 4. *If G contains a chorded 4-cycle containing x, then there exists a chorded 5-cycle in G containing x.*

Proof. Suppose not. Let $C = v_1v_2v_3v_4v_1$ be a chorded 4-cycle in G containing x. Suppose that $v_2v_4 \in E(G)$. Since $n \geq 6$ and G is connected by $\sigma_2(G)$ condition, there exists some $z \in V(G-C)$ such that $zv \in E(G)$ for some $v \in V(C)$. To get a contradiction, we prove the existence of a chorded 5-cycle containing x. We consider the following cases.

Case 1. $x = v_1$ or $x = v_3$.

By symmetry, we may assume that $x = v_1$. We consider the following cases based on the adjacency of z.

Subcase 1. $zv_1 \in E(G)$.

We claim that $v_i \notin N_C(z)$ for all $2 \leq i \leq 4$. If $v_2 \in N_C(z)$, then $zv_2v_3v_4v_1z$ is a 5-cycle with chord v_1v_2 containing x. If $v_4 \in N_C(z)$, then by symmetry, there exists a chorded 5-cycle containing x. If $v_3 \in N_C(z)$, then $zv_3v_4v_2v_1z$ is a 5-cycle with chord v_1v_4 containing x. Thus the claim holds. Since $zv_2 \notin E(G)$, $|N_G(z) \cap N_G(v_2)| \geq 2$ by $\sigma_2(G)$ condition. By the above claim, there exists some $w \in N_{G-C}(z) \cap N_{G-C}(v_2)$. Then $zwv_2v_4v_1z$ is a 5-cycle with chord v_1v_2 containing x.

Subcase 2. $zv_3 \in E(G)$.

Then note that $v_i \notin N_C(z)$ for all $i \in \{1,2,4\}$. Since $zv_1 \notin E(G)$, $N_{G-C}(z) \cap N_{G-C}(v_1) \neq \emptyset$, and it is Subcase 1.

Subcase 3. $zv_2 \in E(G)$ or $zv_4 \in E(G)$.

By symmetry, we may assume that $zv_2 \in E(G)$. We claim that $v_i \notin N_C(z)$ for all $i \in \{1,3,4\}$. If $v_i \in N_C(z)$ for some $i \in \{1,3\}$, then it is easy to find a chorded 5-cycle containing x. Suppose that $v_4 \in N_C(z)$. If $v_1v_3 \in E(G)$, then $zv_2v_3v_1v_4z$ is a 5-cycle with chord v_1v_2 containing x.

Thus $v_1v_3 \notin E(G)$. By $\sigma_2(G)$ condition, $\deg_G(v_1) \geq 3$ or $\deg_G(v_3) \geq 3$. If $\deg_G(v_1) \geq 3$, then $N_{(G-C)-z}(v_1) \neq \emptyset$, and it is Subcase 1. If $\deg_G(v_3) \geq 3$, then $N_{(G-C)-z}(v_3) \neq \emptyset$, and it is Subcase 2. Thus $v_4 \notin N_C(z)$, and the claim holds. Since $zv_1 \notin E(G)$, $N_{G-C}(z) \cap N_{G-C}(v_1) \neq \emptyset$, and it is Subcase 1.

Case 2. $x = v_2$ or $x = v_4$.

By symmetry, we may assume that $x = v_2$. We consider the following cases based on the adjacency of z.

Subcase 1. $zv_1 \in E(G)$ or $zv_3 \in E(G)$.

By symmetry, we may assume that $zv_1 \in E(G)$. Then note that $v_i \notin N_C(z)$ for all $2 \leq i \leq 4$. Since $zv_2 \notin E(G)$, there exists some $w \in N_{G-C}(z) \cap N_{G-C}(v_2)$. Then $zwv_2v_4v_1z$ is a 5-cycle with chord v_1v_2 containing x.

Subcase 2. $zv_2 \in E(G)$.

Then note that $v_i \notin N_C(z)$ for all $i \in \{1, 3\}$. Since $zv_1 \notin E(G)$, there exists some $w \in N_{G-v_2}(z) \cap N_{G-v_2}(v_1)$. If $w \notin V(C)$, then $zv_2v_4v_1wz$ is a 5-cycle with chord v_1v_2 containing x. Thus $w = v_4$, that is, $zv_4 \in E(G)$. If $v_1v_3 \in E(G)$, then $zv_2v_1v_3v_4z$ is a 5-cycle with chord v_1v_4 containing x. Thus $v_1v_3 \notin E(G)$. By $\sigma_2(G)$ condition, $\deg_G(v_1) \geq 3$ or $\deg_G(v_3) \geq 3$. Then both cases are Subcase 1.

Subcase 3. $zv_4 \in E(G)$.

Then note that $v_i \notin N_C(z)$ for all $i \in \{1, 3\}$. Since $zv_1 \notin E(G)$, there exists some $w \in N_{G-v_4}(z) \cap N_{G-v_4}(v_1)$. If $w \notin V(C)$, then $zwv_1v_2v_4z$ is a 5-cycle with chord v_1v_4 containing x. Thus $w = v_2$, that is, $zv_2 \in E(G)$. If $v_1v_3 \in E(G)$, then $zv_2v_3v_1v_4z$ is a 5-cycle with chord v_1v_2 containing x. Thus $v_1v_3 \notin E(G)$. By $\sigma_2(G)$ condition, $\deg_G(v_1) \geq 3$ or $\deg_G(v_3) \geq 3$. Then both cases are Subcase 1. □

If $n = 6$, then G is chorded vertex pancyclic by Claims 2, 3 and 4. Thus we may assume that $n \geq 7$.

Claim 5. *There exists a chorded k-cycle in G containing x for all $6 \leq k \leq n-1$.*

Proof. Since $G \neq K_{n/2,n/2}$ (n is even) by our assumption, G is vertex 4-pancyclic by Theorem 11. Let $6 \leq k \leq n-1$, and consider a chordless k-cycle $C = v_1v_2 \ldots v_kv_1$ in G containing x. Without loss of generality, we may assume that $x = v_1$. Since C is chordless, $v_1v_3 \notin E(G)$. Then there exists $z \in N_{G-C}(v_1) \cap N_{G-C}(v_3)$. Similarly, since $v_2v_6 \notin E(G)$, there exists $w \in N_{G-C}(v_2) \cap N_{G-C}(v_6)$. If $k = n-1$, then $z = w$, and $zv_3v_4 \ldots v_kv_1z$ is a k-cycle with chord zv_6 containing x. Suppose that $6 \leq k \leq n-2$. If $z = w$, then there exists a chorded k-cycle containing x as above. If $z \neq w$, then $zv_3v_2wv_6 \ldots v_kv_1z$ is a k-cycle with chord v_1v_2 containing x. □

Claims 2 – 5 imply that G is chorded vertex pancyclic. This completes the proof of Theorem 14. □

3 Edge Pancyclic Extensions

A natural variation of vertex pancyclic graphs is that of edge pancyclic graphs.

In [12], a sharp minimum degree condition was established for edge pancyclic graphs. The graph $K_{n/2}, n/2$ shows we cannot reduce this minimum degree by one.

Theorem 15 (Randerath et al. [12])**.** *If G is graph of order n with $\delta(G) \geq \frac{n+2}{2}$, then G is edge pancyclic.*

Our next result extends Theorem 15.

Theorem 16. *If G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n+2}{2}$, then G is chorded edge pancyclic.*

Proof. Let $e = x_1x_2$ be an edge of the graph G. Since G is edge pancyclic, by Theorem 15 e must be contained as a cycle-edge in at least one k-cycle for every k, $3 \leq k \leq n$. By the minimum degree condition, it is clear that every cycle of length at least $n/2 + 2$ must be chorded. Assume for some $k < n/2 + 2$ that none of the k-cycles containing e are chorded. Let $C = x_1, x_2, x_3, \ldots, x_k, x_1$ be such a chordless k-cycle with $k \geq 6$ in G. Since $\delta(G) \geq \frac{n+2}{2}$, every pair of vertices in G share at least two common neighbors. Since C is chordless, there exist vertices $w_1 \in N_{G-C}(x_2) \cap N_{G-C}(x_3)$ and $w_2 \in N_{G-C}(x_3) \cap N_{G-C}(x_6)$ such that $w_1 \neq w_2$. Then $x_1, x_2, w_1, x_3, w_2, x_6, \ldots, x_1$ is a k-cycle containing e as a cycle-edge and x_2x_3 as a chord.

Now let $C' = x_1, x_2, x_3, x_1$ be a 3-cycle in G containing e. There exists a vertex $w \in N_{G-C'}(x_2) \cap N_{G-C'}(x_3)$, so x_1, x_2, w, x_3, x_1 is a 4-cycle containing e as a cycle-edge and x_2x_3 as a chord. Notice that x_2 and w have a common neighbor $w' \neq x_3$. Thus $x_1, x_2, w', w, x_3, x_1$ is a 5-cycle containing e as a cycle-edge and x_2x_3 (and x_2w) as a chord. Therefore e is contained in a chorded k-cycle in G for $4 \leq k \leq n$, so G is chorded edge pancyclic. $\square$

In [5], the idea of edge pancyclic graphs was extended to containing paths.

Definition 2. *If G is a graph of order n. We say G is (P, m)*-**pancyclic** *if any path $P = P_k$ is contained on a cycle of every length from m to n.*

Definition 3. *If G is a graph of order n. We say G is* **chorded** (P, m)-**pancyclic** *if any path $P = P_k$ is contained on a chorded cycle of every length from m to n.*

The next result follows easily form Theorem 16.

Corollary 17. *Given a fixed integer k, let G be a graph of order $n \geq k + 2$ containing a path $P = P_k$ and with $\delta(G) \geq \frac{n}{2} + k - 1$. Then G is chorded $(P, k + 2)$-pancyclic.*

Proof. Suppose G is a graph of order n with $\delta(G) \geq \frac{n}{2} + k - 1$ and let $P = P_k$ be a path in G. We obtain a new graph G' by contracting P to a single edge e. This reduces the minimum degree by up to $k - 2$, the number of interior vertices of P. So

$$\delta(G') \geq \frac{n}{2} + k - 1 - (k - 2) = \frac{n}{2} + 1.$$

By Theorem 16, we know that G' is chorded edge pancyclic. Now expand e back to P to re-obtain G. In doing so, each chorded cycle that contained e in G' is now a chorded cycle that contains P in G. Each such chorded cycle will expand by $k - 2$ vertices when it is re-obtained in G. As e was contained in chorded cycles of length 4 to $n - (k - 2)$ in G', we now have that G is now on cycles of all lengths $4 + k - 2 = k + 2$ to n and so G is chorded $(P, k + 2)$-pancyclic. $\square$

The next result is a consequence of a theorem in [5].

Theorem 18 ([5])**.** *Let G be a graph of order $n \geq 5$ and let e be a edge of G. If $\sigma_2(G) \geq n + 1$, then for reach $r \geq 4$, the graph G contains a cycle of length r containing e.*

Note: The above $\sigma_2(G)$ condition is clearly sharp for general n.

Theorem 19. *Let G be a graph of order $n \geq k+2$. If $\sigma_2(G) \geq n+k-1$, then for any path $P = P_k$, the graph G is $(P, k+2)$-pancyclic.*

Proof. The proof is by induction on k. If $k = 2$ the result follows from Theorem 18. Thus, we assume the result follows if $k = t \geq 2$ and we consider $k = t+1$. Then, in G there is a path $P' = P_{t+1}$ and $\sigma_2(G) \geq n+t$. Now we contract one edge of P' obtaining the graph G^* of order $n-1$ containing the contracted path $P^* = P_t$ satisfying

$$\sigma_2(G^*) \geq n+t-2 = (n-1)+(t-1).$$

Thus, G^* is $(P^*, t+2)$-pancyclic. Expanding P^* back to P' we see that every cycle containing P^* now expands to a cycle containing P'. As these cycles had each length from $t+2$ to $(n-1)$ in G^*,we see that P' now lies on cycles of each length from $t+3$ to n in G. Thus, by induction, we see that G is $(P, k+2)$-pancyclic. □

Example 5. *To see the sharpness of the last result, consider the following graph. Take a copy of K_k, $k \geq 3$, and a copy of $K_{n-k} - e$ where $e = ab$ was an edge of the K_{n-k}. Now select a spanning path $P : x_1, x_2, \ldots, x_k$ of the K_k. We now join x_1 to a and b and x_k to a and b. The vertices $x_2, \ldots, x_{k-1}$ are each joined to all of the K_{n-k}. The resulting graph G has $\sigma_2(G) = n+k-2$ and is realized by the degree sum of x_1 and any vertex $w \in K_{n-k}$ where $w \neq a$ and $w \neq b$ as:*

$$deg\ x_1 + deg\ w = k+1+n-k-1+k-2 = n+k-2.$$

But the path P is not on any cycle of length $k+2$ (although it is on a cycle of length $k+1$).

Acknowledgments. The second author is supported by the Heilbrun Distinguished Emeritus Fellowship from Emory University.

References

[1] J. A. Bondy, Pancyclic Graphs. *Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing*, Louisiana State Univ., Baton Rouge, LA, (1971), 167–172.

[2] J. A. Bondy, Pancyclic graphs I, *J. Combin. Theory Ser. B* 11 (1971), 80–84.

[3] G. Chen, R. J. Gould, X. Gu, A. Saito, The chorded pancyclic problem, Preprint.

[4] M. Cream, R. J. Gould, K. Hirohata, A note on extending Bondy's meta-conjecture, *Australasian journal of combinatorics* Vol. 67 (3) (2017), 463–469.

[5] Faudree, R. J., Gould, R. J., Jacobson, M. S., Lesniak,L., Generalizing pancyclic and k-ordered graphs. *Graphs and Combin.* 20(2004), 291–309.

[6] Faudree, R. J., Gould, R. J., Jacobson, M. S., Pancyclic graphs and linear forests. *Discrete Math.* 309(2009), 1178-1189.

[7] D. Finkel, On the number of independent chorded cycles in a graph, *Discrete Math.* 308 (2008), 5265–5268.

[8] Gould, R. J., *Graph Theory*, Dover Publications Inc., Mineola, NY, 2012.

[9] Gould, R. J., Hirohata, K., Horn, P., On independent doubly chorded cycles, *Discrete Math.* 338 (2015), No.11, 2051–2071.

[10] Hendry, G., Extending cycles in graphs. *Discrete Math.* 83(1990), 59–72.

[11] Ore, O., Note on Hamilton circuits, *Amer. Math. Monthly* 67 (1960), 55.

[12] B. Randerath, I. Schiermeyer, M. Tewes, L. Volkmann, Vertex pancyclic graphs, *Discrete Applied Math.* 120 (2002), 219–237.