Chorded Cycles in Chvátal-Erdős Graphs

Allan Bickle*

Department of Mathematics, Purdue University, 610 Purdue Mall, West Lafayette, 47907, IN, USA

Megan Cream

Department of Mathematics, Lehigh University, 27 Memorial Dr W, Bethlehem, 18015, PA, USA

Ronald J. Gould

Department of Mathematics, Emory University, 201 Dowman Dr, Atlanta, 30322, GA, USA

Abstract

Inspired by the famed Chvátal-Erdős Theorem, we say a Chvátal-Erdős graph G is triangle-free, with $\alpha(G) = \kappa(G)$. Denote a Chvátal-Erdős graph with $k = \alpha(G) = \kappa(G)$ as a $k-CE$ graph. We report the results of a search that found all $k-CE$ graphs for $k \leq 6$, and describe several infinite classes of $k-CE$ graphs.

A cycle C is chorded if there is an edge between two vertices of C that is not an edge of C. We show that every sufficiently long cycle in a regular graph has many chords, and every sufficiently long cycle in a $k-CE$ graph has a chord.

A graph is chorded pancyclic if it contains chorded cycles of each length from 4 to $n(G)$, while G is chorded r-pancyclic if there is a chorded cycle of each length from r to $n(G)$. Further, G is doubly chorded r-pancyclic if G contains cycles of all lengths from r to $n(G)$ that have at least two chords each. We show $k-CE$ graphs to be doubly chorded 8-pancyclic and extend this result to r-chorded graphs for $r \geq 3$.

*Corresponding author

Email addresses: aebickle@purdue.edu (Allan Bickle), macd19@lehigh.edu (Megan Cream), rg@mathcs.emory.edu (Ronald J. Gould)

Keywords: Chvátal-Erdős Theorem, chorded cycle, pancyclic, Ramsey number, circulant graph

1. Introduction

In 1972, Chvátal and Erdős [8] proved a fundamental result about hamiltonian graphs by relating the connectivity $\kappa(G)$ and the independence number $\alpha(G)$ of a graph G.

Theorem 1. (**Chvátal-Erdős Theorem**) *For any graph G, if $\alpha(G) \leq \kappa(G)$, then G is hamiltonian.*

This result is tight in that $\alpha(G)$ cannot be lowered to $\alpha(G) - 1$. They futher showed that if $\alpha(G) < \kappa(G)$ then G has the very strong property of being hamiltonian-connected.

This work is fundamental in several ways. First, it depends only on the structural connectivity versus the independence number, rather than on some sort of edge density condition based on degrees. In fact, the Chvátal-Erdős Theorem generated enough other work that a survey of such work can be found in [25].

At about the same time, Bondy [5] proved several results about pancyclic graphs. A graph G is r**-pancyclic** if G contains cycles of all lengths from r to $n(G)$. A **pancyclic** graph is 3-pancyclic. Other natural extensions of pancyclic graphs have also been developed, see for example [6], [12] and [13].

Bondy [5] also made his famed meta-conjecture that almost any condition that implies a graph is hamiltonian also implies it is pancyclic. This inspired further work on graphs with $\alpha(G) \leq \kappa(G)$. In fact, the Chvátal-Erdős paper generated enough further research that two surveys of such work can be found in [18] and [25].

Amar et al. [1] conjectured that if $\alpha(G) \leq \kappa(G)$ and G is not bipartite, then G is 4-pancyclic. Lou [19] considered this conjecture and proved the following.

Theorem 2. (**Lou's Theorem** [19]) *If G is a triangle-free graph of order $n \geq 4$ with $\alpha(G) \leq \kappa(G)$,*
1. G is k-regular with $k = \alpha(G) = \kappa(G)$.
2. G has diameter 2.
3. G is 4-pancyclic, or $G \in \{K_{r,r}, C_5\}$.

We define a **Chvátal-Erdős graph** to be a triangle-free graph with $\alpha(G) = \kappa(G)$. Denote a Chvátal-Erdős graph with $k = \alpha(G) = \kappa(G)$ as a $k-CE$ graph.

This name is inspired by the Chvátal-Erdős Theorem. Graphs with $\alpha(G) = \kappa(G)$ (not necessarily triangle-free) were denoted as such in [1]. In addition, Erdős asked in 1986 for which orders these graphs exist. We will also see that the Chvátal graph (4-regular, triangle-free, 4-chromatic with order 12) is a Chvátal-Erdős graph.

Properties of these graphs are studied in [1, 19]. Several examples of these graphs are given in [1, 6], and they have been studied in relation to independence number [3, 6], fractional chromatic number and toughness [6].

Chorded cycles have recently attracted significant attention. An edge between two vertices of a cycle C is a **chord** if it is not an edge of C. A cycle C is a **chorded cycle** if the vertices of C induce at least one chord. Pósa [23] asked what conditions imply a graph contains a chorded cycle. This question has seen considerable interest and extensions lately (see for example [9], [14], [15] and [17]).

In this paper we consider a merging of pancyclic graphs and chorded cycles. We say a graph G is **t-chorded r-pancyclic** if G contains cycles of each length from r to $n(G)$, with each such cycle having at least t chords. We use **chorded r-pancyclic** for 1-chorded r-pancyclic and **doubly chorded r-pancyclic** for 2-chorded r-pancyclic. In [12], Lou's Theorem was extended as follows.

Theorem 3. [12] *Let $G \neq K_{n/2,n/2}$ be a $k-CE$ graph of order $n \geq 13$. Then G is chorded 8-pancyclic.*

The goal of this paper is to study the case when $\alpha(G) = \kappa(G)$ in more detail to better understand the graphs at the boundary of Theorem 1. We further wish to extend Theorem 3, and in so doing, also extend Lou's Theorem. Thus we will gain more information on strong cycle properties in $k-CE$ graphs.

In Section 2, we use theoretical results and a computer search to describe all $k - CE$ graphs with $k \leq 6$. In Section 3, we describe several infinite classes of Chvátal-Erdős graphs, particularly those that are circulant. In Section 4, we prove two results that guarantee many chords of large cycles in Chvátal-Erdős graphs. In Section 5, we prove several results on extending chorded cycles in Chvátal-Erdős graphs.

We consider only simple graphs in this paper. We use the standard notation of $V(G)$ and $E(G)$ for the vertex set and edge set, and of the graph G. The order of G is $n(G)$; the minimum degree is $\delta(G)$. Let $K_{a,b}$ denote the complete bipartite graph with parts of order a and b. Let C_n and P_n denote the cycle and path with n vertices. Let $N_H(x)$ denote the set of neighbors of the vertex x in the graph (or subgraph) H. Given an orientation of some path or cycle, we denote by x^+ and x^- the successor and predessor of the vertex x following the given orientation. Further, let $x^{+2} = (x^+)^+$ and similarly, let $x^{-2} = (x^-)^-$, etc. Let $d(u, v)$ denote the distance in the graph between vertices u and v. Given a subgraph or vertex subset S, let $G - S$ be the graph obtained by removing S from G. For terms not defined here see [16].

2. Small Chvátal-Erdős Graphs

In this section, we prove basic results on $k-CE$ graphs.

Corollary 4. *The only $1-CE$ graph is K_2. The only $2-CE$ graphs are C_4 and C_5.*

This follows immediately from Theorem 2, since these are the only 2-regular triangle-free graphs with diameter 2.

Next we give sharp bounds on the order of $k-CE$ graphs. The upper bound depends on the **Ramsey number** $R(s, t)$, the smallest order n such that in any 2-decomposition of K_n, the first factor contains K_s or the second factor contains K_t.

Proposition 5. *If G is a $k-CE$ graph, then $2k \leq n(G) \leq R(3, k+1) - 1$. The unique extremal graph for the lower bound is $K_{k,k}$.*

Proof. Let uv be an edge of G. Since G is triangle-free, $N(u)$ and $N(v)$ are disjoint independent sets, so there are at least $2k$ vertices. If there are no other vertices, G is bipartite, so $G = K_{k,k}$.

The Ramsey number $R(3, k+1)$ is the smallest order so that any graph G of this order either contains a triangle, or $\overline{G}$ contains K_{k+1}. Thus $R(3, k+1) - 1$ is the largest order so that there is a triangle-free graph with $\alpha(G) = k$. □

Note that $R(3, k)$ is only known exactly for $k \leq 9$, but it is known that $R(3, k) \leq \frac{k^2}{\log k}$ [24]. The only bipartite $k-CE$ graph is $K_{k,k}$, since any bipartite graph has an independent set with at least half its vertices.

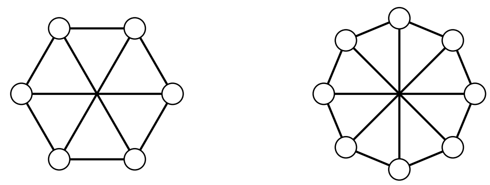

Figure 1: The two $3-CE$ graphs: $K_{3,3}$ and M_8.

Proposition 6. *The only* $3-CE$ *graphs are* $K_{3,3}$ *and the Mobius ladder* M_8 *(see Figure 1).*

Proof. Any $3-CE$ graph is cubic with order at most $R(3,4)-1=8$. The only $3-CE$ graph with order 6 is $K_{3,3}$. There are six cubic graphs with order 8 (see [10, 11]), only one of which (M_8) is triangle-free and not bipartite. Clearly $\alpha(M_8)=\kappa(M_8)$. □

The **circulant graph** $C_n(l_1,\ldots,l_r)$ has vertices $v_1,\ldots v_n$ and all edges v_iv_j whenever $|i-j|=l_k \mod n$, $1\le k\le r$. The values l_k are **jumps**. Note that $C_n=C_n(1)$ and the Mobius ladder $M_n=C_n\left(1,\frac{n}{2}\right)$, n even. Note that $K_{k,k}$ is also circulant with all possible odd jumps present.

Any circulant graph is vertex-transitive. The description of its jumps is usually not unique. For example, $C_{11}(1,3)=C_{11}(1,4)=C_{11}(2,3)=C_{11}(2,5)=C_{11}(4,5)$.

Theorem 7. *The only* $4-CE$ *graphs are* $K_{4,4}$, $C_{10}(1,4)$, $C_{11}(1,3)$, $C_{12}(2,3)$, *the Chvátal Graph, and* $C_{13}(1,5)$.

Proof. Since $R(3,5)=14$, a $4-CE$ graph has order at most 13. A $4-CE$ graph has order at least 8, and the only such graph is $K_{4,4}$. To search orders 9-13, we consult several databases.

The Combinatorial Object Server [10] can be searched for 4-regular triangle-free graphs. These can be displayed in graph6 format. This can be input into House of Graphs [11] to generate drawings of the graphs, which can easily be checked for independence number.

For $n=9$ there are no graphs to check. For $n=10$ there are two graphs to check, $C_{10}(1,3)$ and $C_{10}(1,4)$. The first is bipartite (and has $\alpha=5$), the latter is a $4-CE$ graph. For $n=11$ there are two graphs to check. The first

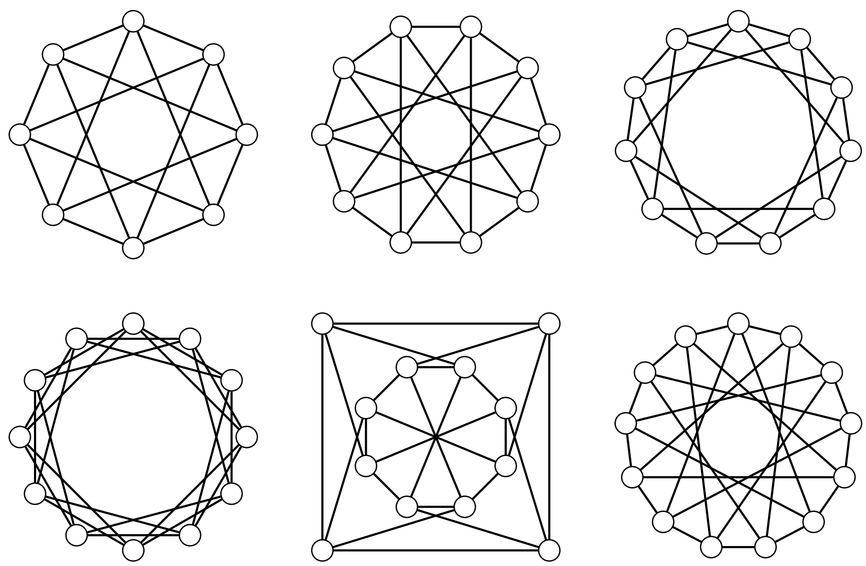

Figure 2: The six $4-CE$ graphs: $K_{4,4}$, $C_{10}\,(1,4)$, $C_{11}\,(1,3)$, $C_{12}\,(2,3)$, the Chvátal Graph, and $C_{13}\,(1,5)$.

has $\alpha = 5$, and the second, $C_{11}\,(1,3)$, works. For $n = 12$ there are 12 graphs to check. There are two, $C_{12}\,(2,3)$ and the Chvátal Graph, that work.

McKay [21] has a database of Ramsey $(3,5)$-graphs, which have no triangle and no independent set with size at least 5. These are displayed in graph6 format, and can be drawn using House of Graphs. For $n = 13$ there is a unique graph, $C_{13}\,(1,5)$, which works. $\square$

Note that [1] observed $K_{4,4}$, $C_{10}\,(1,4)$, and $C_{13}\,(1,5)$ are Chvátal-Erdős graphs. Thus the Chvátal Graph is the smallest $k-CE$ graph that is not circulant, since it has two vertex orbits.

Theorem 8. *There are 13 $5-CE$ graphs.*

Proof. There are no $5-CE$ graphs with odd order since they must be regular. There is one ($K_{5,5}$) with order 10. A search using COS found none of order 12 and one of order 14, $C_{14}\,(1,4,7) = C_{14}\,(5,6,7)$. Any others must have order 16. Brendan McKay conducted search of the database of Ramsey $(3,6)$-graphs [21] for those that are 5-regular. The search found 11 $5-CE$ graphs with order 16. $\square$

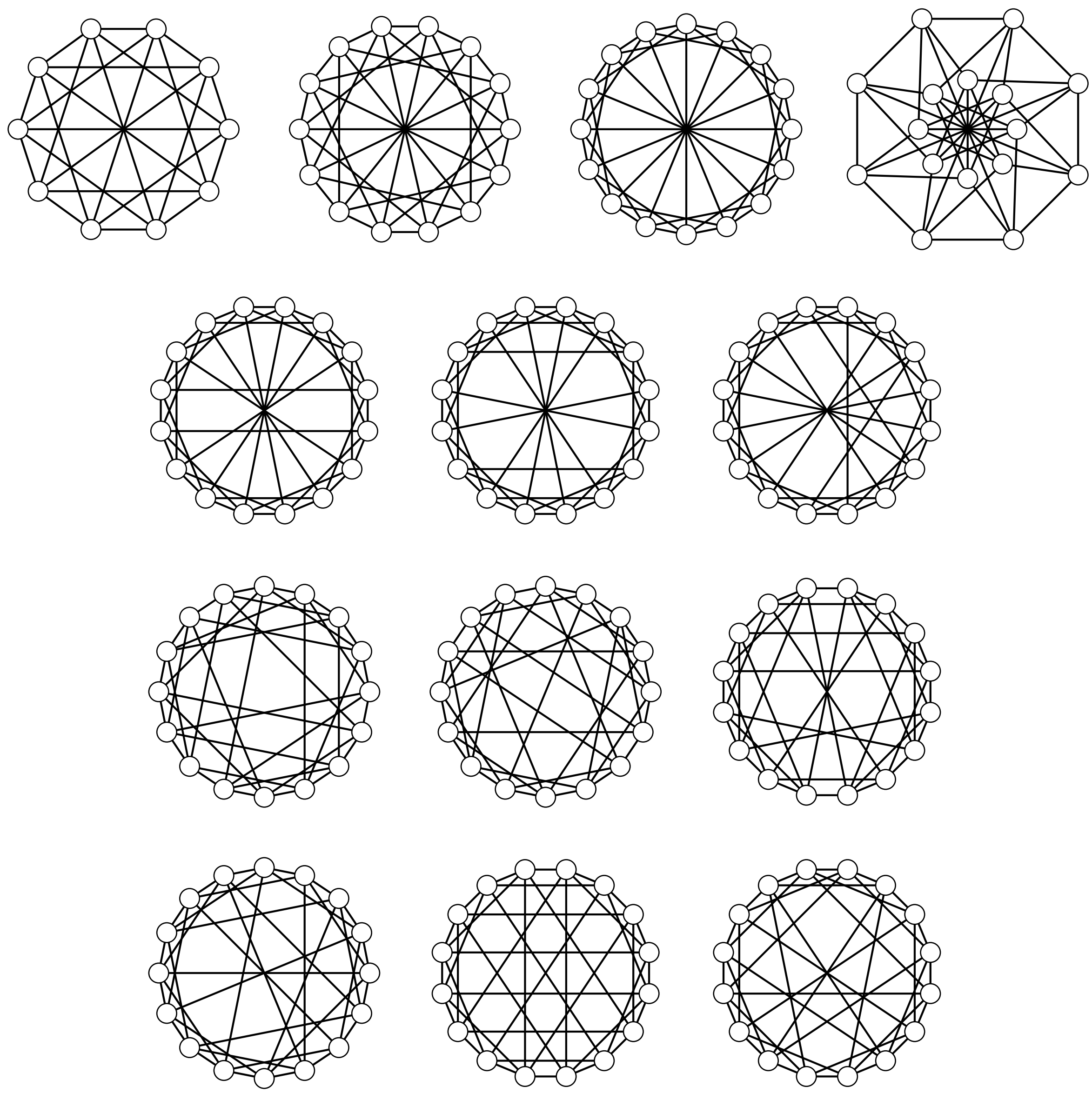

Figure 3: All 13 $5-CE$ graphs. The first row is $K_{5,5}$, $C_{14}\,(1,4,7)$, $C_{16}\,(1,3,8)$, and the Clebsch graph.

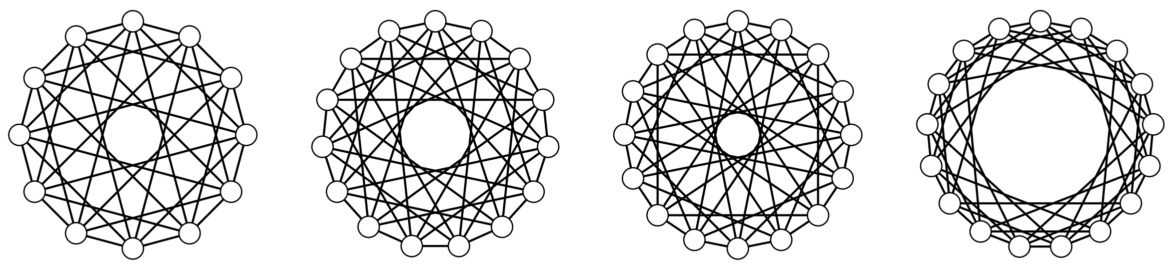

Figure 4: Four $6-CE$ graphs: $K_{6,6}$, $C_{15}(1,4,6)$, $C_{16}(1,4,7)$, and $C_{17}(1,3,5)$.

All $5-CE$ graphs have been added to House of Graphs ($K_{5,5}$ and Clebsch were already present). Those added are numbers 53516 to 53526. This allowed the calculation of various parameters, including the length of the longest induced cycle. Note that of the 13 $5-CE$ graphs, only $K_{5,5}$, $C_{14}(1,4,7)$, and $C_{16}(1,3,8)$ are circulant. The fact that the other $5-CE$ graphs (aside from the Clebsch graph) are not circulant follows from the fact that they have more than one vertex orbit, as calculated by House of Graphs. Note that graphs 53517 and 53520 can be formed by performing a 2-switch on two edges of the circulant graph $C_{16}(1,3,8)$.

A search by Brendan McKay found 602 $6-CE$ graphs. A file containing the graphs in graph6 format is available from the authors on request. All $6-CE$ graphs with at most 18 vertices have been added to House of Graphs ($K_{6,6}$ and $C_{15}(1,4,6)$ were already present). Those added are numbers 53527 to 53532. The $6-CE$ graphs of order 22 are graphs 1991, 2956, and 53583 to 53590. None of these graphs are vertex-transitive, and hence none are circulant.

Table 1 has the number of $k-CE$ graphs of order n. The values in the last two rows come from a search by Brendan McKay.

3. Classes of Circulant $k-CE$ Graphs

Next we consider some general constructions of classes of Chvátal-Erdős graphs. Several basic classes of $k-CE$ graphs are circulant graphs.

$k\backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1																				
2			1	1																	
3					1		1														
4							1		1	1	2	1									
5									1				1		11						
6											1			1	1	1	4	21	232	331	10

Table 1: The number of $k-CE$ graphs of order n.

Note that a connected k-regular circulant graph need not be k-connected. There is a complete characterization of the circulant graphs with $\kappa(G) < k$ in [4]. For example, $C_8(1,3,4)$ is 5-regular with connectivity 4.

Theorem 9. [20] *Any connected K_4-free k-regular vertex-transitive graph G has $\kappa(G) = k$.*

Of course, this includes all connected triangle-free circulant graphs.

Erdős asked in 1986 for which orders Chvátal-Erdős graphs exist. Sidorenko [26] showed they exist for every order $n \notin \{1,3,7,9\}$.

Theorem 10. (Sidorenko [26]) *If $6r-2 \leq n \leq 8r-3$ then $C_n(r,\ldots,2r-1)$ is a $2r-CE$ graph.*

For $r=1$, this gives C_4 and C_5. For $r=2$, this gives the four circulant graphs of orders 10-13. For even k, this gives k circulant $k-CE$ graphs.

Theorem 11. (Andrasfai [2]) *The circulant graph $G = C_{3k-1}\left(k,\ldots,\left\lfloor\frac{3k-1}{2}\right\rfloor\right)$ is a $k-CE$ graph.*

The graphs in the previous theorem are known as **Andrasfai graphs** [2]. They can also be described as $G = C_{3k-1}\left(1,4,7,\ldots,3\left\lceil\frac{k}{2}\right\rceil-2\right)$. This can be seen by multiplying the jumps of the latter by k, or the former by 3.

It is easy to see that $C_{3k-1}\left(k,\ldots,\left\lfloor\frac{3k-1}{2}\right\rfloor\right)$ has the property that any independent set is contained in the neighborhood of a vertex (hence any maximum independent set is the neighborhood of a vertex). Pach [22] and Brouwer [7] proved this, and proved the stronger result that a triangle-free graph with no twin vertices so that each independent set lies in the neighborhood of a vertex is an Andrasfai graph.

Ayat et al [3] proved a similar result to the following theorem without considering connectivity.

Theorem 12. *For $r \geq 1$, $G = C_{8r}(1, 3, \ldots, 2r-1, 4r)$ is a $2r+1-CE$ graph.*

Proof. Note that G is triangle-free since no pair of odd numbers sums to $4r$, and no three odd numbers sum to $8r$. Thus $\kappa(G) = k$ by Theorem 9.

Suppose there is an independent set of $2r+2$ vertices. We can specify the independent set by the consecutive differences in the indices of its vertices. Suppose all these differences are even. Then we have $2r+2$ vertices out of the $4r$ (say) even-indexed vertices. By the Pigeonhole Principle, there must be some pair of vertices whose indices differ by $4r$. But then these vertices are adjacent.

If there is an odd difference, there are at least two odd differences that are at least $2r+1$. Then there are $2r$ other differences that are all at least 2. The sum of all the differences is at least $2(2r+1) + 2r \cdot 2 = 8r + 2 > 8r$, a contradiction. $\square$

The **lexicographic product** of G and H, $G[H]$, has vertex set $V(G) \times V(H)$ and (u, v) adjacent to (u', v') if $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Note that [1] observed $C_4\left[\overline{K}_r\right]$ and $C_5\left[\overline{K}_r\right]$ are Chvátal-Erdős graphs. In fact, $C_4\left[\overline{K}_r\right] = K_{2r,2r}$ and $C_5\left[\overline{K}_2\right] = C_{10}(1, 4)$.

The following result is stated in [26, 1] without proof.

Theorem 13. *If G is a $k-CE$ graph, then $G\left[\overline{K}_r\right]$ is a $kr-CE$ graph.*

Proof. Let G be a $k-CE$ graph, so G is triangle-free, and $k = \alpha(G) = \kappa(G)$. If $v \in V(G)$, let S_v be the corresponding set of r vertices in $G\left[\overline{K}_r\right]$. Now $G\left[\overline{K}_r\right]$ is triangle-free, since otherwise there would be a triangle on the corresponding vertices of G.

If $uv \in E(G)$, then all vertices in S_u and S_v are adjacent. Thus any independent set of $G\left[\overline{K}_r\right]$ contains vertices in at most k sets S_i, so $\alpha\left(G\left[\overline{K}_r\right]\right) = kr$.

Clearly $G\left[\overline{K}_r\right]$ is kr-regular, so $\kappa\left(G\left[\overline{K}_r\right]\right) \leq kr$. There are k independent paths between nonadjacent vertices u and v in G. Thus there are kr independent paths between nonadjacent vertices $u' \in S_u$ and $v' \in S_v$ in G. Similarly, there are kr independent paths between nonadjacent vertices $u', u'' \in S_u$. By Menger's Theorem, $\kappa\left(G\left[\overline{K}_r\right]\right) = kr$. $\square$

A lexicographic product of a circulant graph and an empty graph is itself a circulant graph.

Lemma 14. *If G is a circulant graph with jumps a_i, $1 \le i \le p$, then $G\left[\overline{K}_r\right]$ is a circulant graph with jumps $a_i + jn$, $1 \le i \le p$, $0 \le j \le r-1$.*

Proof. Let the vertices of G be v_l, $1 \le l \le n$. Let the vertices in $G\left[\overline{K}_r\right]$ corresponding to v_l be $v_{l,j}$, $1 \le j \le r$. List the vertices of $G\left[\overline{K}_r\right]$ with the order

$$v_{1,1}, v_{2,1}, \ldots, v_{n,1}, v_{1,2}, v_{2,2}, \ldots, v_{n,2}, \ldots, v_{1,r}, v_{2,r}, \ldots, v_{n,r}.$$

A jump a_i in G corresponds to jumps of length $a_i + jn$, $0 \le j \le r-1$, in $G\left[\overline{K}_r\right]$. Thus, $G\left[\overline{K}_r\right]$ is a circulant graph with the stated jumps. $\square$

Ayat et al. [3] proved that for $r > 1$, there does not exist any $2r$-regular circulant triangle-free graph on $4r+1$ vertices, and there does not exist any $(2r+1)$-regular circulant triangle-free graph on $4r+4$ vertices. The following theorem generalizes these results.

Theorem 15. *A non-bipartite $k-CE$ graph has order $n \ge \frac{5}{2}k$. The unique graph that attains this is $C_5\left[\overline{K}_r\right]$, $r = \frac{k}{2}$.*

Proof. Let G be a non-bipartite $k-CE$ graph of minimum order. Let uv be an edge of G. Since G is triangle-free, $N(u)$ and $N(v)$ are disjoint independent sets of k vertices. Since G is non-bipartite with diameter 2, there is another vertex x, and it has neighbors in each of $N(u)$ and $N(v)$. If x has d neighbors in $N(u) \cup N(v)$, x has $k-d$ other neighbors. Thus x has at least $\frac{d}{2}$ neighbors in (say) $N(u)$, so any vertex in $N(v) \cap N(x)$ has at least $\frac{d}{2}$ neighbors not in $N(u) \cup N(v)$. Thus there are at least $\max\left\{k-d, \frac{d}{2}\right\}$ vertices not in $N(u) \cup N(v)$. This is minimized when $d = k$, so G has at least $2k + \frac{k}{2} = \frac{5}{2}k$ vertices.

Let k be even. Let $V_1 = N(v) \cap N(x)$, $V_2 = N(u) - N(x)$, $V_3 = N(v) - N(x)$, $V_4 = N(u) \cap N(x)$, and $V_5 = V(G) - N(u) - N(v)$. We know that each set has size $r = \frac{k}{2}$, and there are no $V_1 - V_4$ edges, $V_1 - V_3$ edges, or $V_2 - V_4$ edges. For the vertices to have degree k, all $V_1 - V_2$ $V_1 - V_5$ $V_3 - V_4$, and $V_4 - V_5$ edges must be present. Then there are no $V_2 - V_5$ or $V_3 - V_5$ edges, so all $V_2 - V_3$ edges are present. Then $G = C_5\left[\overline{K}_r\right]$. $\square$

The proof uses the facts that $k-CE$ graphs are k-regular, triangle-free, and have diameter 2, but does not use the independence number or connectivity. Thus the theorem actually holds for a larger class of graphs.

Note for $k = 2r+1$ odd, a non-bipartite $k-CE$ graph has order $n \ge \frac{5k+1}{2}$. This can only be attained when r is odd (else k and n are both odd).

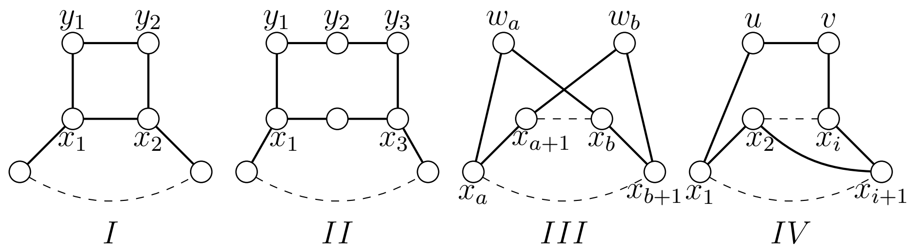

Figure 5: Four structures producing chorded $(k+2)$-cycles. A dashed edge indicates a path with length at least 1.

Note that in $C_5\left[\overline{K}_r\right]$, the longest chordless cycle has length 5. This follows because any longer cycle must contain multiple vertices corresponding to a single vertex of C_5, which forces a chord with a neighboring vertex.

4. Extending Chorded Cycles in $k - CE$ Graphs

In this section we use a different approach to produce chorded cycles in $k-CE$ graphs. This will improve on the results of the previous section when the cycles are relatively small.

Given a cycle $C_r = x_1x_2\dots x_rx_1$ with $r \leq n(G) - 2$, Lou [19] provided several supergraph structures, one of which would contain C_{r+2}. These can be seen in Figure 5 and will be denoted structures I, II, III, and IV.

Lemma 16. [19] *Given a k-CE graph G of order $n(G) \geq r+2$ with $r \geq 4$, if G contains a cycle C_r, then C_r is contained in one of the structures I, II, III, or IV from Figure 5.*

Lemma 17. *Let $r \geq 4$ be an integer. Given a $k-CE$ graph G and a cycle C_r with $r \leq n(G) - 2$, if C_r is contained in one of structures I, III, or IV, then G contains a cycle C_{r+2} with at least one additional chord. Further, if C_r is contained in either of structures III or IV, then there exists a C_{r+2} containing two chords that were not induced by C_r.*

Proof. Let $C_r : x_1, x_2, \dots, x_r, x_1$.
Case 1. Suppose that C_r is contained in structure I. Let adjacent vertices y_1 and y_2 be such that, without loss of generality, y_1x_1 and y_2x_2 are edges of G

completing structure I. Then $C: x_1, y_1, y_2, x_2, x_3, \ldots, x_r, x_1$ is an $(r+2)$-cycle with the chord x_1x_2.
Case 2. Suppose that C_r is contained in structure III. Let vertices w_a and w_b be such that, without loss of generality, w_ax_a and w_ax_b and $w_bx_{a^+}$ and $w_bx_{b^+}$ are edges of G, where $a^+ < b$, completing structure III. Then

$$x_a, w_a, x_b, x_{b^-}, \ldots, x_{a^+}, w_b, x_{b^+}, x_{b^{+2}}, \ldots, x_a$$

is a $(r+2)$-cycle with additional chords $x_ax_{a^+}$ and $x_bx_{b^+}$.
Case 3. Suppose that C_r is contained in structure IV. Without loss of generality assume the edge uv has ux_1 and vx_i as edges to C_r as well as the edge x_2x_{i+1} forming structure IV. Then the cycle

$$x_1, u, v, x_i, x_{i-1}, \ldots, x_2, x_{i+1}, \ldots, x_r, x_1$$

has length $r+2$ and has the additional chords x_1x_2 and x_ix_{i+1} as desired. □

An examination of the k-CE graphs found in Proposition 7 and Theorem 8, shows that every one that is not bipartite contains a chorded 6-cycle and chorded 7-cycle. For $k \geq 5$, the k-CE graphs all have order at least 14. Thus, we state the following slight improvement of Theorem 2 which will be useful later.

Theorem 18. *Let $G \neq K_{n/2,n/2}$ be a $k-CE$ graph of order $n \geq 8$. Then G is chorded 6-pancyclic.*

The next lemma is obvious.

Lemma 19. *Suppose $k \geq 2$. Given a list of at least $2k$ items that is 2-colored red and blue, if there are at most $k-1$ items colored red, then there are at least two consecutive items colored blue.*

Next we prove our final three results on chorded cycles, beginning with a proposition.

Proposition 20. *If $G \neq K_{n/2,n/2}$ is a $k-CE$ graph of order $n \geq 8$, then G contains a doubly chorded 8-cycle and, if $n \geq 10$, a doubly chorded 9-cycle.*

Proof. Suppose the result fails to hold. By Theorem 2, G contains a cycle $C : x_1, x_2, \ldots, x_t, x_1$ where $t \in \{6, 7\}$. However, G must lack a doubly chorded C^* of order 8 or 9. By Lemma 16, C must be contained in one of the structures shown in Figure 5.

Case 1. Suppose a copy of structure I exists containing C. Without loss of generality, let the edge uv have the additional edges ux_1 and vx_2 forming structure I. Note that the 8 or 9-cycle (depending on $t = 6$ or $t = 7$)

$$C^* : x_1, u, v, x_2, x_3, \dots, x_t, x_1.$$

has chord x_1x_2.

Now consider the vertex pairs $\{x_2, x_5\}$ and $\{x_3, x_t\}$. Both pairs are at distance three on C. If either pair is joined by a chord, then C^* has two chords as desired. Thus, by Theorem 2, both pairs must be joined by paths of length two. Suppose x_2, w_2, x_5 and x_3, w_3, x_t are the two distance paths in G, where $w_2, w_3 \notin V(C_6)$. Now if $w_2 = v$ or $w_2 = u$, then the edge w_2x_5 would be the second chord for C^*. Further, $w_2 \neq w_3$ or a triangle would exist in G, a contradiction. Hence, these are distinct vertices not in $V(C) \cup \{u, v\}$. As such these distance paths form structure III on C and by Lemma 17, an 8 or 9-cycle with at least two chords results.

Case 2. Suppose a copy of C_6 is contained in structure III or IV. Then by Lemma 17 an 8 or 9-cycle with at least two chords exists in G.

Case 3. Suppose C is contained in a copy of structure II. Without loss of generality, let the path y_1, y_2, y_3 along with the edges y_1x_1 and y_3x_3 complete the structure. Note that

$$C^* : x_1, y_1, y_2, y_3, x_3, \dots, x_t, x_1$$

is an 8 or 9-cycle in G. If this C induces two chords not incident to x_2, then C^* has the desired two chords. So we may assume this does not happen.

Consider the vertex pairs $\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_t\}$. These pairs are at distance three on C. Thus, by Theorem 2, shorter paths must join each pair. If the first and third pairs are joined by an edge, then two chords exist in C^*. If both pairs are joined by paths of length two, say x_1, w_1, x_4 and x_3, w_3, x_t are the paths, where $w_1, w_3 \notin V(C)$, then $w_1 \neq w_3$ or a triangle would exist.

Suppose $w_j \in \{y_1, y_2, y_3\}$ where $j = 1$ or 3. Then, $w_j \neq y_2$ or a triangke would exist, a contradiction. Therefore suppose $w_1 = y_3$ (or $w_3 = y_1$). The w_1y_4 completes the triangle y_3, x_3, x_4, y_3, a contradiction. Similarily, if $w_3 = y_1$ the edge w_3x_t completes a triangle.

Now,

$$x_1, w_1, x_4, \dots, x_t, w_3, x_3, x_2, x_1$$

is an 8 or 9-cycle with chords x_3x_4 and x_1x_t. Hence, one of these pairs is joined by a chord and the other by a path of length two. Without loss of generality, suppose x_1x_4 is an edge of G and x_3, w_3, x_t the path of length two. This implies that the pair $\{x_2, x_5\}$ must be joined by an edge or structure III would exist. But now,

$$x_2, x_5, x_4, x_3, y_3, y_2, y_1, x_1, x_2$$

is an 8-cycle with chords x_1x_4 and x_2x_3. Further,

$$x_1, x_7, x_6, x_5, x_4, x_3, y_3, y_2, x_2, x_1$$

is a 9-cycle with chords x_1x_4 and x_2x_3.

Thus, in all cases an 8 and 9-cycle with at least two chords exists.

□

Theorem 21. *Let G be a $k-CE$ graph of order $n \geq 8$. Then G is doubly chorded 8-pancyclic, or $G = K_{n/2,n/2}$.*

Proof. Assume $G \neq K_{n/2,n/2}$. Proposition 20 provides the cases for 8-cycles and 9-cycles. By Theorem 18, there must exist an integer r with $8 \leq r \leq n-2$, such that G contains a chorded C_r, but no doubly chorded C_{r+2}. We next show that each of the structures in Figure 5 on a chorded C_r, actually provides a doubly chorded C_{r+2}. Assume that $C_r : x_1, x_2, \ldots, x_r$ is a chorded cycle.

Case 1. If structure I contains C_r, we are done by Lemma 17 as the C_{r+2} contains $V(C_r)$ and hence all its chords as well. Structure 1 provides an extra chord.

Case 2. Suppose G contains a copy of structure II on C_r. By Theorem 18, let $C_r = x_1, x_2, , \ldots, x_r, r \geq 8$, be a chorded r-cycle. By Proposition 20, if $r = 8$ or 9 we are done, so we may assume $r \geq 10$. Without loss of generality, suppose that v_1, v_2, v_3 is a path off C_r and v_1x_1 and v_3x_3 are edges of G forming structure II with C_r.

Now $d(x_3, x_6)$, $d(x_4, x_7)$, $d(x_5, x_8)$ and $d(x_6, x_9)$ are all greater than distance two on C_r, so by Theorem 2(2), each of these pairs of vertices must have shorter distance paths. If two of these pairs had edges joining them, then the cycle $C^* : x_1, v_1, v_2, v_3, x_3, \ldots, x_r, x_1$ would be a $(r+2)$-cycle with those two edges as chords. Thus, at most one of these pairs is joined by an edge. But then structure III containing C_r exists and by Lemma 17, a cycle of length $r+2$ exists with at least two chords. Suppose C_r is contained in either structure III or IV. Then again by Lemma 17, a cycle of length $r+2$ exists with at least two chords.

This completes the proof. $\square$

We now present our final result.

Theorem 22. *Let $G \neq K_{n/2,n/2}$ be a $k-CE$ graph of order $n \geq 8$. Then for $(n-4)/2 \geq t \geq 1$, G is t-chorded $(2t+4)$-pancyclic.*

Proof. Assume that $G \neq K_{n/2,n/2}$. We proceed by induction on t, the number of chords. When $t=1$ the result holds by Theorem 18. Hence, consider $t \geq 2$. By induction we can assume the result holds for $t = m-1 \geq 2$ and we consider the case when $t = m$.

If the result fails for m chords, then there exists a cycle of order at least $2m+4$ that fails to have m chords. By the induction hypothsis there exists a cycle C_p, $2m+2 \leq p \leq n-2$, which is $(m-1)$-chorded. Let C_p : $x_1, x_2, \ldots, x_p, x_1$ be such a cycle with at least $m-1$ chords.

If C_p is contained in any of structures I, III, or IV, we obtain a cycle of order $p+2$ with at least m chords by Lemma 17. Thus, we may assume that C_p is contained in a copy of structure II. Without loss of generality, we may assume that the path y_1, y_2, y_3 off C_p has adjacencies y_1x_1 and y_3x_3 creating structure II.

Now if C_p has at least m chords none of which are incident to x_2, then the cycle

$$C^* : x_1, y_1, y_2, y_3, x_3, x_4, \ldots, x_p, x_1$$

has length $p+2$ and contains at least m chords as desired.

Thus we may assume that there are at most $m-1$ chords in C_p not incident to x_2. Consider the $p-2$ vertex pairs

$$\{x_3, x_6\}, \{x_4, x_7\}, \ldots, \{x_{p-3}, x_p\}, \ldots, \{x_p, x_3\}$$

Each of these pairs is presently at distance three on C_p. But there are only $m-1$ chords possible and there are $p \geq 2m+2$ pairs. Thus, to reduce these distances to at most two, the pairs are either joined by a chord of G or there exists a vertex off C_p that lies on a path of length two between the vertices of the pair. Hence, by Lemma 19, there is some pair of consecutive vertices along C_p that have paths of length two to their paired vertices. Further, these intermediate vertices must be off C_p and must be distinct or a triangle would be formed, a contradiction. But now structure III exists on C_p and by Lemma 17 there exists a cycle of length $p+2$ with at least m chords, completing the induction. $\square$

5. Conclusion

It is clear from recent work that many conditions implying various cycle properties in a graph can be used to show stronger results concerning cycles with chords. This paper provides several examples of this. Broadening the meta-conjecture from [15], it appears that almost any condition that implies some cycle property in a graph also implies a chorded cycle property, possibly with some families of exceptional graphs, and small order exceptions.

Acknowledgement: Thanks to Brendan McKay for searching for $5-CE$ graphs and $6-CE$ graphs.

The second author is supported by the Bianchi - Bugge Award from Emory Univerity Emeritus College.

References

[1] D. Amar, J. Fournier, and A. Germa, Pancyclism in Chvátal-Erdős graphs. *Graphs and Combin.* 7 (1991), 101–112.

[2] B. Andrásfai, Graphentheoretische Extremalprobleme, *Acta Math. Acad. Sci. Hungar.* 15 (1964), 413-438.

[3] S. Masih Ayat, S. Majid Ayat, and M. Ghahramani, The independence number of circulant triangle-free graphs, *AIMS Mathematics* 5 4 (2020), 3741–3750.

[4] F. Boesch and R. Tindell, Circulants and their connectivities, *J. Graph Theory* 8 (1984), 487–499.

[5] J. A. Bondy, Pancyclic graphs I. *J. Combin. Theory Ser. B* 11, (1971), 80–84.

[6] S. Brandt, Sufficient conditions for graphs to contain all subgraphs of a given type. Ph.D. Thesis, Freie Universitat Berlin, 1994.

[7] A. E. Brouwer, Finite Graphs in Which the Point Neighbourhoods Are the Maximal Independent Sets, In From Universal Morphisms to Megabytes: A Baayen Space Odyssey. Amsterdam, Netherlands: Math. Centrum Wisk. Inform., (1994), 231-233.

[8] V. Chvátal and P. Erdős, A note on hamiltonian circuits. *Discrete Math.* 2 (1972), 111-113.

[9] G. Chen, R. J. Gould, X. Gu, A. Saito, Cycles with a chord in dense graphs. *Discrete Math.* 341 8 (2018), 2131–2141.

[10] The Combinatorial Object Server, http://combos.org/index

[11] K. Coolsaet, S. D'hondt and J. Goedgebeur, House of Graphs 2.0: A database of interesting graphs and more, Discrete Applied Mathematics, 325:97-107, 2023. Available at https://houseofgraphs.org

[12] M. Cream and R. J. Gould, Chorded k-pancyclic and weakly k-pancyclic graphs. *Discussiones Math. Graph Theory* 44 (2024), 337–350.

[13] M. Cream and R. J. Gould, Extending vertex and edge pancyclic graphs. *Graphs and Combin.* 34 (2018), 1691–1711.

[14] M. Cream, R. J. Faudree, R. J. Gould, and K. Hirohata, Chorded cycles. *Graphs and Combin.* 32 6 (2016), 2295–2313.

[15] M. Cream, R. J. Gould, and K. Hirohata, A note on extending Bondy's meta-conjecture. *Australas. J. Combin.* 67 (2017), 463–469.

[16] R. J. Gould, **Graph Theory**, Dover Publications Inc., Mineola, NY, 2012.

[17] R. J. Gould, Results and Problems On Chorded Cycles: A Survey, *Graphs and Combin.* 38 (2022), no. 6, 189.

[18] B. Jackson and O. Ordaz, The Chvátal-Erdős condition for paths and cycles in graphs and digraphs: a survey, Discrete Math. 84, 1990, 241-254.

[19] D. Lou, The Chvátal-Erdős condition in triangle free graphs. *Discrete Math.* 152(1996), 253–257.

[20] W. Mader, Uber den Zusammenhang symmetrischer *Graphen. Arch. Math.* (Basel) 21 (1970), 331-336.

[21] Brendan McKay, Ramsey Graphs, https://users.cecs.anu.edu.au/~bdm/data/ramsey.html

[22] J. Pach, Graphs Whose Every Independent Set Has a Common Neighbour, *Discrete Math.* 31 (1981), 217-228.

[23] L. Pósa, Problem no. 127 (Hungarian), *Mat. Lapok*, 254.

[24] S. Radziszowski, Dynamic survey of small Ramsey numbers, Electron. J. Combin., http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1/pdf

[25] A. Saito, Chvatal-Erdos Theorem: Old Theorem with New Aspects. Lecture Notes in Comput. Sci., 4535, Springer, berlin, 2008, 191-200.

[26] A. F. Sidorenko, Triangle-free regular graphs, *Discrete Math.* 91 (1991) 215–217.