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Mittag-Leffler moments and weighted L^∞ estimates for solutions to the Boltzmann equation for hard potentials without cutoff

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Mittag-Leffler moments and weighted L^∞ estimates for solutions to the Boltzmann equation for hard potentials without cutoff

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In this thesis we study analytic properties of solutions to the spatially homogeneous Boltzmann equation for collision kernels corresponding to hard potentials without the angular cutoff assumption, i.e. the angular part of the kernel is non-integrable with prescribed singularity rate. We study behavior in time of such solutions for large velocities i.e. their tails. We do this in two settings - L^1 and L^∞.

In the L^1 setting, we study Mittag-Leffler moments of solutions of the Cauchy problem under consideration. These moments, obtained by integrating the solution against a Mittag-Leffler function, are a generalization of exponential moments since Mittag-Leffler functions asymptotically behave like exponential functions. Mittag-Leffler moments can be also represented as infinite

sums of renormalized polynomial moments. However, instead of considering renormaliztion by integer factorials that would lead to classical exponential moments, we renormalize by Gamma functions with non-integer arguments. By analyzing the convergence of partial sums sequences of these infinite sums, we prove the propagation and generation in time of Mittag-Leffler moments. In the case of propagation, orders of these moments depend on the singularity rate of the angular collision kernel. In the case of generation, the orders depend on the potential rate of the kernel. The proof uses a subtle combination of angular averaging and angular singularity cancellation, to show that partial sums satisfy an ordinary differential inequality with a negative term of the highest order while controlling all positive terms, whose solutions are uniformly bounded in time and number of terms. These techniques apply to both generation and propagation of Mittag-Leffler moments. This part of the thesis is partly based on the joint project with Alonso, Gamba and Pavlović [10].

In the L^∞ setting, we prove that solutions to the Boltzmann equation that satisfy propagation in time of weighted L^1 bounds also satisfy propagation in time of weighted L^∞ bounds. This result is partly based on the joint project with Gamba and Pavlović [36]. To emphasize that the propagation in time of weighted L^∞ bounds relies on the propagation in time of weighted L^1 bounds, we express our main result using certain general weights. Consequently we apply the main result to exponential and Mittag-Leffler weights, for which propagation of weighted L^1 bounds holds. Hence we obtain propagation in time of exponentially or Mittag-Leffler weighted L^∞ bounds on the solution.

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Chapter 1

Introduction

In this chapter we briefly describe main results of the thesis. We also outline the organization of the thesis and gather notation used throughout it.

1.1 A brief description of the results in this thesis

In this thesis we study generation and propagation in time of high energy tail behavior of solutions to the spatially homogeneous Boltzmann equation with collision kernels (or probability transition rates) given by hard potentials without the angular cutoff assumption. This integro-differential equation models the evolution of a probability density function $f(t, v)$ of particles in a rarefied gas. The angular kernel of its integral operator has a non-integrable singularity in many cases. However, most of the mathematical theory of the Boltzmann equation has been developed under the assumption that the angular kernel is integrable (the so called cutoff case). In last 20 years, a lot of attention has been placed on the non-integrable case, and it has been recently shown that in such a regime the solution is regularized. This motivates further study of the non-cutoff case and it is the setting we study in this thesis. The details of the model under the consideration are presented in Section 2.

By the high energy tail behavior we mean decay properties of the solution $f(t, v)$ for large speeds or corresponding energies. A natural way to quantify a tail behavior is by studying weighted L^∞ norms of $f(t, v)$ since finiteness of such norms would imply that the solution has pointwise decay comparable to the inverse of the weight function. Historically, however, (polynomially) weighted L^1 norms were studied first. This is natural thing to do since solutions to the Boltzmann equation are probability density functions $f(t, v)$ whose polynomially weighted L^1 norms, i.e. its statistical moments or observables, play a significant role for further studies of the solution behavior. One can also study more general moments, and from now on, we refer to exponentially weighted L^1 norms as exponential moments.

Definition 1.1.1. A polynomial moment of order q is defined by

$$m_q(t) := \int_{\mathbb{R}^d} f(t, v)\, \langle v \rangle^q \, dv, \tag{1.1}$$

where $\langle v \rangle = \sqrt{1 + |v|^2}$. An exponential moment of order s and rate α is defined by

$$\mathcal{M}_{\alpha,s}(t) := \int_{\mathbb{R}^d} f(t, v)\, e^{\alpha \langle v \rangle^s} \, dv. \tag{1.2}$$

Since the equilibrium state of the Boltzmann equation is a Maxwellian distribution, i.e. a Gaussian distribution in velocity space, we are particularly interested in the study of propagation and generation of the exponential tail behavior of $f(t, v)$, both in L^1 and L^∞ setting. In other words, we study exponentially weighted L^1 norms (i.e. exponential moments) and exponentially weighted L^∞ norms.

By the propagation of the tail behavior we mean the following: given an initial datum with a certain decay rate for large velocities, the corresponding solution to the Boltzmann equation has the tail behavior of the same order with possibly different rates. For example, in the case of exponential moments, i.e. exponentially weighted L^1 norms, propagation in time can be expressed as follows:

$$\int_{\mathbb{R}^d} f(0,v)\, e^{\alpha_0 \langle v \rangle^s} dv < C_0 \;\text{ for some }\; \alpha_0, s > 0 \tag{P-exp-1}$$
$$\Rightarrow \qquad \exists C > 0,\;\; \exists 0 < \alpha \leq \alpha_0,\;\; \forall t \geq 0 : \;\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \langle v \rangle^s}\, dv < C.$$

By generation of the tail behavior we mean the following: given an initial datum with only a first few finite polynomial moments, the corresponding solution to the Boltzmann equation acquires stronger decay property instantaneously. For example, generation of exponential moments of order s means

$$\int_{\mathbb{R}^d} f(0,v)\, \langle v \rangle^q\, dv < C_0, \;\text{ for some }\; q \in \mathbb{N} \tag{G-exp-1}$$
$$\Rightarrow \qquad \exists s, \alpha, C > 0,\;\; \forall t > 0 : \;\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \langle v \rangle^s}\, dv < C.$$

Historically, propagation and generation of polynomial moments has been studied first in the following works [25, 32, 47, 60, 61]. This progress was used, among other things, to develop the existence theory for the angular cutoff Boltzmann equation under minimal conditions on the initial data [47]. Then, in 1972 Arkeryd posed a question [14] of whether it can be shown that if the initial data lies pointwise below a Maxwellian, then the solution remains under a possibly different Maxwellian. This question motivated the study of exponential tails. Even though Arkeryd's question was posed in the L^∞ setting, the first result in this direction was proved in the L^1 setting. Namely, Bobylev [17] proved propagation of Maxwellian moments, i.e. (P-exp-1) with $s = 2$, for the Boltzmann equation for hard spheres potentials. Since this first L^1 result, it took more than two decades until the first L^∞ result by Gamba, Panferov and Villani [35] that addresses the original question of Arkeryd under the cutoff assumption.

Main results of this thesis are related to the questions of both L^1 and L^∞ exponentially weighted norms of solutions to the Boltzmann equation, both studied in the non-cutoff regime. In the next two subsections we briefly explain what has been done prior to our work and what is our contribution in L^1 and L^∞ setting, respectively.

1.1.1 L^1 theory: Exponential and Mittag-Leffler moments

We now briefly review previous results on the exponential moments, and describe our contribution. Details are provided in Chapter 2.

Exponential moments, i.e. exponentially weighted L^1 norms, in the case of hard potentials $\gamma > 0$ were first studied for the Boltzmann equation whose angular kernel is integrable, the so called cutoff case. The first result was by Bobylev [17], who proved propagation (P-exp-1) of exponential moments of order $s = 2$ for the hard spheres model of the Boltzmann equation (Bobylev's prior works [15,16] considered Mawell molecules and used a Fourier transform technieques). This groundbreaking work [17] conceived the idea of controlling exponential moments by proving the summability of the power series expansion whose coefficients are classical polynomial moments of the distribution function $f(t,v)$ renormalized by Gamma functions. This strategy was motivated by formally commuting integration in v-space and the Taylor series representation of the exponential function in v-space

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha\,\langle v\rangle^s}\, dv = \int_{\mathbb{R}^d} f(t,v) \sum_{q=0}^{\infty} \frac{\alpha^q \langle v\rangle^{sq}}{q!} dv \;=\; \sum_{q=0}^{\infty} \frac{\alpha^q\; m_{sq}(t)}{\Gamma(q+1)}. \tag{1.3}$$

Estimating this infinite sum requires estimates on each polynomial moment $m_{sq}(t)$, which is done by establishing an ordinary differential inequality for m_{sq}, which in turn is derived from the corresponding polynomial moment of the collision operator. Gamba, Panferov and Villani [35] extended Bobylev's result to more general cutoff models (variable hard potentials). Mouhot [48] proved generation of exponential moments (G-exp-1) of order $s \leq \gamma/2$, where γ is a potential rate of the collision kernel of the Boltzmann equation. This has been improved by Alonso, Gamba, Cañizo and Mouhot [6], where the generation (G-exp-1) was established for exponential moments of order $s \leq \gamma$. In [6], authors also established propagation of exponential moments (P-exp-1) of general order $s \leq 2$, not just Gaussian moments.

On the other hand, exponential moments for the non-cutoff Boltzmann equation were studied only recently by Lu and Mouhot [44], where generation of exponential moments (G-exp-1) was proved for orders $s \leq \gamma$. However, no results were known about exponential moments of higher orders in the non-cutoff regime.

This brings us to the first part of the thesis, Chapter 3, which is partly based on the joint work with Alonso, Gamba and Pavlović [10], and where we extend the work of [44] to obtain propagation of exponential moments of orders $\gamma < s < 2$ in the non-cutoff case.

One of the novelties of our result is the introduction of Mittag-Leffler moments, which are L^1 norms weighted with Mittag-Leffler functions, and are meant to generalize the concept of exponential moments. Namely, our calculations lead to expressions similar to that of (1.3), yet having $\Gamma(aq + 1)$ with non-integer $a > 1$ in place of factorials $q!$. This motivated us to use Mittag-Leffler functions, as they generalize the Taylor expansion of the exponential function precisely by replacing factorials with non-integer Gamma functions. More precisely, for a parameter $a > 0$, a Mittag-Leffler function is defined via

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}. \tag{1.4}$$

When $a = 1$, this coincides with the Taylor expansion of the classical exponential function e^x. When $a > 0$, it is well known (see e.g. [33], page 208.) that the Mittag-Leffler function $\mathcal{E}_a$ asymptotically behaves like an exponential function of order $1/a$, that is

$$\mathcal{E}_a(x) \sim e^{x^{1/a}}, \qquad \text{as } x \to \infty,$$

and thus

$$\mathcal{E}_{2/s}(\alpha^{2/s} x^2) \sim e^{\alpha x^s}, \qquad \text{as } x \to \infty.$$

Because of this asymptotic behavior, finiteness of $e^{\alpha x^s}$-weighted L^1 norm is equivalent to finiteness of $\mathcal{E}_{2/s}(\alpha^{2/s}x^2)$-weighted L^1 norm.

Studying Mittag-Leffler moments, i.e. Mittag-Leffler weighted L^1 norms, enabled us to extend the range of orders of exponential moments that can be propagated uniformly in time for the non-cutoff case. Namely, we will be able to prove an analogue of (P-exp-1) for orders s larger than γ, where instead of exponential weights we use the corresponding Mitta-Leffler weights, that is

$$\int_{\mathbb{R}^d} f(0,v)\ \mathcal{E}_{2/s}(\alpha_0^{2/s}\langle v\rangle^2)dv < C_0 \ \text{ for some } \ \alpha_0, s > 0 \qquad \text{(P-ML-1)}$$
$$\Rightarrow \qquad \exists C>0,\ \ \exists 0<\alpha\le\alpha_0,\ \ \forall t\ge 0:\ \int_{\mathbb{R}^d} f(t,v)\ \mathcal{E}_{2/s}(\alpha^{2/s}\langle v\rangle^2)\, dv < C.$$

Another important aspect of our main result is that the highest order s of the Mittag-Leffler moment which can be propagated in time as (P-ML-1), depends continuously on the singularity rate of the angular cross-section. The less singular the angular kernel is, the higher order Mittag-Leffler moment can be propagated.

1.1.2 L^∞ theory: Pointwise behavior of tails

We now briefly review previous results on the upper pointwise bounds for solutions to the Boltzmann equation, and describe our contribution. Details are provided in Chapter 4.

After the behavior of weighted L^1-norms is understood, the next natural question is to obtain information about weighted point-wise (in velocity) bounds on solutions to the Boltzmann equation. This has been achieved in the cutoff case for the polynomial weights by Arkeryd [13], and later for exponential weights, again in the Grad's cutoff case, in the work of Gamba, Panferov and Villani [35]. More precisely, in [35] authors prove that if an initial data is below a Maxwellian point-wise, the same is true uniformly in time for the corresponding solution of the homogeneous cutoff Boltzmann equation (for a possibly different Maxwellian), i.e. for $s = 2$ they show

$$f(0,v) \leq e^{-\alpha_0 |v|^s + c_0}, \;\; \text{for some} \;\; \alpha_0, c_0 > 0 \qquad \text{(P-exp-}\infty\text{)}$$
$$\Rightarrow \qquad \exists C > 0, \;\; \exists 0 < \alpha \leq \alpha_0, \forall\, t \geq 0 : \;\; f(t,v) \leq e^{-\alpha |v|^s + c},$$

or equivalently,

$$\| f(0,v) e^{\alpha_0 |v|^s} \|_{L^\infty_v} < C_0, \;\; \text{for some} \;\; \alpha_0, C_0 > 0$$
$$\Rightarrow \qquad \exists C > 0, \;\; \exists 0 < \alpha \leq \alpha_0, \; \forall t \geq 0 : \;\; \| f(t,v) e^{\alpha |v|^s} \|_{L^\infty_v} < C.$$

A remarkable aspect of the work of Gamba, Panferov, Villani [35] is that it established a comparison principle for the Boltzmann equation, and then used it to prove the desired point-wise bounds by exploiting the propagation in time of the corresponding weighted L^1 bounds of the solution. This can be understood in the spirit of an important step in the De-Giorgi-Nash-Moser argument, in the sense that weighted L^1 bounds are used to prove weighted L^∞ bounds.

No results of the type (P-exp-∞), even with s other than two, were available in the non-cutoff setting.

This brings us to the second topic of this thesis, Chapter 4, which is partly based on the joint project with Gamba and Pavlović [36], and where we show propagation in time of weighted L^∞ bounds for solutions to the homogeneous Boltzmann equation in the non-cutoff case. Our results can be understood as an extension of [35] to the non-cutoff case. We too exploit our weighted L^1 bounds in the non-cutoff case to obtain the propagation of the corresponding L^∞ bounds for the non-cutoff Boltzmann equation.

For solutions to the non-cutoff Boltzmann equation, Silvestre, in a recent work [52], established propagation in time of L^∞ bounds (without a weight) on solutions to (not necessarily homogeneous) Boltzmann equation by introducing at the level of the Boltzmann equation techniques motivated by the theory of integro-differential equations, which "replace" the role of the comparison principle of Gamba, Panferov and Villani [35]. Furthermore, Silvestre obtained certain regularity results which can be understood as a companion to regularization mechanisms of solutions to the non-cutoff Boltzmann equation (e.g. [26, 28, 29, 43]) previously obtained via harmonic analysis methods.

Our main result in Chapter 4 can also be understood as a generalization of [52] to exponentially decaying pointwise upper bounds on the solution. In our proof we modify the contradiction argument of Silvestre [52] to take advantage of known results on propagation in time of exponentially or Mittag-Leffler weighted L^1 bounds of the solution.

In order to emphasize the transition from weighted L^1 bounds to weighted L^∞ bounds, we state the main theorem in terms of certain general weights that mimic exponential decay. Consequently we apply this result to cases of exponential and Mittag-Leffler weights, for which propagation in time of weighted

L^1 norms holds, see Corollary 4.1.2. Hence we obtain unconditional propagation in time of exponentially or Mittag-Leffler weighted L^∞ bounds on the solution. More precisely, for a range of s for which (P-exp-1) or (P-ML-1) holds, we obtain pointwise bounds (P-exp-∞) and

$$\int_{\mathbb{R}^d} f(0,v)\ \mathcal{E}_{2/s}(\alpha_0^{2/s}\langle v\rangle^2)dv < C_0 \ \text{ for some } \ \alpha_0, s > 0 \qquad \text{(P-ML-}\infty\text{)}$$
$$\Rightarrow \qquad \exists C > 0,\ \ \exists 0 < \alpha \leq \alpha_0,\ \ \forall t \geq 0 : \ \int_{\mathbb{R}^d} f(t,v)\ \mathcal{E}_{2/s}(\alpha^{2/s}\langle v\rangle^2)\ dv < C.$$

We make a remark here that our result is a priori in the sense that our proof requires sufficiently nice solutions that are not yet know to exist. Silvestre [52] too needed nice solutions. In that paper Schwartz class solutions are good enough, and they are known to exist for certain range of potentials. In our case, however, we would need somewhat stronger solutions due to the added weight in the norms we consider.

1.2 Organization of the thesis

In Chapter 2 we review the Boltzmann equation. Our first main result, related to exponential and Mittag-Leffler moments, is discussed in Chapter 3. The precise statement can be found in Section 3.1, while the relevant history of the problem is recalled in Section 3.2. In Section 3.3 we outline the strategy of the proof, while Section 3.4 contains some combinatorial inequalities used for the proof of the main result. The angular averaging and cancellation are explained in Section 3.5, which is then used in Section 3.6 to derive differential inequalities for polynomial moments. The last two sections contain proofs of propagation of moments and generation of moments, respectively.

Chapter 4 is dedicated to our second main result, namely the pointwise upper bounds of solutions to the Boltzmann equation. Section 4.1 contains the precise statement of the main result and its corollary. Section 4.2 recalls the relevant previous results. Proof of the main theorem is in Section 4.3, while the proof of the corollary is contained in Section 4.4.

1.3 Notation

In this section we gather notation used throughout the thesis. We start by commonly used notation. For any $x, a, b \in \mathbb{R}$ and $v \in \mathbb{R}^d$:

- $\lfloor x \rfloor$ = the largest integer less than or equal to x
- $\langle v \rangle = \sqrt{1 + |v|^2}$
- $a \approx b \quad \iff \quad \exists c, C > 0 : \quad c\, b \leq a \leq C\, b$

Functional spaces:

- $L^1_k = \left\{ f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f \langle v \rangle^k dv = \int_{\mathbb{R}^d} f \left(1 + |v|^2\right)^{k/2} dv < \infty \right\}, \forall k \in \mathbb{R}.$
- $L \log L = \left\{ f : \mathbb{R}^d \to \mathbb{R}^+ : \int_{\mathbb{R}^d} f \ (1 + \log f) \ dv < \infty \right\}$

Special functions:

- Gamma function: $\Gamma(t) = \int_0^\infty x^{t-1}\, e^{-x}\, dx, \quad t \in \mathbb{R}$
- Beta function: $B(x,y) = \int_0^1 t^{x-1}\, (1-t)^{y-1}\, dt, \quad x, y \in \mathbb{R}$

Chapter 2

The Boltzmann equation

The Boltzmann equation, first described by J. C. Maxwell [45] in 1867 and a few years later by L. W. Boltzmann [19, 20], models the evolution of a dilute monoatomic gas containing a large number of particles which interact via predominantly binary collisions. Instead of following each particle in their phase spaces separately, the Boltzmann equation takes a statistical approach and considers the evolution of the density function of the particles, which is denoted by $f(t, x, v)$, for time t, position x and velocity v.

Rigorous derivation of the Boltzmann equation is an active filed of research. First breakthoughs were made by Lanford [42] and King [41], whose works have been later completed by Gallagher, Saint-Raymond and Texiera [34], and Pulvirenti, Saffirio and Simonella [51]. See also [24,53]. These results studied hard spheres and short range potentials, and are valid for short times. Extensions of the time of the validity and the study of long range potentials are still challenging problems.

The evolution of the density $f(t, x, v)$ is influenced by the transport and collision effects. Accordingly, the Boltzmann equation reads

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f, f)(t, x, v). \tag{2.1}$$

2.1 The collision operator $Q(f,f)$

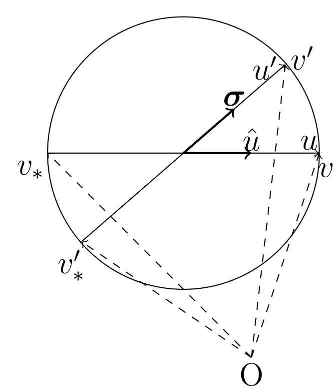
The effect of collisions on the density function $f(t,x,v)$ is captured by the collision operator $Q(f,f)$. Since it is assumed that the gas is rarefied enough that collisions are predominantly binary, the collision operator $Q(f,f)$ is quadratic. More precisely, it is a bilinear integral operator which acts on v and which is local in t and x. It is defined via

$$Q(f,f)(t,x,v) \; = \int_{\mathbb{R}^d}\int_{S^{d-1}} \left(f'f'_* - ff_*\right) B(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_*, \qquad (2.2)$$

where we employ the abbreviated notation

$$f_* = f(t,x,v_*),$$
$$f' = f(t,x,v'),$$
$$f'_* = f(t,x,v'_*).$$

For a pair of particles, vectors v', v'_* denote their pre-collision velocities, while v, v_* are their corresponding post-collision velocities. Relative velocities are denoted by



$$u' = v' - v'_*, \quad u = v - v_*,$$

while the corresponding unit vectors in the direction of these relative velocities are denoted by

$$\sigma = \frac{u'}{|u'|}, \quad \hat{u} = \frac{u}{|u|}.$$

The unit vector $\sigma \in S^{d-1}$ is referred to as the scattering direction, while the angle between two relative velocities, that is, between σ and $\hat{u}$, is denoted by θ, and it is referred to as the scattering angle.

For elastic interactions momentum and energy are conserved, that is,

$$\begin{aligned} &v + v_* = v' + v'_*, \\ &|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2, \end{aligned}$$

which implies the following relation between pre and post collision velocities

$$v' = \frac{v + v_*}{2} + \frac{|u|}{2}\,\sigma, \tag{2.3}$$

$$v'_* = \frac{v + v_*}{2} - \frac{|u|}{2}\,\sigma. \tag{2.4}$$

Also, it is possible to represent these relations as functions of the relative velocity u and the scattering direction σ via

$$\begin{aligned} v' - v &= \frac{1}{2}(|u|\,\sigma - u), \\ v'_* - v_* &= -\frac{1}{2}(|u|\,\sigma - u). \end{aligned}$$

The kernel $B(|u|, \hat{u} \cdot \sigma)$, which due to the physical considerations depends on the magnitude of the relative velocity $|u|$ and the cosine of the scattering angle $\hat{u} \cdot \sigma = \cos\theta$, carries the essential information about the gas as it encodes the likelihood of collisions. It is usually referred to as the collision kernel or the cross section. Already Maxwell [45] calculated the collision kernel in the case of hard spheres (billiard-like model) and for the so-called inverse-power law model. In the latter case the interactions between particles are guided by the forces proportional to an inverse power of their distances $\phi(x) = Cx^{-(p-1)}$, $C > 0$, $p > 2$. In dimension $d = 3$, Maxwell found that the collision kernel then takes the following form:

$$
\begin{aligned}
B(|u|, \hat{u} \cdot \sigma) = |u|^{\gamma}\, b(\hat{u} \cdot \sigma) = |u|^{\gamma}\, b(\cos\theta),& \\
b(\cos\theta)\ \sin\theta \sim C\, \theta^{-1-\nu}, \qquad \theta \to 0^{+},& \\
\nu = \frac{2}{p-1}, \qquad \gamma = \frac{p-5}{p-1}, \qquad p > 2.& \qquad (2.5)
\end{aligned}
$$

As is the case in the above model, we assume that the collision kernel $B(|u|, \hat{u} \cdot \sigma)$ takes a factorized form

$$B(|u|, \hat{u} \cdot \sigma) \;=\; |u|^{\gamma}\, b(\cos\theta), \qquad (2.6)$$

throughout the manuscript. We work with variable hard potentials, that is

$$0 < \gamma \leq 1. \qquad (2.7)$$

For completeness we mention that the parameter γ can more generally be in the range $\gamma \in (-d, 1]$, where d is the dimension of the velocity space. When γ has a negative value, the potential is said to be soft, while the case $\gamma = 0$ is referred to as the Maxwell molecules.

We assume that the angular kernel $b(\hat{u} \cdot \sigma)$ is given by a positive measure over the sphere S^{d-1}. In many models this function is not integrable in σ. For example, in the case of inverse power forces it is never integrable.

2.2 Cutoff vs. non-cutoff

In 1963, Grad [39] proposed considering a bounded angular kernel $b(\cos\theta)$ and pointed out that different cutoff conditions could be implemented too. This is why when the angular kernel is integrable it is often said to satisfy a Grad's angular cutoff condition. On the other hand, its non-integrability is referred to as an angular non-cutoff. The convenience in assuming that the angular kernel is integrable lies in the possibility of splitting the collision operator into two integrals, so called the gain $Q^+(f,f)$ and the loss $Q^-(f,f)$ terms, which then can be analyzed separately

$$Q(f,f) = Q^+(f,f) - Q^-(f,f), \tag{2.8}$$

where

$$\begin{aligned} Q^+(f,f)(t,v) &= \int_{\mathbb{R}^d}\int_{S^{d-1}} f' f'_* \, B(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_*, \\ Q^-(f,f)(t,v) &= f(v)\int_{\mathbb{R}^d}\int_{S^{d-1}} f_* \, B(|u|, \hat{u}\cdot\sigma)\, d\sigma\, dv_*. \end{aligned}$$

For several decades the theory of the Boltmann equation has been developing under angular cutoff conditions with the belief that removing the singularity of the angular kernel should not affect properties of the equation. However, since the late 1990s it has been observed (see for example [26,28,29,43]) that the singularity of $b(\cos\theta)$ carries regularizing properties. This fact, in addition to the technical challenge of not being allowed to separate the gain and the loss term, motivated further study of the non-cutoff regime, which will be the setting we study here. More precisely, in this thesis we work in the following non-cutoff setting. While the angular kernel is non-integrable, we assume that for some $\beta \in (0, 2]$ the following weighted integral is finite

$$\begin{aligned} A_\beta &:= \int_{S^{d-1}} b(\hat{u} \cdot \sigma)\ \sin^\beta \theta\ d\sigma \\ &= V_{d-2} \int_0^\pi b(\cos\theta)\ \sin^\beta \theta\ \sin^{d-2} \theta\ d\theta < \infty, \end{aligned} \tag{2.9}$$

where $V_{d-2} = \frac{\pi^{(d-2)/2}}{\Gamma((d-1)/2)}$ is the volume of the $d-2$ dimensional unit sphere. When $\beta = 0$, this condition would coincide with Grad's cutoff assumption.

The typical non-cutoff assumption in the literature is the condition (2.9) with $\beta = 2$. However, we work in the non-cutoff regime where the parameter $\beta \in (0, 2]$ is allowed to vary and we will see how the strength of the singularity of b influences our main results. In this setting, the splitting (2.8) is not valid, which is one of the technical challenges that non-cutoff setting brings. In order to address this obstacle we exploit angular cancellation properties (for details see Section 3.5).

2.3 Weak formulation of the collision operator

Thanks to the symmetries associated to the collision operator $Q(f, f)$, defined in the strong form (2.2), the collision operator has a weak formulation that is very important for the analytical manipulation of the equation. Indeed, for any test function $\phi(v)$, $v \in \mathbb{R}^d$, one has (see for example [23])

$$\int_{\mathbb{R}^d} Q(f, f)(t, v)\, \phi(v) dv = \frac{1}{2} \iint_{\mathbb{R}^{2d}} f(v)\, f(v_*)\ G_\phi(v, v_*)\, dv_*\, dv, \tag{2.10}$$

where

$$G_\phi(v, v_*) = \int_{S^{d-1}} (\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*))\, B(|u|, \hat{u}\cdot\sigma)\ \ d\sigma. \qquad (2.11)$$

The key aspect of the equation in the weak formulation is expressed in the weight G_ϕ as it carries all the information about collisions through the collision kernel B, which is averaged over the unit sphere against test functions $\Delta\phi = \phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)$.

Crucial estimates on the function G_ϕ referred to in the Boltzmann equation literature as Povzner estimates. In the Grad's cutoff case, positive and negative contributions are treated separately and such estimates are used to estimate the positive part of G_ϕ. A sharp form of angular averaged Povzner estimates from [17, 18, 35] is obtained for general test functions $\phi(v)$ which are positive and convex. They are crucial for the study of moments summability, the main point of this thesis. When $\phi(v) = (1 + |v|^2)^{k/2} = \langle v\rangle^k$, these estimates, originally developed by Povzner [50], yield ordinary differential inequalities for moment estimates that lead to an existence theory and generation and propagation of moments as developed in Elmroth [32], Desvillettes [25] Wennberg [61] and Mischler, Wennberg [47]. These estimates were also obtained in the non-cutoff case by Wennberg [60] for hard potentials. Uniqueness theory to solutions of the Boltzmann equation for hard potentials was first developed by Di Blassio in [30].

When the angular part of the collision kernel is not integrable, i.e. the non-cutoff case, one needs to expand $\Delta\phi$ in terms of $v' - v$ and $v'_* - v_*$, since both are a multiples of $|u|\,\sin\theta/2$. For this strategy to succeed, the spherical integration variable $\sigma \in S^{d-1}$ must be decomposed as $\sigma = \hat{u}\cos\theta + \omega\sin\theta$, corresponding to the polar direction of the relative velocity u, and the azimuthal direction $\omega \in S^{d-1}$ satisfying $u \cdot \omega = 0$. This decomposition also plays a fundamental role in our derivation of the angular averaged Povzner with singularity cancellation in the proof of Lemma 3.5.1.

Remark 2.3.1. We note that the identity (2.10) can also be expressed in a double mixing (weighted) convolutional form (see e.g. [7, 9, 37])

$$\int\limits_{\mathbb{R}^d} Q(f,f)(t,x,v)\,\phi(v)dv = \frac{1}{2}\iint\limits_{\mathbb{R}^{2d}} f(v)f(v-u)\;G_\phi(v,u)\,du\,dv$$

$$G_\phi(v,u)\int_{S^{d-1}} \left(\phi(v') + \phi(v'-u') - \phi(v) - \phi(v-u)\right)B(|u|,\hat{u}\cdot\sigma)\;\;d\sigma$$

since both v' and v'_* can be written as functions of v, u and σ from (2.3), and so the weight function $G_\phi(v,u)$ is an average over $\sigma \in S^{d-1}$.

Remark 2.3.2. A classical consequence of the weak formulation of the Boltzmann equation is Boltzmann's well-known H-theorem. Taking $\phi = \log f$ as a test function in v and exploiting symmetries of the collision operator, yields the following estimate on the entropy dissipation functional $D(f)$ which is defined by $\int_{\mathbb{R}^d} Q(f,f)\;\log f\;dv = -D(f)$:

$$\int_{\mathbb{R}^d} Q(f,f)\;\log f\;dv \tag{2.12}$$

$$= -\frac{1}{4}\iiint_{\mathbb{R}^{3d}\times\mathbb{S}^d} (f'f'_* - f\;f_*)\;\log\frac{f'f'_*}{f\;f_*}\;B(|u|,\cos\theta)\;d\sigma\;dv_*\;dv$$

$$\leq 0. \tag{2.13}$$

The last inequality holds due to since

$$(\log x - \log y)(x - y) \geq 0, \qquad \text{for any } \; x, y > 0.$$

Therefore, Boltzmann's H functional (or entropy) defined as

$$H(f) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \; \log f \; dvdx \tag{2.14}$$

satisfies

$$\begin{aligned} \frac{dH}{dt} &= \int_{\mathbb{R}^d \times \mathbb{R}^d} Q(f, f) \log f dvdx \\ &= - \int_{\mathbb{R}^d} D(f) \; dx \leq 0. \end{aligned} \tag{2.15}$$

The inequality (2.15) encodes the time-irreversibility property of the Boltzmann equation as it states that $H(t)$ functional is non-increasing. Moreover, Boltzmann's inequality (2.15) becomes equality if and only if f is a Maxwellian distribution. In other words, the statistical equilibrium of the Boltzmann equation is Maxwellian distribution (or Gaussian distribution).

2.4 Homogeneous case

Another significant simplification of the Boltzmann equation, which still keeps the problem challenging, is the situation where the probability density that is being modeled does not depend on the spatial variable x. We work in this setting, which is referred to as the spatially homogeneous regime. In this case, the Cauchy problem for the Boltzmann equation simplifies to

$$\begin{cases} \partial_t f(t, v) \; = \; Q(f, f)(t, v), \\ f(0, v) \; = \; f_0(v). \end{cases} \tag{2.16}$$

In summary, we study the spatially homogeneous Boltzmann equation (2.16) with the collision kernel (2.6) corresponding to hard potentials $\gamma \in (0, 1]$ and the non-cutoff condition (2.9) with $\beta \in (0, 2]$. We study the tail behavior of solutions to the spatially homogeneous Boltzmann equation (2.16) with hard potentials (2.6)-(2.7) and the angular non-cutoff assumption (2.9).

2.5 Existence theory in the non-cutoff case

The first existence result in the non-cutoff regime was due to Arkeryd [12], where the existence of weak solutions was established for $\gamma > -1$. Goudon [38] and Villani [55] extended Arkeryd's proof to the range $\gamma > -2$. More recently, Alexandre, Morimoto, Ukai, Xu and Yang [4] proved that these weak solutions are in fact of Schwartz class provided that moments of all orders remain finite. This condition is known to be satisfied. In fact, for hard potentials exponential moments of certain order are generated instantaneously and remain uniformly bounded in time.

Villani [55] introduced H-solutions, which allow the study of the very soft potentials $\gamma \in (-4, -2)$ via the entropy production. Alexandre and Villani [5] extended the concept of renormalized solutions of DiPerna-Lions [31] to the non-cutoff case. Lu and Mouhot [44] obtained existence of the measure weak solutions. Ukai [54] established local existence in the Gevrey class (in v) spaces. In the perturbative regime around the equilibrium state (for not necessarily homogeneous equation), Gressman and Strain [40] and Alexandre, Morimoto, Ukai, Xu and Yang [3] established global existence of classical solutions.

Chapter 3

L^1 theory: Mittag-Leffler moments

In this chapter we present details of our first main result regarding exponential and Mittag-Leffler moments, which is partly based on the joint work with Alonso, Gamba and Pavlović [10]. We start with the precise statement of the result, which is followed by an overview of the previous relevant results. Next we outline the strategy of the proof and present the main tools. Finally, we show details in deriving an ordinary differential inequality for polynomial moments and use that in the last two subsection to complete the proof of propagation of Mittag-Leffler moments and generation of exponential moments.

3.1 Statement of the main result

Our main result in this chapter consists of two parts. First, under the non-cutoff assumption (2.9) with $\beta = 2$, we provide a new proof of the generation of exponential moments of order $s \in (0, \gamma]$, where $0 < \gamma \leq 1$ is the potential rate in the collision kernel (2.6). Second, we show the propagation in time of the Mittag-Leffler moments of order $s \in (\gamma, 2)$. When $s \in (\gamma, 1]$, non-cutoff (2.9) with $\beta = 2$ is assumed. When $s \in (1, 2)$, the angular kernel is assumed to be less singular.

Before we state the theorem, we remind the reader of the following notation

$$L^1_k = \{f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f\langle v\rangle^k dv < \infty\}.$$

This is the natural Banach space to solve the Boltzmann equation.

We also recall the definition of the weak solution, whose existence in three dimensions and in the non-cutoff case (2.9) with $\beta \in (0,2]$ is proved in [12, 55].

Definition 3.1.1. Let $f_0 \geq 0$ be a function defined in $\mathbb{R}^d$ with finite mass, energy and entropy

$$\int_{\mathbb{R}^d} f_0(v)\ \left(1 + |v|^2 + \log(1 + f_0(v))\right)\ dv < +\infty. \tag{3.1}$$

Then we say f is a *weak solution* to the Cauchy problem (2.16) if it satisfies the following conditions:

- $f \geq 0,\ f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^d)) \cap L^1([0,T]; L^1_{2+\max\{\gamma,0\}})$
- $f(0,v) = f_0(v)$
- $\forall t \geq 0$: $\int f(t,v)\psi(v)dv = \int f_0(v)\psi(v)dv$, for $\psi(v) = 1, v_1, ..., v_d, |v|^2$
- $f(t,\cdot) \in L\log L$ and $\forall t \geq 0 : \int f(t,v)\log f(t,v)dv \leq \int f_0(v)\log f_0 dv$
- $\forall \phi(t,v) \in C^1(\mathbb{R}^+, C_0^\infty(\mathbb{R}^3))$, $\forall t \geq 0$ we have that

$$\int_{\mathbb{R}^d} f(t,v)\phi(t,v)dv - \int_{\mathbb{R}^d} f_0(v)\phi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^d} f(\tau,v)\partial_\tau\phi(\tau,v)dv$$
$$= \int_0^t d\tau \int_{\mathbb{R}^d} Q(f,f)(\tau,v)\phi(\tau,v)dv.$$

Finally, we recall that a Mittag-Leffler function, for a parameter $a > 0$, is defined via

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}. \tag{3.2}$$

When $a = 1$, it coincides with the Taylor expansion of e^x, while for any $a > 0$ (see e.g. [33], page 208.), it asymptotically behaves like an exponential function of order $1/a$, that is

$$\mathcal{E}_a(x) \sim e^{x^{1/a}}, \qquad \text{as}\, x \to \infty.$$

Since $\langle v \rangle^2$ is the building block of our calculations, we prefer to have x^2 as the argument of the Mittag-Leffler function when generalizing $e^{\alpha\, x^s}$,

$$\mathcal{E}_{2/s}(\alpha^{2/s}\, x^2) \sim e^{\alpha\, x^s}, \qquad \text{for}\, x \to \infty. \tag{3.3}$$

Hence, they satisfy the following, with some positive constants c, C

$$c\, e^{\alpha x^s} \;\leq\; \mathcal{E}_{2/s}(\alpha^{2/s} x^2) \;\leq\; C\, e^{\alpha x^s}. \tag{3.4}$$

This motivates our definition of Mittag-Leffler moments.

Definition 3.1.2 (Mittag-Leffler moment)**.** A Mittag-Leffler moment of order s and rate $\alpha > 0$ of a function f is introduced via

$$\int_{\mathbb{R}^d} f(t,v) \;\; \mathcal{E}_{2/s}(\alpha^{2/s}\, \langle v \rangle^2)\, dv. \tag{3.5}$$

Remark 3.1.1. In the rest of the paper we will use the fact that Mittag-Leffler moments can be represented as the following sum, which follows from (3.2)

$$\int_{\mathbb{R}^d} f(t,v) \;\; \mathcal{E}_{2/s}(\alpha^{2/s}\, \langle v \rangle^2)\, dv \;\; = \;\; \sum_{q=0}^{\infty} \frac{m_{2q}(t)\, \alpha^{2q/s}}{\Gamma(\frac{2}{s}\, q + 1)}. \tag{3.6}$$

The following theorem is based in part on the joint work with Alonso, Gamba and Pavlović [10].

Theorem 3.1.1 (Generation and Propagation of Mittag-Leffler moments)**.** *Suppose f is a weak solution to the Boltzmann equation* (2.16) *with the collision kernel of the form* (2.6) *for hard potentials* (2.7)*, corresponding to the initial data $f_0 \in L^1_2 \cap L \log L$.*

(a) (Generation of exponential moments) If the angular kernel satisfies the non-cutoff condition (2.9) *with $\beta = 2$, then the exponential moment of order γ is generated with a rate $\alpha \min\{t, 1\}$. More precisely, there are positive constants C, α, depending only on b, γ and initial mass and energy, such that*

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \, \min\{t,1\}\, |v|^\gamma}\, dv \,\leq\, C, \quad \text{for } \; t \geq 0. \tag{3.7}$$

(b) (Propagation of Mittag-Leffler moments) Let $s \in (0, 2)$ and suppose that the Mittag-Leffler moment of order s of the initial data f_0 is finite with a rate α_0, that is, for some $M_0 > 0$,

$$\int_{\mathbb{R}^d} f_0(v) \;\; \mathcal{E}_{2/s}(\alpha_0^{2/s} \, \langle v \rangle^2)\; dv \leq M_0. \tag{3.8}$$

Suppose also that the angular cross-section satisfies assumption (2.9)

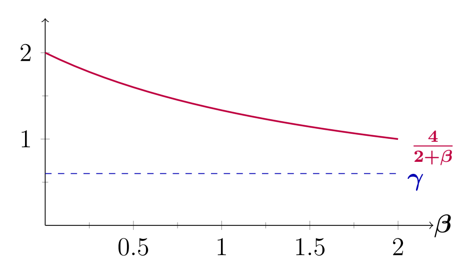
$$\begin{aligned} &\text{with } \; \beta = 2, && \text{if } \; s \in (0, 1] \\ &\text{with } \; \beta \leq \frac{4}{s} - 2, && \text{if } \; s \in (1, 2). \end{aligned} \tag{3.9}$$

Then, there exist positive constants C, α, depending only on M_0, α_0, b, γ and initial mass and energy such that the Mittag-Leffler moment of order s and rate α remains uniformly bounded in time, that is

$$\int_{\mathbb{R}^d} f(t,v) \ \mathcal{E}_{2/s}(\alpha^{2/s} \langle v \rangle^2) \, dv \leq C, \quad \textit{for } \ t \geq 0. \tag{3.10}$$

Remark 3.1.2. The angular singularity condition $\beta = \frac{4}{s} - 2$ in the case of Mittag-Leffler moments of order $s \in (1,2)$, continuously changes from $\beta = 2$ (for $s = 1$) to $\beta = 0$ (for $s = 2$). Hence condition $\beta = \frac{4}{s} - 2$ continuously interpolates between the most singular kernel typically considered in the literature, which is (2.9) with $\beta = 2$, and the Grad's cutoff condition, which corresponds to (2.9) with $\beta = 0$. This also tells us that in the most singular case one can propagate exponential moments of order $s \leq 1$, while in the Grad's cutoff case one can propagate exponential moments of order $s \leq 2$ (to be completely rigorous, Theorem 3.1.1 goes up to $\beta > 0$, i.e. $s < 2$, but [6] already established the case $\beta = 0$ i.e. $s = 2$). In other words, the less singular the angular kernel is, the higher the order the exponential moment propagates in time.

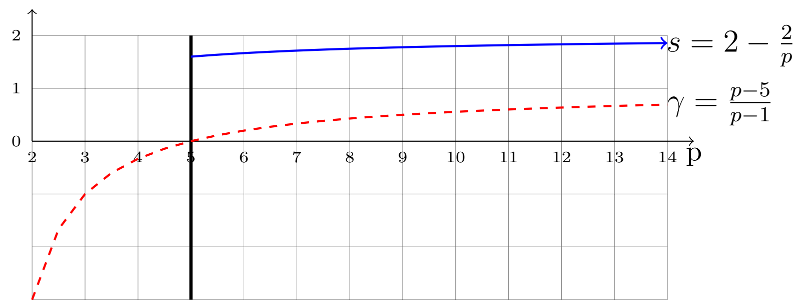
Remark 3.1.3. The propagation result of the theorem can be interpreted in two ways. First, for a Mittag-Leffler (or exponential) moment of order s to be propagated, the singularity of b should be such that it satisfies (2.9) with $\beta = \frac{4}{s} - 2$. On the other hand, given an angular kernel b that satisfies condition (2.9) with a parameter $\beta \in (0,2]$, one can propagate Mittag-Leffler



(and exponential) moments of order $s \leq \frac{4}{\beta+2}$.

Remark 3.1.4. Since Mittag-Leffler functions asymptotically behave like exponentials (3.4), finiteness of the exponential moment of order s is equivalent to finiteness of the corresponding Mittag-Leffler moment. Therefore, Theorem 3.1.1 (b) implies the propagation of classical exponential moments.

Remark 3.1.5. In the case of the inverse-power law model described via (2.5), in which hard potentials cor-



respond to $p > 5$, the non-cutoff condition (2.9) is satisfied for $\beta > \nu$. Hence, Mittag-Leffler moments of orders $s < 2 - \frac{2}{p}$ can be propagated in time. In the graph below the y-axis represents the order of the exponential tails. The dashed red line marks the highest order of exponential moments that can be generated, while the blue line marks the highest order of Mittag-Leffler moments that can be propagated in time. This graph visually confirms that our propagation result indeed goes beyond the rate of potentials γ.

3.2 Relevant previous results

In this subsection we provide a detailed overview of the previous results on exponential moments for the Boltzmann equation with hard potentials.

After the theory of polynomial moments had developed for some time, it was the work of Bobylev [17] that made the first contribution to the theory of exponential moments. There the Taylor series expansion of the exponential weight was used to make the connection between exponential and polynomial moments by representing exponential moments as infinite sums of polynomial moments weighted by Gamma functions

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \langle v \rangle^s}\, dv = \int_{\mathbb{R}^d} f(t,v) \sum_{q=0}^{\infty} \frac{\alpha^q \langle v \rangle^{sq}}{q!} dv \;=\; \sum_{q=0}^{\infty} \frac{\alpha^q\, m_{sq}(t)}{\Gamma(q+1)}. \qquad (3.11)$$

Bobylev [17] proved that the spatially homogeneous Boltzmann equation for hard spheres, i.e. with $\gamma = 1$ and constant angular kernel b, has the property that if the initial data has finite exponential moment (3.11) of order $s = 2$ and rate α_0, then its unique solution has the same property with the same order $s = 2$ and rate α for some $\alpha < \alpha_0$ depending on a few moments of the initial state. To achieve such bounds for the sum (3.11), Bobylev found ordinary differential inequalities for polynomial moments m_{sq} by exploiting the weak formulation of the Boltzmann equation and the Povzner inequality type estimates. This was then used to show that the sum (3.11) can be estimated term-by-term by a summable geometric series.

A few years later, Bobylev, Gamba and Panferov [18] established the angular averaged Povzner inequality for elastic or inelastic collisions, by a reduced argument that could be extended to a non-constant, bounded angular part in the collision kernel. They showed that stationary solutions of the spatially homogeneous inelastic Boltzmann equation for hard spheres (i.e. $\gamma =$

1), with stochastic heating sources corresponding to diffusion, shear flow and homogeneous cooling states, have bounded exponential moments in the sense that (1.3) was satisfied, with a constant in time rate $\alpha > 0$ and order $s < 2$. In the case of stochastic diffusing heating with a drift source the order was $s = 2$.

The uniform propagation in time of Maxwellian moments for solutions to the homogeneous Boltzmann equation in d dimensions with intramolecular potentials corresponding to values of $\gamma \in (0, 1]$ and an angular kernel $b \in L^{1+}(S^{d-1})$, was established by Gamba, Panferov and Villani [35]. More precisely, they showed the propagation in time of estimates (1.3) with $s = 2$, and rates $\alpha < \alpha_0$ depending on the rate α_0 of the initial data and few moments of the initial state. In that manuscript the authors also gave a proof of propagation of L^∞-Maxwellian weighted bounds, uniformly in time. This is a revealing fact which implies that any solution of the elastic initial value problem for the d-dimensional Boltzmann equation, with variable hard potentials and integrable angular cross section $b \in L^{1+}(S^d)$, decays in $L^\infty(\mathbb{R}^d)$ like a Maxwellian with a constant rate α_2, uniformly in time, as long as the initial state has finite L^∞ exponentially weighted norm with a rate α_0. The constant α_2 depends on the first few moments of the initial state, and it is smaller than α, where $\alpha < \alpha_0$ is the rate of the Maxwellian weight from the L^1 propagation result. Their results follow from the application of a maximum principle of parabolic type, due to the dissipative nature of the collision integral, and estimates of the classical Carleman representation of the gain (positive) part of the collision operator that depend on the L^1-Maxwellian weighted bounds uniformly propagated in time.

These techniques were also used by Alonso and Gamba [8] to show both propagation of L^1-Maxwellian and L^∞-Maxwellian weighted estimates for all derivatives of the solution to the initial value problem to space homogeneous Boltzmann equations under the same conditions as in [35]. In addition, Alonso and Lods [11] used these techniques to show the Haff law of decay rate to homogeneous cooling states for the inelastic Boltzmann equation for rarefied granular flows.

The techniques from [17,18,35] were also used by Mouhot [48] to establish, for the elastic case under the same assumptions on the angular function as in [18], the instantaneous generation of L^1-exponential bounds uniformly in time, with only $L^1_2 \cap L^2$ initial data, with the exponential of order up to $s = \gamma/2$, with γ being the variable hard potential exponent, and a time dependent rate $\alpha(t)$.

Recently Alonso, Cañizo, Gamba and Mohout [6] introduced a new technique (based on analyzing partial sums corresponding to the infinite sum appearing in (3.11)), to prove the generation of exponential moments with orders up to $s = \gamma$ and the propagation of exponential moments with orders $0 < s \leq 2$. This was done under the Grad's cutoff assumption of just $b \in L^1(S^d)$. It is interesting that these results do not rely on the rate of Povzner estimates for angular averaging, which was the case in the above mentioned works.

The only existing result in the non-cutoff case was established by Lu and Mouhot [44], where they showed only generation of exponential moments of order up to $s = \gamma$. No information was available about exponential moments of higher orders $s \in (\gamma, 2]$, which is what together with the partial sum technique of Alonso, Cañizo, Gamba and Mohout [6] motivated our project.

3.3 A strategy of the proof

Here we outline our strategy for proving the propagation of moments, Theorem 3.1.1 (b). Proof of the generation of moments is similar. Details are provided in Section 3.7 and Section 3.8. The proof is inspired by the recent work of Alonso, Cañizo, Gamba, Mouhot [6], where the partial sum technique was developed to show propagation and generation of exponential moments (1.3) for the Grad's cutoff case.

Our goal is to prove that a solution $f(t, v)$ of the Boltzmann equation for hard potentials and an angular non-cutoff condition admits L^1-Mittag-Leffler moments (3.1.2) of order s and some rate $\alpha(t)$. Our proof is based on studying partial sums of these moments. To this end, we work with the n-th partial sum defined as

$$E_a^n(\alpha, t) := \sum_{q=0}^{n} \frac{m_{2q}(t)\ \alpha^{aq}}{\Gamma(aq+1)}, \tag{3.12}$$

where

$$a = \frac{2}{s}.$$

Given $s \in (0,2)$, we need to prove that there exits a positive rate $\alpha(t)$ so that $E_a^n(t)$ is uniformly bounded in time and in n, so that the sequence of finite sums converges as $n \to \infty$. The value of α and the bound of the partial sums are found and shown to depend on parameters of the collision kernel and properties of the initial data. This uniform bound of $E_a^n(t)$ is proved by a "continuity argument". Define the time T_n by:

$$T_n := \sup \left\{ t \geq 0 \mid E_a^n(\alpha, \tau) < 4M_0, \text{ for all } \tau \in [0,t) \right\}, \tag{3.13}$$

where the constant M_0 is the one from the initial condition (3.8).

If T_n is well-defined and positive (which will be checked later), then $E_a^n(t) < 4M_0$ holds for $t \in [0, T_n)$. We will prove that in fact the inequality is true on the closed interval $[0, T_n]$, so by continuity of $E_a^n(t)$ it actually holds on a slightly larger interval. Unless $T_n = +\infty$, this would contradict the fact that T_n was the largest time for which the bound holds.

In order to achieve all of this, we derive a differential inequality for $E_a^n = E_a^n(\alpha, t)$. The proof proceeds in the following steps. All inequalities that follow are valid on the closed interval $[0, T_n]$.

Step 1. Derive ODI for polynomial moments. The first step is to obtain differential inequalities for moments $m_{2q}(t)$, by studying the balance

$$m'_{2q}(t) = \int_{\mathbb{R}^d} Q(f,f)(t,v) \, \langle v \rangle^{2q} dv, \tag{3.14}$$

which is a consequence of the Boltzmann equation. The right hand side requires finding the estimates of the weak formulation of the collision operator

(2.10) with test functions $\phi(v) = \langle v \rangle^{2q}$. Consequently, we need to estimate the angular integration

$$\int_{\mathbb{S}^{d-1}} \left(\langle v' \rangle^{2q} + \langle v'_* \rangle^{2q} - \langle v \rangle^{2q} - \langle v_* \rangle^{2q} \right) b(\cos\theta) d\sigma. \tag{3.15}$$

This will require the key tool of the proof - the angular averaged Povzner estimate for the non-cutoff case. This is the key ingredient of the proof and is explained in details in Section 3.5. These estimates will lead to the following differential inequalities for polynomial moments:

$$\begin{aligned} m'_{2q} \;\; \leq \;\; & -K_1\, m_{2q+\gamma} \; + \; K_2\, m_{2q} \\ & + K_3\, \varepsilon_q\, q\,(q-1) \sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q-2}{k-1} \left(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k}\, m_{2(q-k)+\gamma} \right), \end{aligned} \tag{3.16}$$

where $K_1 = A_2 C_\gamma$, with A_2 as defined in (3.30) and C_γ depending on the rate of potentials γ. Similarly K_2 and K_3 depend on these parameters as well. The key property of this inequality is that the highest order moment of the right-hand side comes with a negative sign which is crucial for moment propagation and generation. Another important aspect of this differential inequality is the presence of the factor $q(q-1)$ in the last term, which was absent in the Grad's cutoff case. Because of it, it will be of great importance to know the decay rate for ε_q.

Step 2. Derive ODI for partial sums - part 1. The second step consists in the derivation of a differential inequality for partial sums $E_a^n = E_a^n(\alpha, t)$ obtained by adding n inequalities corresponding to (3.16) for the renormalized polynomial moments $m_{2q}(t)\alpha^{aq}/\Gamma(aq+1)$. This will yield

$$\frac{d}{dt}E_a^n \leq c_{q_0} + \Big(- K_1\, I_{a,\gamma}^n + K_1\, c_{q_0} + K_2\, E_a^n + \varepsilon_{q_0}\, q_0^{2-a}\, K_3\, C\, E_a^n\, I_{a,\gamma}^n \Big) . \quad (3.17)$$

In particular we obtain an ordinary differential inequality for the partial sum E_a^n that depends on a shifted partial sum $I_{a,\gamma}^n$, defined by

$$I_{a,\gamma}^n(\alpha, t) = \sum_{q=0}^{n} \frac{m_{2q+\gamma}(t)\ \alpha^{aq}}{\Gamma(aq+1)}. \quad (3.18)$$

The derivation of the last term in the right hand side of (3.17) requires a decay property of combinatoric sums of Beta functions. These estimates are presented in detail in Lemma 3.4.4 and Lemma 3.4.5. The constants K_1, K_2 and K_3 only depend on the singularity conditions (2.9), and so they are independent of n and on any moment q. The constant c_{q_0} depends only on a finite number q_0 of moments of the initial data. The choice of q_0 is crucial to control the long time behavior of solutions to inequality (3.17), and it is done so that $\varepsilon_{q_0}\ q_0^{2-a}\, K_3 < K_1/2$, after using the decay property of ε_{q_0} (3.32) in Lemma 3.5.1.

Step 3. Derive ODI for partial sums - part 2. Finally, after showing that $I_{a,\gamma}^n(\alpha, t)$ is bounded below by the sum of two terms depending linearly on $E_a^n(\alpha, t)$ and on mass m_0, and nonlinearly on the rate α, we obtain the following differential inequality for partial sums in the case of propagation of Mittag-Leffler moments

$$\frac{d}{dt}E_a^n(t) \leq -\frac{K_1}{2\,\alpha^{\frac{\gamma}{2}}}E_a^n(t) + \frac{K_1\, m_0\, e^{\alpha^{a-1}}}{2\alpha^{\frac{\gamma}{2}}} + \mathcal{K}_0 \quad \textit{(Propagation estimate).}$$

The constant $\mathcal{K}_0$ depends on parameters characterizing q_0, c_{q_0} and K_i, $i = 1, 2, 3.$. In addition, for the generation case, we obtain

$$\frac{d}{dt} E^n_\gamma \leq -\frac{1}{t}\left(\frac{K_1(E^n_\gamma - m_0)}{2\alpha} - C_{q_0}\right) + \mathcal{K}_0 \quad \textit{(Generation estimate).}$$

Thus, the differential inequalities (3.17) are reduced to linear ones. Both inequalities have desired uniform bounds for a sufficiently small parameter α, which is independent of n and time t, and will depend on q_0, which depends only on data parameters.

3.4 Useful tools for the proof

In this subsection we gather several inequalities related to binomial coefficients and binomial sums. The first two lemmas focus on elementary polynomial inequalities that will be used to derive ordinary differential inequalities for polynomial moments in Section 3.6.

Lemma 3.4.1 (Polynomial inequality I). *Let $b \leq a \leq \frac{s}{2}$. Then for any $x, y \geq 0$,*

$$x^a y^{s-a} + x^{s-a} y^a \ \leq\ x^b y^{s-b} + x^{s-b} y^b. \tag{3.19}$$

Remark 3.4.1. This lemma is useful for comparing products of moments. Namely, as its consequence, we have that for a fixed s, the sequence $\{m_k\, m_{s-k}\}_k$ is decreasing in k, for $k = 1, 2, ..., \lfloor s/2 \rfloor :=$ Integer Part of $s/2$. For example, if $s \geq 4$, then $m_2 m_{s-2} \leq m_1 m_{s-1}$.

Proof: Note that a, b and s satisfy $a - b \geq 0$ and $s - a - b \geq 0$. Therefore

$$\left(y^{a-b} - x^{a-b}\right) x^b y^b \left(y^{s-a-b} - x^{s-a-b}\right) \geq 0,$$

which is easily checked to be equivalent to the inequality (3.19). □

Lemma 3.4.2 (Polynomial inequality II, Lemma 2 in [18])**.** *Assume* $p > 1$*, and let* $k_p = \lfloor (p+1)/2 \rfloor$*. Then for all* $x, y > 0$ *the following inequalities hold*

$$\sum_{k=1}^{k_p-1} \binom{p}{k} (x^k y^{p-k} \,+\, x^{p-k} y^k) \,\le\, (x+y)^p \,-\, x^p \,-\, y^p \,\le\, \sum_{k=1}^{k_p} \binom{p}{k} (x^k y^{p-k} \,+\, x^{p-k} y^k).$$

Remark 3.4.2. Using this lemma, it is easy to see a coarse, but useful estimate

$$\sum_{k=0}^{k_p} \binom{p}{k} (x^k y^{p-k} \,+\, x^{p-k} y^k) \,\le\, 2(x+y)^p. \tag{3.20}$$

Next, we recall the basic definitions and properties of the Gamma function $\Gamma(x)$ and the Beta function $B(x,y)$, which are useful for our further estimates. They are defined via

$$\Gamma(x) \,=\, \int_0^\infty t^{x-1}\, e^{-t}\, dt, \qquad \text{and} \qquad B(x,y) \,=\, \int_0^1 t^{x-1}\, (1-t)^{y-1}\, dt. \tag{3.21}$$

Two fundamental properties of these well-know functions are

$$\Gamma(x+1) = x\, \Gamma(x), \qquad \text{and} \qquad B(x,y) = \frac{\Gamma(x)\, \Gamma(y)}{\Gamma(x+y)} \tag{3.22}$$

The following classic result for estimates of generalized Laplace transforms will be needed to estimate the combinatoric sums of Beta functions to be shown in the subsequent Lemma 3.4.4.

Lemma 3.4.3. *Let* $0 < \alpha, R < \infty$*,* $g \in C([0,R])$ *and* $S \in C^1([0,R])$ *be such that* $S(0) = 0$ *and* $S'(x) < 0$ *for all* $x \in [0,R]$*. Then for any* $\lambda \ge 1$ *we have*

$$\int_0^R x^{\alpha-1}\, g(x)\, e^{\lambda S(x)}\, dx = \Gamma(\alpha)\, \left(\frac{1}{-\lambda\, S'(0)} \right)^{\alpha}\, (g(0) + o(1)).$$

The proof of this estimate is a direct application of the Laplace's method for asymptotic expansion of integrals that can be found in [49], page 81, Theorem 7.1.

The next two lemmas estimate a combinatoric sum of Beta functions. These estimates are inspired by [18, Lemma 4] and [44, Lemma 3.3]. However, in our context, the arguments of Beta functions are shifted, so we compute exact decay rates for our situation. These estimates are crucial to control the growth in q of the ordinary differential inequality of partial sums of renormalized moments.

The first lemma will be used for the proof of propagation of moments with $a = 2/s$, while the second will be used for the generation of moments with $s = \gamma$.

Lemma 3.4.4 (First estimate on combinatoric sums of Beta Functions). *Let $q \geq 3$ and $k_q = [(q+1)/2]$. Then for any $a > 1$ we have*

$$\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \leq \frac{C_a}{(aq)^{1+a}}, \tag{3.23}$$

where the constant C_a depends only on a.

Proof: Reindexing the summation by changing $k-1$ into k and rearranging the integral forms defining Beta functions, yields

$$\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1)$$

$$= \sum_{k=0}^{k_q-1} \binom{q-2}{k} \; B(a(k+1)+1, a(q-k-1)+1)$$

$$= \frac{1}{2} \int_0^1 \sum_{k=0}^{k_q-1} \binom{q-2}{k} \left(x^{a(k+1)} \; (1-x)^{a(q-k-1)} \; + \; x^{a(q-k-1)} \; (1-x)^{a(k+1)} \right) \, dx$$

$$= \frac{1}{2} \int_0^1 x^a (1-x)^a \sum_{k=0}^{k_{q-2}} \binom{q-2}{k} \left(x^{ak} \; (1-x)^{a(q-2-k)} \; + \; x^{a(q-2-k)} \; (1-x)^{ak} \right) \, dx$$

$$= \frac{1}{2} \int_0^1 x^a (1-x)^a \sum_{k=0}^{k_p} \binom{p}{k} \left(x^{ak} \; (1-x)^{a(p-k)} \; + \; x^{a(p-k)} \; (1-x)^{ak} \right) \, dx$$

after setting $q-2=p$ in the last integral. In particular using the estimate (3.20), the right hand side of the above sum is estimated by

$$\frac{1}{2} \int_0^1 x^a (1-x)^a \; 2 \; (x^a + (1-x)^a)^p \; dx = \int_0^1 x^a (1-x)^a \, (x^a + (1-x)^a)^{q-2} \; dx$$
$$= 2 \int_0^{1/2} x^a \; g(x) \; e^{qS(x)} \; dx,$$

where $g(x) = (1-x)^a \, (x^a + (1-x)^a)^{-2}$ and $S(x) = \log(x^a + (1-x)^a)$, for $x \in [0, 1/2]$. Finally, applying Lemma 3.4.3 for these $g(x)$ and $S(x)$ as indicated, and noting that $g(0) = 1$ and $S'(0) = -a$, yields the desired estimate

$$\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \; \leq \; C_a \; \Gamma(a+1) \; \left(\frac{1}{aq} \right)^{a+1} . \qquad (3.24)$$

□

Lemma 3.4.5 (Second estimate on combinatoric sums of Beta Functions)**.** *Let* $0 < s \leq 1$ *and* $q \geq 3$*. Denote* $k_p = [(p+1)/2]$*, for any* $p \in \mathbb{R}$*. Then, there* *exits a constant C, independent of q, such that*

$$\sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{s}}} \binom{\frac{q}{2}-\frac{2}{s}}{k-1} B(2k+1,\ q-2k+1) \leq \frac{C}{q^3}. \tag{3.25}$$

Proof: First we note a simple property of binomial coefficients. For any integer $k \in \mathbb{N}_0$ and any real numbers $\tilde{a}, a \in \mathbb{R}$ that satisfy $\tilde{a} \geq a \geq k$,

$$\binom{a}{k} \leq \binom{\tilde{a}}{k}. \tag{3.26}$$

This is easily proved by noting that the binomial coefficient $\binom{a}{k}$ (and similarly $\binom{\tilde{a}}{k}$) can be computed as

$$\binom{a}{k} = \frac{a\ (a-1)\ (a-2)\ldots(a-k+1)}{k!}.$$

Next, since $s \leq 1$,

$$\frac{q}{2} - \frac{2}{s} \leq \frac{q}{2} - 2. \tag{3.27}$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{s}}} \binom{\frac{q}{2}-\frac{2}{s}}{k-1} B(2k+1,\ q-2k+1) \\
&\quad\leq \sum_{k=1}^{1+k_{\frac{q}{2}-2}} \binom{\frac{q}{2}-2}{k-1} B(2k+1,\ q-2k+1) \\
&\quad= \sum_{k=1}^{k_{\frac{q}{2}}} \binom{\frac{q}{2}-2}{k-1} B\left(2k+1,\ 2\left(\frac{q}{2}-k\right)+1\right).
\end{aligned} \tag{3.28}$$

Now applying (3.23) yields (3.25). □

3.5 Angular averaging lemma

In this section we prove the key ingredient of the proof of Theorem 3.1.1, namely the angular averaging estimate for the non-cutoff case. This lemma gives an estimate of the weight function G_ϕ in the weak formulation (2.10) when the test function is a monomial $\phi(v) = \langle v\rangle^{rq}$. We denote this weight function by

$$G_{rq} := G_{\langle v\rangle^{rq}} := \int_{S^{d-1}} \left(\langle v'\rangle^{rq} + \langle v'_*\rangle^{rq} - \langle v\rangle^{rq} - \langle v_*\rangle^{rq}\right) B(|u|, \hat{u}\cdot\sigma) \; d\sigma. \tag{3.29}$$

Due to the presence of the non-integrable angular singularity, subtle cancellations between the gain and the loss terms need to be exploited.

Lemma 3.5.1. *Suppose that the angular kernel $b(\cos\theta)$ satisfies the non-cutoff condition (2.9) with $\beta = 2$. Let $r, q > 0$. Then the weight function satisfies*

$$\begin{aligned} G_{rq}(v, v_*) \le |v - v_*|^\gamma \Big[& - A_2 \left(\langle v\rangle^{rq} + \langle v_*\rangle^{rq}\right) \quad + A_2 \left(\langle v\rangle^{rq-2}\langle v_*\rangle^2 + \langle v\rangle^2\langle v_*\rangle^{rq-2}\right) \\ & + \varepsilon_{qr/2}\, A_2 \frac{qr}{2}\left(\frac{qr}{2} - 1\right) \langle v\rangle^2\langle v_*\rangle^2 \left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{\frac{qr}{2}-2} \Big], \end{aligned} \tag{3.30}$$

where the constant $A_2 = |S^{d-2}| \int_0^\pi b(\cos\theta)\, \sin^d\theta \, d\theta$ is finite by (2.9). The sequence $\varepsilon_{qr/2} =: \varepsilon_{\mathbf{q}}$, defined as

$$\varepsilon_{\mathbf{q}} := \frac{2}{A_2}|S^{N-2}| \int_0^\pi \left(\int_0^1 t \left(1 - \frac{\sin^2\theta}{2} t\right)^{\mathbf{q}-2} dt \right) b(\cos\theta)\; \sin^d\theta \; d\theta, \tag{3.31}$$

has the following decay properties. If $b(\cos\theta)$ satisfies the non-cutoff assumption (2.9) with $\beta \in (0, 2]$, then

$$0 \; < \; \varepsilon_{\mathbf{q}}\, \mathbf{q}^{1-\frac{\beta}{2}} \; \to 0, \qquad \text{as } \mathbf{q} \to \infty. \tag{3.32}$$

Remark 3.5.1. The sequence $\varepsilon_{\mathbf{q}}$ is the same as in [44]. Its decay properties (3.32) are also proved in [44], after invoking angular averaging and the dominated convergence theorem. Condition (3.32) is crucial for finding the highest order s of Mittag-Leffler moment that can be propagated in time.

Remark 3.5.2. The decay rate of $\varepsilon_{\mathbf{q}}$ is fundamental for the success of summability arguments, yet is not relevant for the generation and propagation of polynomial moments. In the Grad's cutoff case when term-by-term techniques were used, the corresponding constant had a rate $\varepsilon_q \approx q^{-r}$, with r depending on the integrability of b, see [17, 18, 35]. When the partial sum technique was employed in [6], the precise rate was not needed any longer. Here however, in the non-cutoff case, the knowledge of the precise decay rate of $\varepsilon_{\mathbf{q}}$ becomes important again because of the extra power of q in the last term of the right-hand side of (3.5.1).

Our proof of Lemma 3.5.1, while inspired by the one given in [44], produces an improvement that enable us, among other things, to obtain exponential and Mittag-Leffler moments up to order $s < 2$. This improvement is a direct consequence of the following estimate on symmetrized convex binomial expansions. namely the angular averaged Povzner estimate for the non-cutoff case.

Lemma 3.5.2. [Symmetrized convex binomial expansions estimate]

Let $a, b \geq 0$*,* $t \in [0,1]$ *and* $p \in (0,1] \cup [2,\infty)$*. Then*

$$\Big(ta + (1-t)b\Big)^p + \Big((1-t)a + tb\Big)^p - a^p - b^p$$
$$\leq -2t(1-t)\Big(a^p + b^p\Big) + 2t(1-t)\Big(ab^{p-1} + a^{p-1}b\Big). \tag{3.33}$$

Proof: Suppose $p \geq 2$. The case $p \in (0,1]$ can be done analogously. Due to the symmetry of the inequality (3.33), we may without the loss of generality assume that $a \geq b$. Since all the terms have homogeneity p, the inequality (3.33) is equivalent to showing

$$F(z) \geq 0, \qquad \forall z \geq 1,$$

where $F(z)$ is defined by

$$F(z) := \Big(1{-}2t(1{-}t)\Big)\big(z^p{+}1\big) + 2t(1{-}t)\big(z{+}z^{p-1}\big) - \Big(tz{+}(1{-}t)\Big)^p - \Big((1{-}t)z{+}t\Big)^p.$$

It is easy to check that

$$F''(z) = (p-1)\Bigg[p\Big(1-2t(1-t)\Big)z^{p-2} + 2t(1-t)(p-2)z^{p-3}$$
$$- pt^2\Big(tz + (1-t)\Big)^{p-2} - p(1-t)^2\Big((1-t)z + t\Big)^{p-2}\Bigg].$$

As $tz + (1-t)$ and $(1-t)z + t$ are two convex combinations of z and 1, and since $z \geq 1$, we have that $tz + (1-t) \leq z$ and $(1-t)z + t \leq z$. Since $p \geq 2$, this implies $(tz + (1-t))^{p-2} \leq z^{p-2}$ and $((1-t)z + t)^{p-2} \leq z^{p-2}$. Therefore,

$$\begin{aligned}\frac{F''(z)}{p-1} &\geq p\big(1-2t(1-t)\big)z^{p-2} + 2t(1-t)(p-2)z^{p-3} - pt^2z^{p-2} - p(1-t)^2z^{p-2}\\ &= 2t(1-t)(p-2)z^{p-3}\\ &\geq 0.\end{aligned}$$

Thus, $F''(z) \geq 0$ for $z \geq 1$. So, $F'(z)$ is increasing. Since $F'(1) = 0$, we have that $F'(z) \geq 0$ for $z \geq 1$. Finally using the fact that $F(1) = 0$, we conclude $F(z) \geq 0$ for $z \geq 1$. $\square$

We are now ready to prove the new form of the angular averaged lemma.

Proof of Lemma 3.5.1 Recall the definition of the weight G_{rq}

$$G_{rq}(v, v_*) := |v - v_*|^\gamma \int_{S^{d-1}} b(\cos\theta) \sin^{d-2}\theta \, \Delta\langle v\rangle^{rq} \, d\sigma, \tag{3.34}$$

where $\Delta\langle v\rangle^{rq} = \langle v'\rangle^{rq} + \langle v'_*\rangle^{rq} - \langle v\rangle^{rq} - \langle v_*\rangle^{rq}$.

This integral is rigorous even in cases when $\int_{S^{d-1}} B(|u|, \cos\theta) \; d\sigma$ is unbounded, by an angular cancellation. A natural way of handling the cancellation is to decompose $\sigma \in S^{d-1}$ into $\theta \in [0, \pi]$ and its corresponding azimuthal variable $\omega \in S^{d-2}$, i.e.

$$\sigma = \cos\theta \, \hat{u} + \sin\theta \, \omega,$$

where $S^{d-2}(\hat{u}) = \{\omega \in S^{d-1} : \omega \cdot \hat{u} = 0\}$.

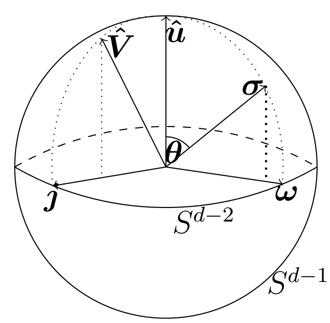


Figure 3.1: Decomposition of σ.

This decomposition allows handling the lack of integrability concentrated at the origin of the polar direction $\theta = 0$. To see this, a specific way of decomposing $\langle v'\rangle^2$ and $\langle v'_*\rangle^2$ that separates the part that depends on ω is convenient. More precisely, $\langle v'\rangle^2$ and $\langle v'_*\rangle^2$ are decomposed into a sum of a convex combination of the local energies proportional to a function of the polar angle θ, and another term depending on both the polar angle and ω

$$\langle v'\rangle^2 = E_{v,v_*}(\theta) \; + \; P(\theta,\omega), \tag{3.35}$$

$$\langle v'_*\rangle^2 = E_{v,v_*}(\pi-\theta) \; - \; P(\theta,\omega).$$

Here

$$P(\theta,\omega) = |v\times v_*| \, \sin\theta \, (j\cdot\omega),$$

where the vector $j \in S^{d-2}$ is obtained by projecting the center of mass $V = \frac{v+v_*}{2}$, or its unit direction $\hat{V}$. $P(\theta,\omega)$ is a null form in ω by averaging, i.e.

$$\int_{S^{d-2}} P(\theta,\omega) d\omega = 0,$$

and $E_{v,v_*}(\theta)$ is a convex combination of $\langle v\rangle^2$ and $\langle v'_*\rangle^2$ given by

$$E_{v,v_*}(\theta) \; = \; t\,\langle v\rangle^2 \; + \; (1-t)\,\langle v_*\rangle^2, \qquad \text{where} \quad t \; = \; \sin^2\frac{\theta}{2}.$$

To verify the representation of, for example $\langle v'\rangle^2$, in (3.35), recall that

$$v' = \frac{v+v_*}{2} + \frac{1}{2}|u|\sigma.$$

Hence,

$$\begin{aligned}
\langle v'\rangle^2 &= 1 + \frac{|v+v_*|^2}{4} + \frac{|v-v_*|^2}{4} + \frac{1}{2}|u|\sigma \\
&= 1 + \frac{|v|^2+|v_*|^2}{2} + \frac{1}{2}|u|(v+v_*)\cdot(\hat{u}\cos\theta + \omega\sin\theta) \\
&= 1 + \frac{|v|^2+|v_*|^2}{2} + \frac{1}{2}(v+v_*)\cdot(v-v_*)\cos\theta + \frac{1}{2}|u||V|\sin\theta(\hat{V}\cdot\omega) \\
&= 1 + |v|^2\cos^2\frac{\theta}{2} + |v_*|^2\sin^2\frac{\theta}{2} + \frac{1}{2}|u||V|\sin\theta(j\cdot\omega)\sin\alpha \\
&= \langle v\rangle^2\cos^2\frac{\theta}{2} + \langle v_*\rangle^2\sin^2\frac{\theta}{2} + |v\times v_*|\sin\theta(j\cdot\omega),
\end{aligned}$$

which coincides with the representation of $\langle v'\rangle^2$ in (3.35).

The decomposition (3.35), of local energies into a convex combination and a null form in ω, make the weight function $G_{rq}(v, v_*)$ well defined for every v and v_* for sufficiently smooth test functions ($\phi \in C^2(\mathbb{R}^d)$) even under the non-cutoff assumption (2.9) with $\beta = 2$. In fact Taylor expansions associated to $\langle v'\rangle^{rq}$ are a sum of a power of $E_{v,v_*}(\theta)$, plus a null form in the azimuthal direction, plus a residue proportional to $\sin^2\theta$ that will secure the integrability of the angular cross section with respect to the scattering angle θ.

While some of these estimates are found also in [44], we still provide details below for completeness. Indeed, we Taylor expand $\langle v'\rangle^{rq}$ around $E(\theta)$ up to the second order to obtain

$$\begin{aligned}\langle v'\rangle^{rq} &= \Big(E_{v,v_*}(\theta) \,+\, h\sin(\theta)\,(j\cdot\omega)\Big)^{\frac{rq}{2}} \\ &= \big(E_{v,v_*}(\theta)\big)^{rq/2} + \frac{rq}{2}\,\big(E_{v,v_*}(\theta)\big)^{\frac{rq}{2}-1} h\,\sin\theta\,(j\cdot\omega) \\ &+ \frac{rq}{2}\left(\frac{rq}{2}-1\right) h^2\,\sin^2\theta\,(j\cdot\omega)^2 \int_0^1 (1-t)\big[E(\theta) + t\,h\,\sin\theta\,(j\cdot\omega)\big]^{\frac{rq}{2}-2}\,dt.\end{aligned} \tag{3.36}$$

A similar identity can be obtained for $\langle v'_*\rangle^{rq}$.

Since the collision cross section is independent of the azimuthal direction ω, and since $\int_{S^{d-2}} j\cdot\omega\; d\omega = 0$, we can write $G_{rq}(v, v_*)$ as the sum of two integrals on the S^{d-1} sphere, whose first integrand contains the zero-order term of the Taylor expansion of both $\langle v'_*\rangle^{rq}$ and $\langle v'\rangle^{rq}$ subtracted by their corresponding un-primed forms, while the second integrand is just the second order term of the Taylor expansion (3.37)

$$
\begin{aligned}
G_{rq}(v,v_*) &= I_1 \;+\; I_2 \qquad (3.37)\\
&= \int_0^{\pi}\int_{S^{d-2}} \Big(E_{v,v_*}(\theta)^{rq/2} + E_{v,v_*}(\pi-\theta)^{rq/2} - \langle v\rangle^{rq} - \langle v_*\rangle^{rq}\Big)\, b(\cos\theta)\;\sin^{d-2}\theta\, d\omega\, d\theta\\
&\quad + \frac{rq}{2}\Big(\frac{rq}{2}-1\Big)\, h^2 \int_0^{\pi} \sin^d\theta\, b(\cos\theta)\int_{S^{d-2}} (j\cdot\omega)^2 \int_0^1 (1-t)\\
&\quad \Big(\big[E_{v,v_*}(\theta) \;+\; t\,h\,\sin\theta\,(j\cdot\omega)\big]^{\frac{rq}{2}-2} + \big[E_{v,v_*}(\pi-\theta) \;-\; t\,h\,\sin\theta\big]^{\frac{rq}{2}-2}\Big)\, dt d\omega d\theta.
\end{aligned}
$$

At this point we use polynomial inequality (3.33) to estimate the first integral I_1. We use it with $a=\langle v\rangle^2$, $b=\langle v_*\rangle^2$ and $t=\cos^2\frac{\theta}{2}$, and this yields

$$
\begin{aligned}
I_1 \;&\le\; \big|S^{d-2}\big| \int_0^{\pi} -\frac{\sin^2\theta}{2}\,\Big(\langle v\rangle^{rq} + \langle v_*\rangle^{rq}\Big)\, b(\cos\theta)\;\sin^{d-2}\theta\, d\theta\\
&\qquad + \int_0^{\pi} \frac{\sin^2\theta}{2}\,\Big(\langle v\rangle^{rq-2}\langle v_*\rangle^2 + \langle v\rangle^2\langle v_*\rangle^{rq-2}\Big)\, b(\cos\theta)\;\sin^{d-2}\theta\, d\theta\\
&=\; -A_2\Big(\langle v\rangle^{rq} + \langle v_*\rangle^{rq}\Big) \;+\; A_2\Big(\langle v\rangle^{rq-2}\langle v_*\rangle^2 + \langle v\rangle^2\langle v_*\rangle^{rq-2}\Big).
\end{aligned}
\qquad (3.38)
$$

The constant A_2 was defined after (3.30).

For the second order term I_2, we use that $(j\cdot\omega)^2 \le 1$ and $h=|v\times v_*|\le \langle v\rangle\,\langle v_*\rangle$, and that (see [44])

$$
\big|E_{v,v_*}(\theta) + th\sin\theta\;(j\cdot\omega)\big| \le \Big(\langle v\rangle^2 + \langle v_*\rangle^2\Big)\left(1-\frac{t}{4}\sin^2\theta\right), \qquad (3.39)
$$

to conclude

$$
\begin{aligned}
I_2(r) \le \frac{rq}{2}\Big(\frac{rq}{2}-1\Big)\;\langle v\rangle^2\langle v_*\rangle^2\,\big|S^{d-2}\big|\int_0^{\pi}\sin^d\theta\, b(\cos\theta)\cdot\\
\cdot\int_0^1 2(1-t)\,\big(\langle v\rangle^2+\langle v_*\rangle^2\big)^{\frac{rq}{2}-2}\,\left(1-\frac{1-t}{4}\sin^2\theta\right)^{\frac{rq}{2}-2}\; dt\, d\theta.
\end{aligned}
$$

After a simple change of variables ($t \mapsto 1-t$) and recalling the definition of constant $\varepsilon_{rq/2}$ in (3.31), we see that

$$I_2(r) \leq \varepsilon_{rq/2}\, A_2 \frac{rq}{2}\left(\frac{rq}{2}-1\right) \langle v\rangle^2 \langle v_*\rangle^2 \left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{\frac{rq}{2}-2}. \tag{3.40}$$

Putting together the estimate for I_1 and for I_2, we obtain the desired estimate on the weight $G_{rq}(v, v_*)$. $\square$

3.6 Ordinary differential inequalities for moments

In this section we present two differential inequalities for polynomial moments (Proposition 3.6.1) which will be essential for the proof of Theorem 3.1.1. We also state and prove a result about generation of polynomial moments in the non-cutoff case (Proposition 3.6.2). Before we state the proposition, we recall the "floor function" of a real number, which in the case of a positive real number $x \in \mathbb{R}^+$ coincides with the integer part of x

$$\lfloor x \rfloor := \text{integer part of } x. \tag{3.41}$$

Proposition 3.6.1. *Suppose all the assumptions of Theorem 3.1.1 are satisfied. Let $q \in \mathbb{N}$, and define $k_p = \lfloor \frac{p+1}{2} \rfloor$ for any $p \in \mathbb{R}$ to be the integer part of $(p+1)/2$. Then for some constants $K_1, K_2, K_3 > 0$ (depending on γ, $b(\cos\theta)$, dimension d, and initial mass and energy) we have the following two ordinary differential inequalities for polynomial moments of the solution f to the Boltzmann equation*

(a) *The "$m_{\gamma k}$ version" needed for the generation of exponential moments*

$$m'_{\gamma q}(t) \leq -K_1\, m_{\gamma q+\gamma} \;+\; K_2\, m_{\gamma q} + K_3\, \varepsilon_{\frac{q\gamma}{2}}\, \frac{q\gamma}{2}\left(\frac{q\gamma}{2}-1\right)\cdot \tag{3.42}$$
$$\cdot \sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{\gamma}}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} \left(m_{2\gamma k+\gamma}\, m_{\gamma q-2\gamma k} + m_{2\gamma k}\, m_{\gamma q-2\gamma k+\gamma}\right).$$

(b) *The "m_{2k} version" needed for propagation of Mittag-Leffler moments*

$$m'_{2q} \leq -\,K_1\, m_{2q+\gamma} \;+\; K_2\, m_{2q} \tag{3.43}$$
$$+\; K_3\, \varepsilon_q\, q(q-1) \sum_{k=1}^{k_q} \binom{q-2}{k-1} \left(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k}\, m_{2(q-k)+\gamma}\right).$$

In both cases, the constant $K_1 = A_2 C_\gamma$, where A_2 was defined in (3.30) *and C_γ, to be defined in the proof below, only depends on the γ rate of the hard potentials. Similarly K_2 and K_3, also depend on data, through the dependence on A_2 and C_γ.*

Proof: We begin the proof by analyzing m_{rq} with a general monomial weight $\langle v\rangle^{rq}$. Then by setting $r=\gamma$ we shall derive (a) and by setting $r=2$ we shall obtain (b).

Recall that after multiplying the Boltzmann equation (2.16) by $\langle v\rangle^{rq}$, the weak formulation (2.10) yields

$$m'_{rq}(t) = \frac{1}{2} \iint_{\mathbb{R}^{2d}} f f_*\; G_{rq}(v, v_*)\, dv\, dv_*. \tag{3.44}$$

In fact, since a polynomial $\phi(v) = \langle v\rangle^{rq}$ is not admissible test function in the definition of a weak solution (Definition 3.1.1), an approximation argument is needed to rigorously show (2.10). Such a procedure is standard and can be found for example in [44, Section 4].

The weight function G_{rq} can be estimated as in Proposition 3.5.1, which yields

$$\begin{aligned} m'_{rq}(t) \leq & -\frac{A_2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\ f_*\ |v-v_*|^\gamma\ \Big(\langle v\rangle^{rq}+\langle v_*\rangle^{rq}\Big)dvdv_* \\ & + \frac{A_2}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\ f_*\ |v-v_*|^\gamma\ \Big(\langle v\rangle^{rq-2}\langle v_*\rangle^2+\langle v\rangle^2\langle v_*\rangle^{rq-2}\Big)dvdv_* \\ & + \frac{A_2}{2}\,\varepsilon_{\frac{q\gamma}{2}}\,\frac{rq}{2}\left(\frac{rq}{2}-1\right)\iint_{\mathbb{R}^{2d}} f\ f_*\,|v-v_*|^\gamma\,\langle v\rangle^2\langle v_*\rangle^2\,\Big(\langle v\rangle^2+\langle v_*\rangle^2\Big)^{\frac{rq}{2}-2}dvdv_*. \end{aligned} \tag{3.45}$$

We estimate $|v-v_*|^\gamma$ via elementary inequalities

$$|v-v_*|^\gamma \leq C_\gamma^{-1}\big(\langle v\rangle^\gamma+\langle v_*\rangle^\gamma\big) \qquad \text{and} \qquad |v-v_*|^\gamma \geq C_\gamma\langle v\rangle^\gamma-\langle v_*\rangle^\gamma, \tag{3.46}$$

where $C_\gamma = \min\{1, 2^{1-\gamma}\}$ (see for example [6]). As an immediate consequence

$$\begin{aligned} |v-v_*|^\gamma & \Big(\langle v\rangle^{rq}+\langle v_*\rangle^{rq}\Big) \\ & \geq \Big(C_\gamma\langle v\rangle^\gamma-\langle v_*\rangle^\gamma\Big)\langle v\rangle^{rq}+\Big(C_\gamma\langle v_*\rangle^\gamma-\langle v\rangle^\gamma\Big)\langle v_*\rangle^{rq} \\ & = C_\gamma\Big(\langle v\rangle^{rq+\gamma}+\langle v_*\rangle^{rq+\gamma}\Big)-\Big(\langle v\rangle^{rq}\langle v_*\rangle^\gamma+\langle v\rangle^\gamma\langle v_*\rangle^{rq}\Big), \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} |v-v_*|^\gamma & \Big(\langle v\rangle^{rq-2}\langle v_*\rangle^2+\langle v\rangle^2\langle v_*\rangle^{rq-2}\Big) \\ & \leq C_\gamma^{-1}\Big(\langle v\rangle^\gamma+\langle v_*\rangle^\gamma\Big)\ \Big(\langle v\rangle^{rq-2}\langle v_*\rangle^2+\langle v\rangle^2\langle v_*\rangle^{rq-2}\Big) \\ & \leq 2C_\gamma^{-1}\Big(\langle v\rangle^{rq}\langle v_*\rangle^\gamma+\langle v\rangle^\gamma\langle v_*\rangle^{rq}\Big), \end{aligned} \tag{3.48}$$

where the last inequality uses Lemma 3.4.1.

Combining (3.45) with (3.47) and (3.48) we obtain

$$
\begin{aligned}
m'_{rq}(t) \leq &- \frac{A_2}{2}\, C_\gamma \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f f_* \Big(\langle v \rangle^{rq+\gamma} + \langle v_* \rangle^{rq+\gamma} \Big) dv dv_* \\
&+ \frac{A_2}{2}(1+2C_\gamma^{-1}) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\ f_* \ \Big(\langle v \rangle^{rq} \langle v_* \rangle^{\gamma} + \langle v \rangle^{\gamma} \langle v_* \rangle^{rq} \Big) dv dv_* \\
&+ \frac{A_2\ \varepsilon_{rq/2}}{2\, C_\gamma}\, \frac{rq}{2} \Big(\frac{rq}{2} - 1 \Big) \cdot \\
&\qquad \cdot \iint_{\mathbb{R}^{2d}} f f_* \Big(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \Big)\ \langle v \rangle^2 \langle v_* \rangle^2\ \Big(\langle v \rangle^2 + \langle v_* \rangle^2 \Big)^{\frac{rq}{2}-2} dv dv_* \\
\leq &- A_2\ C_\gamma m_0(t) m_{rq+\gamma}(t)\ +\ A_2 (1+2C_\gamma^{-1}) m_\gamma(t) m_{rq}(t) \\
&+ \frac{A_2\ \varepsilon_{rq/2}}{2\, C_\gamma}\, \frac{rq}{2} \Big(\frac{rq}{2} - 1 \Big) \cdot \\
&\qquad \cdot \iint_{\mathbb{R}^{2d}} f f_* \Big(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \Big) \langle v \rangle^2 \langle v_* \rangle^2 \Big(\langle v \rangle^2 + \langle v_* \rangle^2 \Big)^{\frac{rq}{2}-2} dv dv_* .
\end{aligned}
$$

The mass is conserved $m_0(t) = m_0(0)$, and since $0 < \gamma \leq 1$, monotonicity of moments and conservation of energy implies $m_\gamma(t) \leq m_2(0)$. Using these facts in the above estimate yields

$$
\begin{aligned}
m'_{rq}(t)\ \leq\ -K_1\, m_{rq+\gamma}(t) + K_2\, m_{rq}(t) + \frac{K_3}{2}\, \varepsilon_{rq/2}\, \frac{rq}{2} \Big(\frac{rq}{2} - 1 \Big) \\
\iint_{\mathbb{R}^{2d}} f f_* \Big(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \Big) \langle v \rangle^2 \langle v_* \rangle^2 \Big(\langle v \rangle^2 + \langle v_* \rangle^2 \Big)^{\frac{rq}{2}-2} dv dv_* ,
\end{aligned}
\tag{3.49}
$$

where $K_1 = A_2\, C_\gamma\, m_0(0)$, $\quad K_2 = A_2\, (1 + 2C_\gamma^{-1})\, m_2(0)$ and $K_3 = \frac{A_2}{C_\gamma}$. Thus, these constants depend on the initial mass and energy, on the rate of the potential γ and the constant A_2 determined by (2.9).

From here, we proceed to prove (a) and (b) separately.

(a) Setting $r = \gamma$ in (3.49), applying the following elementary polynomial inequality which is valid for $\gamma \in (0, 1]$

$$
\Big(\langle v \rangle^2 + \langle v_* \rangle^2 \Big)^{\frac{\gamma q}{2}-2}\ \leq\ \Big(\langle v \rangle^{2\gamma} + \langle v_* \rangle^{2\gamma} \Big)^{\frac{q}{2}-\frac{2}{\gamma}}, \tag{3.50}
$$

and using the polynomial Lemma 3.4.2 yields

$$\begin{aligned} m'_{\gamma q}(t) \leq & - K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + \frac{K_3}{2}\, \varepsilon_{\gamma q/2}\, \frac{\gamma q}{2} \left(\frac{\gamma q}{2} - 1\right) \\ & \iint_{\mathbb{R}^{2d}} f\, f_* \Big(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \Big)\, \langle v \rangle^2 \langle v_* \rangle^2\, \Big(\langle v \rangle^{2\gamma} + \langle v_* \rangle^{2\gamma} \Big)^{\frac{q}{2} - \frac{2}{\gamma}}\, dv dv_* \\ \leq & - K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + \frac{K_3}{2}\, \varepsilon_{\gamma q/2}\, \frac{\gamma q}{2} \left(\frac{\gamma q}{2} - 1\right) \iint_{\mathbb{R}^{2d}} f\, f_*\, \Big(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \Big) \\ & \sum_{k=0}^{k_{\frac{q}{2} - \frac{2}{\gamma}}} \binom{\frac{q}{2} - \frac{2}{\gamma}}{k} \Big(\langle v \rangle^{2\gamma k + 2} \langle v_* \rangle^{\gamma q - 2\gamma k - 2} + \langle v \rangle^{\gamma q - 2\gamma k - 2} \langle v_* \rangle^{2\gamma k + 2} \Big)\, dv dv_* \\ \leq & - K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + K_3\, \varepsilon_{\gamma q/2}\, \frac{\gamma q}{2} \left(\frac{\gamma q}{2} - 1\right) \cdot \\ & \sum_{k=0}^{k_{\frac{q}{2} - \frac{2}{\gamma}}} \binom{\frac{q}{2} - \frac{2}{\gamma}}{k} \Big(m_{2\gamma k + 2 + \gamma}\, m_{\gamma q - 2\gamma k - 2} + m_{\gamma q - 2\gamma k - 2 + \gamma}\, m_{2\gamma k + 2} \Big)\, dv dv_*. \end{aligned}$$

Finally, re-indexing k to $k-1$ and applying Lemma 3.4.1 yields

$$\begin{aligned} m'_{\gamma q}(t) \leq & - K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + K_3\, \varepsilon_{\gamma q/2}\, \frac{\gamma q}{2} \left(\frac{\gamma q}{2} - 1\right) \\ & \sum_{k=1}^{1 + k_{\frac{q}{2} - \frac{2}{\gamma}}} \binom{\frac{q}{2} - \frac{2}{\gamma}}{k - 1} \Big(m_{2\gamma k + \gamma}\, m_{\gamma q - 2\gamma k} + m_{\gamma q - 2\gamma k + \gamma}\, m_{2\gamma k} \Big)\, dv dv_*. \end{aligned}$$

which completes proof of (a).

(b) Now, we set $r = 2$ in (3.49) and apply Lemma 3.4.2 to obtain

$$m'_{2q}(t) \leq \; - K_1 \; m_{2q+\gamma} + K_2 \; m_{2q} + \; K_3 \; \varepsilon_q \; q(q-1) \iint_{\mathbb{R}^{2d}} f \; f_* \Big(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \Big)$$

$$\langle v \rangle^2 \langle v_* \rangle^2 \sum_{k=0}^{k_{q-2}} \binom{q-2}{k} \left(\langle v \rangle^{2k} \; \langle v_* \rangle^{2(q-2)-2k} + \langle v \rangle^{2(q-2)-2k} \langle v_* \rangle^{2k} \right) \; dv dv_*$$

$$= - \; K_1 \; m_{2q+\gamma} + K_2 \; m_{2q} + \; K_3 \; \varepsilon_q \; q(q-1) \; \iint_{\mathbb{R}^{2d}} f \; f_* \; \left(\langle v \rangle^{\gamma} + \langle v_* \rangle^{\gamma} \right)$$

$$\sum_{k=0}^{k_{q-2}} \binom{q-2}{k} \Big(\langle v \rangle^{2k+2} \langle v_* \rangle^{2q-2k-2} + \langle v \rangle^{2q-2k-2} \langle v_* \rangle^{2k+2} \Big) \; dv dv_*$$

$$= - \; K_1 \; m_{2q+\gamma} + K_2 \; m_{2q} + \; K_3 \; \varepsilon_q \; q(q-1) \sum_{k=1}^{k_q} \binom{q-2}{k-1} \big(m_{2k+\gamma} \, m_{2q-2k} + m_{2k} \, m_{2q-2k+\gamma} \big).$$

The last equality is obtained by re-indexing k to $k-1$ and using that $1 + k_{q-2} = k_q$. This completes proof of (b). □

Proposition 3.6.2 (Polynomial moment bounds for the non-cutoff case)**.** *Suppose all the assumptions of Theorem 3.1.1 are satisfied. Let f be solution to the homogeneous Boltzmann equation* (2.16) *associated to the initial data f_0.*

1. *Let the initial mass and energy be finite, i.e. $m_2(0)$ bounded, then for every $p > 0$ there exists a constant $\mathbf{B}_{rp} \geq 0$, depending on 2^{rp}, γ, $m_2(0)$ and A_2 from condition* (2.9)*, such that*

$$m_{rp}(t) \; \leq \; \mathbf{B}_{rp} \, \max\{1, t^{-rp/\gamma}\}, \quad \textit{for all} \;\; r \in \mathbb{R}^+ \;\; \textit{and} \; t \geq 0 \, . \qquad (3.51)$$

2. *Furthermore, if $m_{rp}(0)$ is finite, then the control can be improved to*

$$m_{rp}(t) \; \leq \; \mathbf{B}_{rp}, \quad \textit{for all} \;\; r \in \mathbb{R}^+ \;\; \textit{and} \; t \geq 0. \qquad (3.52)$$

Proof: These statements can be shown by studying comparison theorems for initial value problems associated with ordinary differential inequalities of the type

$$y'(t) + Ay^{1+c}(t) \le By(t),$$

and comparing them to classical Bernoulli's differential equations for the same given initial $y(0)$. In our context, these inequalities are a result of estimating moments for variable hard potentials, i.e. $\gamma > 0$ as indicated in (2.7). Comparison with Bernoulli type differential equations was classically used in the Grad's angular cutoff case in [6, 47, 60, 61]. Also it was used in the proof of propagation of L^1 exponential tails for the derivatives of the solution of the Boltzmann equation by means of geometric series methods in [8, 18, 35].

In fact, the extension to the non-cutoff case follows in a straightforward way from the moments estimates in Proposition 3.6.1. This was also used in [44] to establish generation of moments, yet for completeness purposes we include the proof here.

Applying Lemma 3.4.2 to the binomial factor in the last term of the estimate (3.49), distributing all products in that term and noticing that the resulting products of moments are each less than m_{rq} yields

$$m'_{rp} \;\le\; B_{rp} m_{rp} - K_1 m_{rp+\gamma} \tag{3.53}$$

where $K_1 = K_1(\gamma, m_0(0), A_2)$ with $0 < \gamma \le 1$, and A_2 from the angular integrability condition (2.9); and $B_{rp} = B_{rp}(K_2, 2^{rp}K_3)$, where K_2 and K_3 also depend on the initial data and collision kernel through γ and A_2.

Since $\gamma > 0$, an application of the classical Jensen's inequality with the convex function $\varphi(x) = x^{1+\gamma/(rp)}$ yields

$$m_{rp+\gamma}(t) \;\geq\; m_0^{-\gamma/(rp)}(0)\, m_{rp}^{1+\gamma/(rp)}(t), \quad \text{for all } t \geq 0\,.$$

Applying this estimate to the negative term in (3.53) yields

$$m'_{rp} \;\leq\; B_{rp} m_{rp} - K_4 m_{rp}^{1+\gamma/(rp)}, \tag{3.54}$$

where $K_4 = K_1 m_0^{-\gamma/(rp)}(0)$. Therefore, as in [60], we set $y(t) := m_{rp}(t),\; A := K_4,\; B := B_{rp}$ and $c = \gamma/(rp)$. and look for an upper solution by considering the associated Bernoulli ODE

$$y'(t) = By(t) - Ay^{1+c}(t).$$

Thus for any $t > 0$

$$\begin{aligned}
m_{rp}(t) &\leq \left[m_{rp}^{-\gamma/(rq)}(0)\, e^{-t\,\gamma B/(rp)} + \frac{A}{B}(1 - e^{-t\,\gamma B/(rp)}) \right]^{-rp/\gamma} \\
&\leq \left[\frac{A}{B}(1 - e^{-t\,\gamma B/(rp)}) \right]^{-rp/\gamma} \\
&\leq \left(\frac{A}{B} \right)^{-rp/\gamma} \begin{cases} \left(\frac{\gamma B}{rp} e^{-\gamma B/rp} \right)^{-rp/\gamma} \; t^{-rp/\gamma}, & t < 1, \\ (1 - e^{-\,\gamma B/(rp)})^{-rp/\gamma}, & t \geq 1. \end{cases} \\
&\leq \mathbf{B}_{rp} \max\{1, t^{-rp/\gamma}\},
\end{aligned} \tag{3.55}$$

where $\mathbf{B}_{rp} := \left(\frac{K_4}{B_{rp}}\right)^{-rp/\gamma} \max\left\{ \left(\frac{\gamma B_{rp}}{rp} e^{-\gamma B_{rp}/rp} \right)^{-rp/\gamma},\, (1 - e^{-\,\gamma B_{rp}/(rp)})^{-rp/\gamma} \right\}$.

Now, if $m_{rp}(0)$ is finite, then the continuity of $m_{rp}(t)$ as function of time and the bound for strictly positive times we just obtained in (3.55) implies

$$m_{rp}(t) \leq \mathbf{B}_{rp}. \tag{3.56}$$

for possibly different constants $\mathbf{B}_{rp}$. We finally stress that constants $\mathbf{B}_{rp}$ depend on $2^{rp}, \gamma, m_2(0)$ and A_2 from condition (2.9). $\square$

3.7 Proof of Mittag-Leffler moments' propagation

Proof of Theorem 3.1.1 (b). Let us recall representation (3.6) of the Mittag-Leffler moment of order s and rate α in terms of infinite sums

$$\int_{\mathbb{R}^d} f(t,v) \;\; E_{2/s}(\alpha^{2/s} \langle v \rangle^2) \, dv \;\; = \;\; \sum_{q=0}^{\infty} \frac{m_{2q}(t) \, \alpha^{2q/s}}{\Gamma(\frac{2}{s}\, q + 1)}. \tag{3.57}$$

We introduce abbreviated notation $a = \frac{2}{s}$. Since $s \in (0,2)$, we have

$$1 < a := \frac{2}{s} < \infty. \tag{3.58}$$

We consider the n-th partial sum, denoted by E_a^n, and the corresponding sum, denoted by $I^n_{a,\gamma}$, in which polynomial moments are shifted by γ. In other words, we consider

$$E_a^n(\alpha,t) = \sum_{q=0}^{n} \frac{m_{2q}(t) \; \alpha^{aq}}{\Gamma(aq+1)}, \qquad I^n_{a,\gamma}(\alpha,t) = \sum_{q=0}^{n} \frac{m_{2q+\gamma}(t) \; \alpha^{aq}}{\Gamma(aq+1)}.$$

For each $n \in \mathbb{N}$, define

$$T_n := \sup \left\{ t \geq 0 \mid E_a^n(\alpha, \tau) < 4M_0, \text{ for all } \tau \in [0,t) \right\}, \tag{3.59}$$

where the constant M_0 is the one from the initial condition (3.8).

The number T_n is well-defined and positive. Indeed, since α will be chosen to be, at least, smaller than α_0, then at time $t = 0$ we have

$$E_a^n(0) = \sum_{q=0}^{n} \frac{m_{2q}(0) \; \alpha^{aq}}{\Gamma(aq+1)} \; < \; \sum_{q=0}^{\infty} \frac{m_{2q}(0) \; \alpha_0^{aq}}{\Gamma(aq+1)} = \int f_0(v) \mathcal{E}_{2/s}(\alpha_0^{2/s} \langle v \rangle^2) \, dv < 4\, M_0,$$

uniformly in n. Therefore, since partial sums are continuous functions of time (they are finite sums and each $m_{2q}(t)$ is also continuous function in time t), $E_a^n(\alpha, t) < 4M_0$ holds for t on some positive time interval denoted $[0, t_n)$ with $t_n > 0$ (and hence $T_n > 0$).

Next, we look for an ordinary differential inequality that the partial sum $E_a^n(\alpha, t)$ satisfies, following the steps presented in Subsection 3.3. We start by splitting $\frac{d}{dt} E_a^n(\alpha, t)$ into the following two sums, where index q_0 will be fixed later, and then apply the moment differential inequality (3.43)

$$\begin{aligned} \frac{d}{dt} E_a^n(\alpha, t) &= \sum_{q=0}^{q_0-1} \frac{m'_{2q}(t)\, \alpha^{aq}}{\Gamma(aq+1)} + \sum_{q=q_0}^{n} \frac{m'_{2q}(t)\ \alpha^{aq}}{\Gamma(aq+1)} \\ &\leq \sum_{q=0}^{q_0-1} \frac{m'_{2q}(t)\, \alpha^{aq}}{\Gamma(aq+1)} - K_1 \sum_{q=q_0}^{n} \frac{m_{2q+\gamma}(t)\, \alpha^{aq}}{\Gamma(aq+1)} + K_2 \sum_{q=q_0}^{n} \frac{m_{2q}(t)\, \alpha^{aq}}{\Gamma(aq+1)} \\ &\quad + K_3 \sum_{q=q_0}^{n} \frac{\varepsilon_q\, q(q-1)\, \alpha^{aq}}{\Gamma(aq+1)} \sum_{k=1}^{k_q} \binom{q-2}{k-1} \Big(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k} m_{2(q-k)+\gamma} \Big) \\ &=: S_0 - K_1 S_1 + K_2 S_2 + K_3 S_3. \end{aligned} \tag{3.60}$$

We estimate each of the four sums S_0, S_1, S_2 and S_3 separately, with the goal of comparing each of them to the functions $E_a^n(\alpha, t)$ and $I_{a,\gamma}^n(\alpha, t)$. We remark that the most involved term is S_3. It resembles the corresponding sum in the Grad's cutoff case [6], with a crucial difference that our sum S_3 has two extra powers of q, namely $q(q-1)$. Therefore, a sharp calculation is required to control the growth of S_3 as a function of the number q of moments. This is achieved by an appropriate renormalization of polynomial moments within S_3 and also by invoking the decay rate of associated combinatoric sums of Beta functions developed in Section 3.4.

The term S_0 can be bounded by a constant that depends on q_0, the initial data and the parameters of the collision cross section. Indeed, from Lemma 3.6.2, the propagated polynomial moments can be estimated as follows:

$$m_p \leq \mathbf{B}_p \quad \text{and} \quad m'_p \leq B_p \, \mathbf{B}_p, \qquad \text{for any } p > 0, \tag{3.61}$$

where the constant $\mathbf{B}_p$ defined in (3.55) depends on γ, the initial p-polynomial moment $m_p(0)$ and A_2 from condition (2.9).

In particular, for $0 < \gamma < 1$, we can fix q_0, to be chosen later, such that the constant

$$c_{q_0} := \max_{p \in I_{q_0}} \{\mathbf{B}_p, \, B_p \, \mathbf{B}_p\}, \quad \text{with} \quad I_{q_0} = \{0, \ldots, 2q_0 + 1\} \tag{3.62}$$

depends only on q_0, γ, A_2 from condition (2.9), and the initial polynomial moments $m_q(0)$, for $q \in I_{q_0}$. Thus, due to the monotoncity of L^1_k norms with respect to k, both the $2q$-moments and its derivatives, as well as the shifted moments of order $2q + \gamma$, are controlled by c_{q_0} as follows:

$$m_{2q}(t), \, m_{2q+\gamma}(t), \, m'_{2q}(t) \leq c_{q_0}, \quad \text{for all } \; q \in \{0, 1, 2, ...q_0\}. \tag{3.63}$$

Therefore, for q_0 fixed, to be chosen later, S_0 is estimated by

$$\begin{aligned} S_0 \; &:= \sum_{q=0}^{q_0 - 1} \frac{m'_{2q} \; \alpha^{aq}}{\Gamma(aq+1)} \; \leq \; c_{q_0} \sum_{q=0}^{q_0-1} \frac{\alpha^{aq}}{\Gamma(aq+1)} \\ &\leq \; c_{q_0} \sum_{q=0}^{q_0-1} \frac{(\alpha^a)^q}{\Gamma(q+1)} \; \leq \; c_{q_0} \, e^{\alpha^a} \; \leq 2 \, c_{q_0} \, , \end{aligned} \tag{3.64}$$

for the parameter α small enough to satisfy

$$\alpha < (\ln 2)^{1/a} \; , \quad \text{or equivalently,} \;\; e^{\alpha^a} \leq 2. \tag{3.65}$$

The second term S_1 is crucial, as it brings the negative contribution that will yield uniform in n and global in time control to an ordinary differential inequality for $E_a^n(\alpha, t)$. In fact, S_1 is controlled from below by $I_{a,\gamma}^n(\alpha, t)$ as follows:

$$S_1 := \sum_{q=q_0}^{n} \frac{m_{2q+\gamma}\, \alpha^{aq}}{\Gamma(aq+1)} = I_{a,\gamma}^n - \sum_{q=0}^{q_0-1} \frac{m_{2q+\gamma}\, \alpha^{aq}}{\Gamma(aq+1)}.$$

So using (3.63) and the estimate just obtained for S_0 in (3.64), yields the bound from below

$$S_1 \geq I_{a,\gamma}^n - c_{q_0} \sum_{q=0}^{q_0-1} \frac{\alpha^{aq}}{\Gamma(aq+1)} \geq I_{a,\gamma}^n - 2c_{q_0}. \tag{3.66}$$

The sum S_2 is a part of the partial sum E_a^n, so

$$S_2 \leq E_a^n. \tag{3.67}$$

Finally, we estimate S_3 and show that it can be bounded by the product of $E_a^n(\alpha, t)$ and $I_{a,\gamma}^n(\alpha, t)$. We work out the details of the first term in the sum $S_3 := S_{3,1} + S_{3,2}$, that is the one with $m_{2k+\gamma}\, m_{2(q-k)}$. The other sum with $m_{2k}\, m_{2(q-k)+\gamma}$ can be bounded by following a similar strategy. In order to generate both the partial sum $E_a^n(\alpha, t)$ and the shifted one $I_{a,\gamma}^n(\alpha, t)$, we make use of the following well known relations between Gamma and Beta functions.

$$\begin{aligned} B(ak+1, a(q-k)+1) &= \frac{\Gamma(ak+1)\,\Gamma(a(q-k)+1)}{\Gamma(\,(ak+1)+(a(q-k)+1)\,)} \\ &= \frac{\Gamma(ak+1)\,\Gamma(a(q-k)+1)}{\Gamma(aq+2)}\,. \end{aligned} \tag{3.68}$$

Therefore, multiplying and dividing products of moments $m_{2k+\gamma}m_{2(q-k)}$ in $S_{3,1}$, by $\Gamma(ak+1)\Gamma(a(q-k)+1)$ yields

$$
\begin{aligned}
S_{3,1} :=& \sum_{q=q_0}^{n} \frac{\varepsilon_q\, q\,(q-1)\,\alpha^{aq}}{\Gamma(aq+1)} \sum_{k=1}^{k_q} \binom{q-2}{k-1} m_{2k+\gamma}\, m_{2(q-k)} \\
=& \sum_{q=q_0}^{n} \varepsilon_q\, q\,(q-1) \sum_{k=1}^{k_q} \binom{q-2}{k-1} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)}\, \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \\
& \qquad B(ak+1, a(q-k)+1)\, \frac{\Gamma(aq+2)}{\Gamma(aq+1)} .
\end{aligned}
$$

Note that the factors $\frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)}$ and $\frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)}$ are the building blocks of $I^n_{a,\gamma}(\alpha, t)$ and $E^n_a(\alpha, t)$, respectively.

Next, since $\Gamma(aq+2)/\Gamma(aq+1) = aq+1$, using the inequality $\sum_k a_k\, b_k \le \sum_k a_k \sum_k b_k$, it follows that

$$
\begin{aligned}
S_{3,1} \le& \sum_{q=q_0}^{n} \varepsilon_q\,(aq+1)\,q\,(q-1) \left(\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \right) \\
& \left(\sum_{k=1}^{k_q} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)}\, \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \right) .
\end{aligned}
\tag{3.69}
$$

Next we show that the factor

$$
(aq+1)\,q\,(q-1) \left(\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \right)
$$

on the right hand side of (3.69) grows at most as q^{2-a}. Indeed, using Lemma 3.4.4, the sum of the Beta functions is bounded by $C_a(aq)^{-(1+a)}$. Therefore,

$S_{3,1}$ is estimated by

$$S_{3,1} \le C_a \sum_{q=q_0}^{n} \varepsilon_q \, q^{2-a} \left(\sum_{k=1}^{k_q} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \right), \tag{3.70}$$

where C_a is a (possibly different) constant that depends on a. Now, by Lemma 3.5.1, the factor $\varepsilon_q \, q^{2-a}$ decreases monotonically to zero as $q \to \infty$ provided that the angular kernel $b(\cos\theta)$ satisfies (2.9) with $\beta = 2a - 2$. This indeed was an assumption (3.9) in the theorem. Hence,

$$\varepsilon_q \, q^{2-a} \le \varepsilon_{q_0} \, q_0^{2-a}, \qquad \text{for any } q \ge q_0 , \tag{3.71}$$

and thus the term $S_{3,1}$ is further estimated by

$$S_{3,1} \le C_a \, \varepsilon_{q_0} \, q_0^{2-a} \sum_{q=q_0}^{n} \sum_{k=1}^{k_q} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)}.$$

Finally, inspired by [6], we bound this double sum by the product of partial sums $E_a^n \, I_{a,\gamma}^n$. To achieve that, change the order of summation to obtain

$$\begin{aligned} S_{3,1} &\le C_a \, \varepsilon_{q_0} \, q_0^{2-a} \sum_{k=0}^{k_n} \sum_{\max\{q_0, 2k-1\}}^{n} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \\ &\le C_a \, \varepsilon_{q_0} \, q_0^{2-a} \sum_{k=0}^{k_n} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \sum_{\max\{q_0, 2k-1\}}^{n} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \\ &\le C_a \, \varepsilon_{q_0} \, q_0^{2-a} \, I_{a,\gamma}^n \, E_a^n , \end{aligned} \tag{3.72}$$

obtaining the expected control of $S_{3,1}$. As mentioned above the estimate of the companion sum $S_{3,2}$ follows in a similar way, so we can assert

$$S_3 \le C_a \, \varepsilon_{q_0} \, q_0^{2-a} \, E_a^n(t) \, I_{a,\gamma}^n(t). \tag{3.73}$$

Next we obtain an ordinary differential inequality for $E_a^n(t)$ depending only on data parameters and $I_{a,\gamma}^n(t)$. Indeed, combining (3.64), (3.66), (3.67) and (3.72) with (3.60) yields

$$\frac{d}{dt}E_a^n \;\leq\; -K_1\, I_{a,\gamma}^n \,+\, 2\, c_{q_0}(1+K_1) \,+\, K_2\, E_a^n \,+\, \varepsilon_{q_0}\; q_0^{2-a}\, C_a\, K_3\, I_{a,\gamma}^n\, E_a^n\,. \quad (3.74)$$

Since, by the definition of time T_n, the partial sum E_a^n is bounded by the constant $4M_0$ on the time interval $[0,T_n]$, we can estimate, uniformly in n, the following two terms in (3.74)

$$2\, c_{q_0}(1+K_1) + K_2\, E_a^n \leq 2\, c_{q_0}(1+K_1) + 4K_2\, M_0 =: \mathcal{K}_0, \quad (3.75)$$

where $\mathcal{K}_0$ depends only on the initial data and q_0 (still to be determined).

Thus, factoring out $E_{a,\gamma}^n$ from the remaining two terms in (3.74) yields

$$\begin{aligned} \frac{d}{dt}E_a^n \;&\leq\; -\, I_{a,\gamma}^n \left(K_1 \,-\, \varepsilon_{q_0}\; q_0^{2-a}\, C_a\, K_3\, E_a^n\right) \,+\, \mathcal{K}_0 \\ &\leq\; -\, I_{a,\gamma}^n \left(K_1 \,-\, 4\varepsilon_{q_0}\; q_0^{2-a}\, C_a\, K_3\, M_0\right) \,+\, \mathcal{K}_0, \end{aligned} \quad (3.76)$$

where in the last inequality we again used that, by the definition of T_n, we have $E_a^n \leq 4M_0$ on the closed interval $[0,T_n]$. Now, since $\varepsilon_{q_0}\; q_0^{2-a}$ converges to zero as q_0 tends to infinity (by Lemma 3.5.1 as $b(\cos\theta)$ satisfies (2.9) with $\beta = 2a-2$), we can choose large enough q_0 so that

$$K_1 \,-\, 4\varepsilon_{q_0}\; q_0^{2-a}\, C_a\, K_3\, M_0 > \frac{K_1}{2}. \quad (3.77)$$

For such choice of q_0 we then have

$$\frac{d}{dt}E_a^n \leq -\,\frac{K_1}{2}\, I_{a,\gamma}^n \,+\, \mathcal{K}_0\,. \quad (3.78)$$

The final step consists in finding a lower bound for $I^n_{a,\gamma}$ in terms of E^n_a. The following calculation follows from a revised form of the lower bound given in [6],

$$
\begin{aligned}
I^n_{a,\gamma}(t) := \sum_{q=0}^{n} \frac{m_{2q+\gamma}\ \alpha^{aq}}{\Gamma(aq+1)} \ &\geq\ \sum_{q=0}^{n} \int_{\langle v\rangle \geq \frac{1}{\sqrt{\alpha}}} \frac{\langle v\rangle^{2q+\gamma}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\ dv \\
\geq\ & \tfrac{1}{\alpha^{\gamma/2}} \sum_{q=0}^{n} \int_{\langle v\rangle \geq \frac{1}{\sqrt{\alpha}}} \frac{\langle v\rangle^{2q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\ dv \\
=\ & \tfrac{1}{\alpha^{\gamma/2}} \left(\sum_{q=0}^{n} \int_{\mathbb{R}^d} \frac{\langle v\rangle^{2q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\ dv\ -\ \sum_{q=0}^{n} \int_{\langle v\rangle < \frac{1}{\sqrt{\alpha}}} \frac{\langle v\rangle^{2q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\ dv \right) \\
\geq\ & \tfrac{1}{\alpha^{\gamma/2}} \left(E^n_a(t)\ -\ \sum_{q=0}^{n} \int_{\mathbb{R}^d} \frac{\alpha^{-q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\ dv \right) \\
\geq\ & \tfrac{1}{\alpha^{\gamma/2}} \left(E^n_a(t)\ -\ m_0 \sum_{q=0}^{\infty} \frac{\alpha^{q(a-1)}}{\Gamma(aq+1)} \right) \\
>\ & \tfrac{1}{\alpha^{\frac{\gamma}{2}}}\, E^n_a(t)\ -\ \tfrac{1}{\alpha^{\frac{\gamma}{2}}}\, m_0\, e^{\alpha^{a-1}}.
\end{aligned}
\tag{3.79}
$$

Therefore, applying inequality (3.79) to (3.78) yields the following linear differential inequality for the partial sum E^n_a

$$
\frac{d}{dt} E^n_a(t)\ \leq\ -\frac{K_1}{2\,\alpha^{\frac{\gamma}{2}}} E^n_a(t)\ +\ \frac{K_1\, m_0\, e^{\alpha^{a-1}}}{2\alpha^{\frac{\gamma}{2}}}\ + \mathcal{K}_0.
$$

Then, by the maximum principle for ordinary differential inequalities,

$$
\begin{aligned}
E^n_{2/s}(t) = E^n_a(t)\ &\leq\ M_0\ +\ \frac{2\,\alpha^{\gamma/2}}{K_1} \left(\frac{K_1\, m_0\, e^{\alpha^{a-1}}}{2\alpha^{\frac{\gamma}{2}}}\ +\ \mathcal{K}_0. \right) \\
&=\ M_0\ +\ m_0\, e^{\alpha^{a-1}}\ +\ \tfrac{2\,\alpha^{\gamma/2}}{K_1}\, \mathcal{K}_0 \\
&<\ 4M_0,
\end{aligned}
$$

provided that α is chosen sufficiently small so that

$$m_0\, e^{\alpha^{a-1}} \;+\; \frac{2\,\alpha^{\gamma/2}}{K_1}\,\mathcal{K}_0 \;<\; 3M_0\,. \tag{3.80}$$

Such choice of α is possible since $m_0\, e^{\alpha^{a-1}} \;+\; \frac{2\,\alpha^{\gamma/2}}{K_1}\,\mathcal{K}_0 \to m_0 < M_0$ as $\alpha \to 0$. Thus, by choosing α sufficiently small, (3.80) holds. Let us denote an α for which (3.80) holds by α_1.

In conclusion, if q_0 is chosen according to (3.77), and hence depending only on the initial data, initial Mittag-Leffler moment, γ and A_2 from (2.9), and if $\alpha = \min\{\alpha_0, (\ln 2)^{1/a}, \alpha_1\}$, where α_1 satisfies (3.80), we have that the *strict* inequality $E^n_a(t) < 4M_0$ holds on the *closed* interval $[0, T_n]$ uniformly in n. Therefore, invoking the global continuity of $E^n_a(t)$ once more, the set of time t for $E^n_a(t) < 4M_0$ holds on a slightly larger half-open time interval $[0, T_n + \mu)$, with $\mu > 0$. This would contradict maximality of the definition of T_n, unless $T_n = +\infty$. Hence, we conclude that $T_n = +\infty$ for all n. Therefore, we in fact have that

$$E^n_a(\alpha, t) < 4M_0, \quad \text{for all } t \geq 0, \;\; \text{for all } n \in \mathbb{N}.$$

Thus, by letting $n \to +\infty$, we conclude that $\mathcal{E}^\infty_a(\alpha, t) < 4M_0$ for all $t \geq 0$. That is,

$$\int_{\mathbb{R}^d} f(t,v)\;\; \mathcal{E}_{2/s}(\alpha^{2/s}\,\langle v\rangle^2)\; dv \;<4M_0, \quad \text{for all } t \geq 0. \tag{3.81}$$

Estimate (3.81) shows that the solution of the Boltzmann equation with finite initial Mittag-Leffler moment of order s and rate α_0, will propagate Mittag-Leffler moments with the same order s and rate α satisfying $\alpha = \min\{\alpha_0, (\ln 2)^{1/a}, \alpha_1\}$. This concludes the proof part**(b)** of Theorem 3.1.1. $\square$

Part**(a)** of Theorem 3.1.1 concerns the generation of Mittag-Leffler or exponential moments. This is proven in the next section.

3.8 Proof of exponential moments' generation

Proof of Theorem 3.1.1 (a). The notation and strategy are similar to those in the proof of Theorem 3.1.1 (b), contained in Section 3.7. The goal is to find a positive and bounded real valued number α such that the solution $f(v,t)$ of the Boltzmann equation will have an exponential moment, of order γ and rate $\alpha \min\{t,1\}$, generated for every positive time t, from the fact that the initial data $f_0(v)$ has finite energy given by $M_0^* := m_2(0)$.

The proof works with exponential weights of order γ. From this viewpoint, the difference compared to the propagation of Mittag-Leffler moments result obtained in the previous section is that the propagation result had to be established for every order $s \in (0,2)$, while now the generation of Mittag-Leffler moments of order s and rate α implies generation of such moments for all smaller orders $0 < s$. Thus, it suffices to consider just the order $\boldsymbol{s = \gamma}$.

For an arbitrary positive number α, we denote the n-th partial sum of the exponential moment of order γ by $E_\gamma^n(\alpha t, t)$ and the corresponding one in which polynomial moments are shifted by γ by $I_{\gamma,\gamma}^n(\alpha t, t)$, that is

$$E_\gamma^n(\alpha t, t) = \sum_{q=0}^{n} \frac{m_{\gamma q}(t)\ (\alpha t)^q}{\Gamma(q+1)} = \sum_{q=0}^{n} \frac{m_{\gamma q}(t)\ (\alpha t)^q}{q!}, \tag{3.82}$$

$$I_{\gamma,\gamma}^n(\alpha t, t) = \sum_{q=0}^{n} \frac{m_{\gamma q+\gamma}(t)\ (\alpha t)^q}{\Gamma(q+1)} = \sum_{q=0}^{n} \frac{m_{\gamma q+\gamma}(t)\ (\alpha t)^q}{q!}. \tag{3.83}$$

The sum $E^n_\gamma(\alpha t, t)$ is the partial sum of the exponential moment of order γ with rate α of the probability density f in the Mittag-Leffler representation.

Define the time T^*_n as follows

$$T^*_n := \min\left\{1,\ \ \sup\left\{t \geq 0 \mid E^n_\gamma(\alpha\tau, \tau) < 4M^*_0, \quad \text{for all } \tau \in [0, t)\ \right\}\right\}, \quad (3.84)$$

where now the constant M^*_0 is the sum of the initial conserved mass and energy, i.e. $M^*_0 := M^*_0(t) = \int f(v,t)\langle v\rangle^2 dv = \int f_0(v)\langle v\rangle^2 dv$. Since polynomial moments are generated instantenously for the hard potential case, even for the angular non-cutoff case (see [60]), thus every finite sum $E^n_a(\alpha t, t)$ is well defined and continuous in time. Note that for $t = 0$, we have that $E^n_\gamma(\alpha 0, 0) = m_0 < 4M^*_0$. Then, as in the previous case, continuity in time of partial sums $E^n_a(\alpha t, t)$ implies that $E^n_a(\alpha t, t) < 4M^*_0$ holds for t on some positive time interval $[0, t^*_n)$, which implies that $T^*_n > 0$. In addition, the definition (3.84) implies that $T^*_n \leq 1$ for all $n \in \mathbb{N}$.

As we did in the previous section for the proof of propagation of Mittag-Leffler moments, we search for an ordinary differential inequality for $E^n_\gamma(\alpha t, t)$, depending only on data parameters and on $I^n_{\gamma,\gamma}(\alpha t, t)$, for a positive and bounded real valued α to be found and characterized.

To this end, we start by computing

$$\begin{aligned} \frac{d}{dt}E^n_\gamma(\alpha t, t) &= \alpha \sum_{q=1}^{n} \frac{m_{\gamma q}(t)\,(\alpha t)^{q-1}}{(q-1)!} + \sum_{q=0}^{n} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!} \qquad (3.85) \\ &= \alpha \sum_{q=1}^{n} \frac{m_{\gamma q}(t)\,(\alpha t)^{q-1}}{(q-1)!} + \sum_{q=0}^{q_0-1} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!} + \sum_{q=q_0}^{n} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!}, \end{aligned}$$

where index q_0 will be fixed later. The first sum in this identity is reindexed by from $q-1$ to q and estimated by $I^n_{\gamma,\gamma}(\alpha t, t)$ (defined in (3.83)), as follows:

$$\sum_{q=0}^{n-1} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} \leq \sum_{q=0}^{n} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} = I^n_{\gamma,\gamma}(\alpha t, t).$$

Next, replacing the term $m'_{\gamma q}(t)$, but just on the sums starting from q_0, by the upper bound given via (3.42), for $\alpha > 0$, and for

$$k_{q^*} := \lfloor \frac{q}{4} - \frac{1}{\gamma} + \frac{3}{2} \rfloor \;:=\; \text{integer part of } \; \frac{q}{4} - \frac{1}{\gamma} + \frac{3}{2}\,, \tag{3.86}$$

we have

$$\begin{aligned} \frac{d}{dt} E^n_\gamma(\alpha t, t) \;\leq\; & \alpha I^n_{\gamma,\gamma}(\alpha t, t) \;+\; \sum_{q=0}^{q_0-1} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!} \\ & -\; K_1 \sum_{q=q_0}^{n} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} \;+\; K_2 \sum_{q=q_0}^{n} \frac{m_{\gamma q}(t)\,(\alpha t)^q}{q!} \\ & +\; K_3 \sum_{q=q_0}^{n} \frac{\varepsilon_{\gamma q/2} \frac{\gamma q}{2} (\frac{\gamma q}{2} - 1)\,(\alpha t)^q}{q!} \sum_{k=1}^{k_{q*}} \binom{\frac{q}{2} - \frac{2}{\gamma}}{k-1} \\ & \qquad \Big((m_{2\gamma k+\gamma}(t)\, m_{\gamma q - 2\gamma k}(t) \;+\; m_{2\gamma k}(t)\, m_{\gamma q - 2\gamma k+\gamma}(t) \Big) \\ & =:\; \alpha I^n_{\gamma,\gamma}(\alpha t, t) \;+\; S_0 - K_1\, S_1 \;+\; K_2\, S_2 \;+\; K_3\, S_3\,. \end{aligned} \tag{3.87}$$

We stress that the positive constant $K_1 \,=\, A_2\, m_0(0)\, C_\gamma$ depends only on the collision cross section with A_2 defined in (3.30), inital mass $m_0(0)$ and C_γ only depending on $0 < \gamma \leq 1$. In the sequel, we will estimate the terms in (3.87) to show that the negative one dominates, for a choice of α and q_0 that depend only on the initial and the collision kernel.

The bounds of the term S_0 depends on the initial data and the parameters of the collision cross section. Indeed, from Lemma 3.6.2, setting $r = \gamma$ in (3.55), the generated polynomial moments can be estimated by

$$m_{\gamma q}(t) \leq \mathbf{B}_{\gamma q} \max_{t>0}\{1, t^{-q}\}, \tag{3.88}$$
$$m'_{\gamma q}(t) \leq B_{\gamma q} m_{\gamma q}(t) \leq B_{\gamma q}\, \mathbf{B}_{\gamma q} \max_{t>0}\{1, t^{-q}\},$$

where the constant $\mathbf{B}_{\gamma q}$, now from (3.55), also depends on $m_2(0)$, γ, q and A_2 from condition (2.9). Next, for q_0 fixed, to be chosen later, set

$$c^*_{q_0} := \max_{q \in \{0,\ldots,q_0-1\}} \{\mathbf{B}_{\gamma q},\, B_{\gamma q}\, \mathbf{B}_{\gamma q}\}\,, \tag{3.89}$$

and then, both the $2q$-moments and its derivatives are controlled in terms of $c^*_{q_0}$ as follows:

$$m_{\gamma q}(t), m'_{\gamma q}(t) \leq c^*_{q_0} \max_{t>0}\{1, t^{-q}\}, \quad \text{for all } q \in \{0, \ldots, q_0 - 1\}\,. \tag{3.90}$$

Thus we can estimate S_0, for a fixed q_0 to be defined later, by

$$S_0 := \sum_{q=0}^{q_0-1} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!}$$
$$\leq c^*_{q_0} \max_{t>0}\{1, t^{-q}\} \sum_{q=0}^{q_0-1} \frac{(\alpha t)^q}{q!}$$
$$\leq c^*_{q_0} \max_{t>0}\{t^q, 1\} \sum_{q=0}^{q_0-1} \frac{\alpha^q}{q!} \tag{3.91}$$
$$\leq c^*_{q_0} e^{\alpha} \leq 2\, c^*_{q_0}, \tag{3.92}$$

uniformly in $t \in [0, T^*_n] \subset [0, 1]$, for any $\alpha \leq \ln 2$. To obtain inequality (3.91) we used $t \leq T^*_n \leq 1$.

The sum S_2 is a part of the partial sum E^n_γ, hence

$$S_2 := \sum_{q=q_0}^{n} \frac{m_{\gamma q}(\alpha t)^q}{q!} \leq E^n_\gamma(\alpha t, t). \tag{3.93}$$

The sum S_1 needs to be bounded from below because of the negativity of the term $K_1\, S_1$. To this end, using again the time dependent estimates for moments from Proposition 3.6.2, the estimate from below follows for $t \in (0, T^*_n] \subset (0, 1]$ as

$$\begin{aligned} S_1 \;&:=\; \sum_{q=q_0}^{n} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} \;=\; I^n_{\gamma,\gamma}(\alpha t, t) - \sum_{q=0}^{q_0-1} \frac{m_{\gamma q+\gamma}(\alpha t)^q}{q!} \\ &\geq\; I^n_{\gamma,\gamma}(\alpha t, t) - c^*_{q_0} \sum_{q=0}^{q_0-1} \frac{\max_{0<t\leq 1}\{1, t^{-(\gamma q+\gamma)/\gamma}\}(\alpha t)^q}{q!} \\ &\geq\; I^n_{\gamma,\gamma}(\alpha t, t) - c^*_{q_0} \sum_{q=0}^{q_0-1} \frac{t^{-q-1}(\alpha t)^q}{q!} \\ &=\; I^n_{\gamma,\gamma}(\alpha t, t) - \frac{c^*_{q_0}}{t} \sum_{q=0}^{q_0-1} \frac{\alpha^q}{q!} \\ &\geq\; I^n_{\gamma,\gamma}(\alpha t, t) - \frac{c^*_{q_0}}{t} e^{\alpha} \\ &\geq\; I^n_{\gamma,\gamma}(\alpha t, t) - \frac{2c^*_{q_0}}{t}. \end{aligned} \tag{3.94}$$

The estimate for the double sum term in S_3 uses an analogous treatment to the one in the previous section to obtain Mittag-Leffler moment's propagation. More precisely, we set $S_3 := S_{3,1} + S_{3,2}$, and use the identity (3.22) written in the following format:

$$\Gamma(2k+1)\Gamma(q-2k+1) = B(2k+1, q-2k+1)\,\Gamma(q+2) \tag{3.95}$$

to obtain

$$S_{3,1} := \sum_{q=q_0}^{n} \varepsilon_{\gamma q/2} \frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right) \sum_{k=1}^{k_{q*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(t)\,(\alpha t)^{q-2k}}{\Gamma(q-2k+1)}$$
$$B(2k+1, q-2k+1)\frac{\Gamma(q+2)}{\Gamma(q+1)}$$
$$\leq\ \varepsilon_{\gamma q_0/2} \sum_{q=q_0}^{n} (q+1)\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\left(\sum_{k=1}^{k_{q*}} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(t)\,(\alpha t)^{q-2k}}{\Gamma(q-2k+1)}\right)$$
$$\left(\sum_{k=1}^{k_{q*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} B(2k+1, q-2k+1)\right). \quad (3.96)$$

The last inequality was obtained via the inequality $\sum_k a_k b_k \leq \sum_k a_k \ \sum_k b_k$, and the fact that ε_q decreases in q. Again, using the estimate of Lemma 3.4.5, the sum of the Beta functions is bounded by Cq^{-3}, with C a uniform constant independent of q. Therefore,

$$(q+1)\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\left(\sum_{k=1}^{k_{q*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} B(2k+1, q-2k+1)\right)$$
$$\leq\ (q+1)\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right) q^{-3}\ \leq\ C_\gamma\,, \quad (3.97)$$

uniformly in q. Then, estimating the right hand side of (3.96) by the estimate (3.97) just above, yields

$$S_{3,1} \leq K_3\, C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{q=q_0}^{n} \left(\sum_{k=1}^{k_{q*}} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(t)\,(\alpha t)^{q-2k}}{\Gamma(q-2k+1)}\right). \quad (3.98)$$

Finally, as was the case for the propagation estimates in the previous section, changing the order of summation in the right hand side of (3.98) yields a control by a factor $E^n_\gamma(\alpha t, t)\, I^n_{\gamma,\gamma}(\alpha t, t)$ as follows. Recalling the definition of

k_{q_*} from (3.86), and evaluating it for n instead of q yields

$$
\begin{aligned}
S_{3,1} &\leq C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{k=0}^{\left[\frac{n}{4}+\frac{3}{2}-\frac{1}{\gamma}\right]} \sum_{q=\max\{q_0,4k-2\}}^{n} \frac{m_{2\gamma k+\gamma}\ (\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(\alpha t)^{q-2k}}{\Gamma(q-2k+1)} \\
&= C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{k=0}^{\left[\frac{n}{4}+\frac{3}{2}-\frac{1}{\gamma}\right]} \frac{m_{2\gamma k+\gamma}(t)\ (\alpha t)^{2k}}{\Gamma(2k+1)} \left(\sum_{q=\max\{q_0,4k-2\}}^{n} \frac{m_{\gamma q-2\gamma k}(t)(\alpha t)^{q-2k}}{\Gamma(q-2k+1)} \right) \\
&\leq C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{k=0}^{\left[\frac{n}{4}+\frac{3}{2}-\frac{1}{\gamma}\right]} \frac{m_{2\gamma k+\gamma}(t)\ (\alpha t)^{2k}}{\Gamma(2k+1)}\, E_\gamma^n(\alpha t, t) \\
&\leq C_\gamma\, \varepsilon_{\gamma q_0/2}\, I_{\gamma,\gamma}^n(\alpha t, t)\, E_\gamma^n(\alpha t, t)\,.
\end{aligned}
$$

Analogous estimate can be obtained for $S_{3,2}$, so overall we have

$$
S_3 \ \leq\ 2C_\gamma \varepsilon_{\gamma q_0/2}\, I_{\gamma,\gamma}^n(\alpha t, t)\ E_\gamma^n(\alpha t, t). \tag{3.99}
$$

Therefore, combining estimates (3.92), (3.94), (3.93) and (3.99) with (3.87) yields the following differential inequality for $E_\gamma^n = E_\gamma^n(\alpha t, t)$ depending on $I_{\gamma,\gamma}^n = I_{\gamma,\gamma}^n(\alpha t, t)$,

$$
\frac{d}{dt} E_\gamma^n \leq 2c_{q_0}^* + \left(-K_1\, I_{\gamma,\gamma}^n + K_1\, \frac{2\, c_{q_0}^*}{t} + K_2\, E_\gamma^n + 2\varepsilon_{\gamma q_0/2} C_\gamma\, K_3 E_\gamma^n\, I_{\gamma,\gamma}^n \right) + \alpha I_{\gamma,\gamma}^n.
$$

This inequality is the analog to the one in (3.74) for the propagation argument. Since the partial sum $E_\gamma^n(\alpha t, t)$ is bounded by $4M_0^*$ on the interval $[0, T_n^*]$, uniformly in n and $T_n^* \leq 1$, the right hand side of the above inequality is controlled by

$$
\frac{d}{dt} E_\gamma^n(\alpha t, t) \leq -I_{\gamma,\gamma}^n(\alpha t, t) \Big(K_1 - 8M_0^*\, \varepsilon_{\gamma q_0/2} C_\gamma K_3 - \alpha \Big) + 4M_0^*\, K_2 + \frac{2\, K_1\, c_{q_0}^*}{t} + 2c_{q_0}^*.
$$

Next, since $t \leq T_n^* \leq 1$, then $t^{-1} \geq 1$, thus

$$\frac{d}{dt} E_\gamma^n(\alpha t, t) \leq -I_{\gamma,\gamma}^n(\alpha t, t)\Big(K_1 - 8M_0^* \, \varepsilon_{\gamma q_0/2} C_\gamma K_3 - \alpha\Big) + \frac{\mathcal{K}_{q_0}}{t},$$

with $0 < \mathcal{K}_{q_0} = 2c_{q_0}^* + 4M_0^* K_2 + 2K_1 c_{q_0}^*$ only depending on data parameters, including q_0, independent of n.

Finally, since $\varepsilon_{\gamma q_0/2}$ converges to zero as q_0 goes to infinity, we can choose large enough q_0 and small enough α so that

$$K_1 \,-\, 8\varepsilon_{q_0}\, q_0^{2-a}\, K_3 - \alpha > \frac{K_1}{2}\,, \tag{3.100}$$

which yields

$$\frac{d}{dt} E_\gamma^n(\alpha_1 t, t) \,\leq -\,\frac{K_1}{2}\, I_{\gamma,\gamma}^n(\alpha t, t) \,+\, \frac{\mathcal{K}_{q_0}}{t}\;. \tag{3.101}$$

Therefore, the final step consists in finding a lower bound for $I_{\gamma,\gamma}^n(\alpha t, t)$ in terms of $E_\gamma^n(\alpha t, t)$ as follows

$$\begin{aligned} I_{\gamma,\gamma}^n(\alpha t, t) \;&=\; \sum_{q=0}^{n} \frac{m_{\gamma(q+1)}(t)\;(\alpha t)^q}{q!} \;=\; \sum_{q=1}^{n+1} \frac{m_{\gamma q}(t)\;(\alpha t)^q}{q!}\frac{q}{\alpha t} \\ &\geq\; \frac{1}{\alpha t}\sum_{q=3}^{n} \frac{m_{\gamma q}(t)\;(\alpha t)^q}{q!} \;=\; \frac{E_\gamma^n(t, \alpha t) - M_0^*}{\alpha t}. \end{aligned} \tag{3.102}$$

Combining (3.101) and (3.102) yields

$$\frac{d}{dt} E_\gamma^n(\alpha t, t) \leq -\frac{1}{t}\left(\frac{K_1(E_\gamma^n - M_0^*)}{2\alpha} - \mathcal{K}_{q_0}\right) = -\frac{K_1}{2\alpha t}\left(E_\gamma^n - M_0^* - \frac{2\alpha}{K_1}\mathcal{K}_{q_0}\right).$$

Then choosing a small enough α such that

$$M_0^* + \frac{2\alpha}{K_1}\mathcal{K}_{q_0} < 2M_0^* \qquad \text{or, equivalently,} \qquad \alpha < \frac{K_1 M_0^*}{2\mathcal{K}_{q_0}}\,, \tag{3.103}$$

yields

$$\frac{d}{dt}E_\gamma^n(\alpha t, t) \leq -\frac{K_1}{2\alpha t}\left(E_\gamma^n(\alpha t, t) - 2M_0^*\right). \tag{3.104}$$

Therefore, we set $\alpha = \min\{\ln 2, \alpha_1\}$, having in mind (3.92), and with α_1 satisfying the condition (3.103) that depends on the initial data, γ, the collision kernel and A_2 from the integrability condition (2.9). This α is a positive real number. For such α, the estimate (3.104) holds.

Then, by a comparison argument, whenever $E_\gamma^n(\alpha t, t) > 2M_0^*$, we have $\frac{d}{dt}E_\gamma^n < 0$, and so $E_\gamma^n(\alpha t, t)$ decreases in t. Since at the initial time the partial sum is less that the threshold, i.e. $E_\gamma^n(0, 0) = m_0 < 2M_0^*$ and since it is continuous for all times, we have that the *strict* inequality $E_\gamma^n(\alpha t, t) \leq 2M_0^* < 4M_0^*$ holds uniformly on the closed interval $[0, T_n^*]$. By continuity of the partial sum, this strict inequality $E_\gamma^n(\alpha t, t) < 4M_0^*$ then holds on a slightly larger interval, which would contradict maximality of T_n^* from the definition (3.84), unless $T_n^* = 1$. Hence, we conclude that $T_n^* = 1$ for all n. Therefore, we in fact have that

$$E_\gamma^n(\alpha t, t) < 4M_0^*, \quad \text{for all } t \in [0, 1] \ \text{ for all } n \in \mathbb{N}.$$

Thus, by letting $n \to +\infty$, we conclude that $E_\gamma^\infty(\alpha t, t) < 4M_0^*$ for all $t \in [0, 1]$. That is,

$$\int_{\mathbb{R}^d} f(t, v)\ \ \mathcal{E}_{2/\gamma}((\alpha t)^{2/\gamma} \langle v \rangle^2)\, dv\ < 4M_0, \quad \text{for all } t \in [0, 1]. \tag{3.105}$$

Then, note that the above inequality implies that at the time $t = 1$, the Mittag-Leffler moment of order γ and rate $\alpha t = \alpha$ is finite. Now, starting the argument from $t = 1$ on, we bring ourselves into the setting of the propagation and conclude that for $t \geq 1$, the Mittag-Leffler moment of the same order γ and potentially smaller α than the one found on time interval $[0, 1]$, remain uniformly bounded for all $t \geq 1$.

In conclusion,

$$\int_{\mathbb{R}^d} f(t,v) \;\; \mathcal{E}_{2/\gamma}((\alpha t)^{2/\gamma} \langle v \rangle^2) \, dv \; < C, \quad \text{for all } t \in [0,1], \tag{3.106}$$

and

$$\int_{\mathbb{R}^d} f(t,v) \;\; \mathcal{E}_{2/\gamma}(\alpha^{2/\gamma} \langle v \rangle^2) \, dv \; < C, \quad \text{for all } t \geq 1. \tag{3.107}$$

Therefore, we conclude that for all $t \geq 0$, we have

$$\int_{\mathbb{R}^d} f(t,v) \;\; \mathcal{E}_{2/\gamma}((\alpha \min\{1,t\})^{2/\gamma} \langle v \rangle^2) \, dv \; < C. \tag{3.108}$$

In particular, this asserts that the solution of the Boltzmann equation with an initial mass and energy, will develop Mittag-Leffler moments, or equivalently, exponential high energy tails of order γ with rate $r = \alpha \min\{t, 1\}$. Therefore the proof of Theorem 3.1.1 is now complete.

□

Chapter 4

L^∞ theory: Pointwise behavior of tails

In this chapter we present our result on the pointwise upper bounds of solutions to the Boltzmann equation, which is based in part on the joint project with Gamba and Pavlović [36]. We begin by stating the main theorem and its corollary. We then discuss the main tools of the proof. Finally we present the proof of the main theorem and the corollary.

4.1 Statement of the main result

In this section we state our main result - an a priori estimate on the propagation in time of weighted L^∞ bounds of solutions to the homogeneous Boltzmann equation in the non-cutoff setting. As is often the case with results in the L^∞ setting, the assumption on the angular cross section $b(\cos\theta)$ is not given by the integral behavior (2.9). Instead, its singular behavior is described pointwise

$$b(\cos\theta) \approx (\sin\theta)^{-(d-1)-\nu}, \quad \text{with } \nu \in (0, 2]. \tag{4.1}$$

A kernel that satisfies (4.1) will automatically satisfy the integral condition (2.9) if and only if $\nu < \beta$.

Due to symmetries of the collisional kernel $Q(f,f)$, its value remains the same if B is replaced with $\tilde{B}$, provided that

$$B(|u|,\theta) \,+\, B(|u|,\theta+\pi) \,=\, \tilde{B}(|u|,\theta) \,+\, \tilde{B}(|u|,\theta+\pi).$$

In the case when both B and $\tilde{B}$ are factorized, i.e. $B(|u|,\theta) = |u|^{\gamma}\, b(\theta)$ and $\tilde{B}(|u|,\theta) = |u|^{\gamma}\, \tilde{b}(\theta)$ with the same parameter γ, this condition reduces to

$$b(\theta) \,+\, b(\theta+\pi) \,=\, \tilde{b}(\theta) \,+\, \tilde{b}(\theta+\pi). \tag{4.2}$$

Given $b(\theta)$ as in (4.1), there are many ways to construct $\tilde{b}$ that satisfies (4.2). A frequent choice is to set

$$\tilde{b}(\cos\theta) = \begin{cases} 2b(\cos\theta), & \text{if } \cos\theta > 0 \\ 0, & \text{if } \cos\theta < 0, \end{cases}$$

thus reducing the support of the angular kernel to half of the sphere. We, however, will use the following behavior on half spheres, as was the case in [52]

$$\tilde{b}(\cos\theta) \approx \begin{cases} |\sin\theta|^{-(d-1)-\nu}, & \text{if } \cos\theta > 0 \\ |\sin\theta|^{1+\gamma+\nu}, & \text{if } \cos\theta < 0. \end{cases} \tag{4.3}$$

This particular choice is tailored for the proof of Lemma 4.2.3. From now on, with the abuse of notation, we write $b(\cos\theta)$ instead of $\tilde{b}(\cos\theta)$.

Our main result is valid for exponential and Mittag-Leffler weight functions, and in both cases the proof relies on the corresponding weighted L^1 bounds. To emphasize this, and to make the presentation clean, we state the result for a general weight, which is defined in such a way as to mimic the exponential asymptotic behavior. So, the weight w, a function of velocity v, is defined via two parameters $\alpha > 0$ and $p \in (0, 2]$. One can think of α and p as describing the exponential behavior $e^{\alpha\langle v\rangle^p}$. More precisely, we assume that the weight function $w(v; \alpha, p)$ has the following properties:

(P1) $w(v; \alpha, p)$ is strictly positive, radially increasing in v, increasing in α.

(P2) For every $\alpha, \alpha', p > 0$ there exists a constant $C = C(\alpha, \alpha', p)$ and $c_2 = c_2(p)$, so that for every $v \in \mathbb{R}^d$

$$w(v; \alpha, p)\, w(2v; \alpha', p) \;\leq\; C\, w(v; \alpha + c_2\alpha', p).$$

(P3) Given $\delta \in [0, 1]$, and $\alpha, \alpha', p > 0$ and $k \geq 0$, there exist constants $C = C(\delta, k, \alpha, \alpha', p)$ and $D = D(\delta, k, \alpha, \alpha', p)$ so that $\forall v \in \mathbb{R}^d$

$$\text{If } \delta\alpha < \alpha', \quad \text{then } \frac{w(v; \alpha, p)^\delta}{w(v; \alpha', p)} \leq \frac{C}{\langle v\rangle^k}$$
$$\text{If } \delta\alpha > \alpha', \quad \text{then } \frac{w(v; \alpha, p)^\delta}{w(v; \alpha', p)} \geq D\,\langle v\rangle^k.$$

(P4) For every $\alpha, p > 0$ there is a constant $C = C(\alpha, p)$, so that $\forall v \in \mathbb{R}^d$

$$\left|\nabla_v \left(\frac{1}{w(v; \alpha, p)}\right)\right| \leq C\langle v\rangle.$$

Before we state the main theorem, we define a "*w-suitable solution*" to the Boltzmann equation as the one for which the modification of some of the techniques of Silvestre [52] can be applied. It needs to be in Schwartz class and w-weighted L^1 space, both locally in time. Moreover, for every time f the w-weighted L^1 norm of the solutions needs to be attained for some velocity v (depending on time t).

Definition 4.1.1. For a weight function $w(v)$, we say that a weak solution $f(t, v)$ (see Definition 3.1.1) to the Cauchy problem (2.16) with the cross section satisfying (2.6) and (2.9) is a w-*suitable* if

(i) for every $t > 0, T > t$: $\quad f \in L^\infty([t, T]; \mathcal{S}(\mathbb{R}^d))$

(ii) for every $t > 0$, the norm $\|f(t, v)\ w(v)\|_{L^\infty_v}$ is finite (not necessarily uniformly in time) and the norm is attained for some v.

Remark 4.1.1. For any $\alpha_1, \alpha_2, p > 0$ with $\alpha_2 < \alpha_1$ we have that if f is $w(\cdot; \alpha_1, p)$-suitable solutions, then it is also $w(\cdot; \alpha_2, p)$-suitable solution. To prove this claim, it suffices to show that if f satisfies condition (ii) in Definition 4.1.1 with $w(\cdot; \alpha_1, p)$, then condition (ii) also holds for the weight $w(\cdot; \alpha_2, p)$. So, suppose that for every time t, $\|f(t, v)\ w(v; \alpha_1, p)\|_{L^\infty_v}$ is finite (not necessarily uniformly in time) and the norm is attained for some v. Then, thanks to the property (P3), for every t we have

$$\begin{aligned} f(t, v)\ w(v; \alpha_2, p) &= f(t, v)\ w(v; \alpha_1, p) \frac{w(v; \alpha_2, p)}{w(v; \alpha_1, p)} \\ &\leq C f(t, v)\ w(v; \alpha_1, p) \frac{1}{\langle v \rangle^2} \\ &\leq \frac{C}{\langle v \rangle^2} \|f(t, v)\ w(v; \alpha_1, p)\|_{L^\infty_v}. \end{aligned} \tag{4.4}$$

Therefore, $\|f(t, v)\ w(v; \alpha_2, p)\|_{L^\infty_v}$ is finite for every t. Now, to see that this supremum is achieved, fix an arbitrary time t_0, suppose that

$$\|f(t_0, v)\ w(v; \alpha_2, p)\|_{L^\infty_v} = C_0$$

and suppose on contrary that this supremum is not attained. That is, suppose that there is a sequence $\{v_n\}_n$ so that

$$\|f(t_0,v_n)\, w(v_n;\alpha_2,p)\|_{L^\infty_v} < C_0$$
$$\|f(t_0,v_n)\, w(v_n;\alpha_2,p)\|_{L^\infty_v} \to C_0, \quad \text{as } \; n \to \infty.$$

Velocities v_n cannot be inside of a ball B_R of finite radius R, because then they would converge to some $v_* \in B_R$, and at that point we would have that $\|f(t_0,v_*)\, w(v_*;\alpha_2,p)\|_{L^\infty_v} = C_0$, which would contradict the assumption that the supremum is not achieved. Hence, there exists a subsequence, which we still call v_n, so that $|v_n| \to \infty$ and

$$C_0/2 < \|f(t_0,v_n)\, w(v_n;\alpha_2,p)\|_{L^\infty_v} < C_0.$$

This contradicts the decay in (4.4) as $f(t_0,v_n)\, w(v_n;\alpha_2,p) \le C\langle v_n\rangle^{-2} \to 0$, so the lower bound could not hold. This concludes the proof of the remark.

In the case of hard potentials that we consider, Alexandre, Morimoro, Ukai, Xu and Yang [4] proved that if all polynomial moments are finite, then the weak solution is of Schwartz class. Their result holds even for certain range of negative values γ. More precisely,

Theorem (Alexandre-Morimoto-Ukai-Xu-Yang [4]). *Suppose the cross section B is in the form* (2.6) *and satisfies* (4.1)*, and suppose that $\gamma > \max\{-\nu,-1\}$. Let f be a weak solution to the Cauchy problem* (2.16)*. For $0 \le T_0 < T_1$, if f satisfies*

$$|v|^l f \in L^\infty([T_0,T_1];L^1(\mathbb{R}^d)), \quad \textit{for any } \; l \in \mathbb{N}, \tag{4.5}$$

then

$$f \in L^{\infty}([t_0, T_1]; \mathcal{S}(\mathbb{R}^d)]), \tag{4.6}$$

for any $t_0 \in (T_0, T_1)$.

In the case of hard potentials, the condition (4.5) is automatically satisfied since the exponential moment of order γ is generated and remains uniformly bounded in time. Therefore, the weak solutions is really of the Schwartz class and the condition (i) of Definition 3.1.1. is satisfied.

We are now ready to state out main result, which is in part based on the joint work with Gamba and Pavlović [36].

Theorem 4.1.1. *(Propagation of* L^{∞}_{w} *tails) Let* $\alpha_0 > 0$, $p \in (0, 2]$ *and let* $w(v; \alpha_0, p)$ *be a weight function that satisfies properties* $(P1) - (P4)$. *Suppose* f *is a* $w(\cdot; \alpha_0, p)$*-suitable solution to the Cauchy problem* (2.16) *with the cross section* (2.6) *with* $0 < \gamma \leq 1$, *the angular kernel* (4.3) *with* $\nu \in (0, 1]$ *and the initial data* $f_0(v)$ *which has finite mass, energy and entropy.*

Suppose that propagation of w*-moments of* f *holds. More precisely, suppose that for every* $\alpha > 0$ *there exists* $0 < \alpha_1 < \alpha$ *and a constant* $C_1 > 0$ *(uniform in time) so that*

$$\begin{aligned} &\text{if } \; \|f_0(v)\, w(v; \alpha, p)\|_{L^1_v} < \infty, \\ &\qquad \text{then } \; \|f(t, v)\, w(v; \alpha_1, p)\|_{L^1_v} < C_1, \quad \forall t \geq 0. \end{aligned} \tag{4.7}$$

Then for any given $\alpha_0 > 0$, there exists $0 < \alpha_2 < \alpha_0$ and a constant C (uniform in time, depending on C_1, p, α_0, initial data and the cross section) so that

$$\begin{aligned} &\text{if } \ \|f_0(v)\, w(v;\alpha_0,p)\|_{L^\infty_v} < \infty, \\ &\qquad \text{then } \ \|f(t,v)\, w(v;\alpha_2,p)\|_{L^\infty_v} < C, \quad \forall t \geq 0. \end{aligned} \tag{4.8}$$

In Section 4.4 we provide examples of functions that satisfy properties (P1)-(P4). They will include exponentials and Mittag-Leffler functions, for which it has already been established that the corresponding moments (i.e. weighted L^1 bounds) propagate in time, and thus satisfy the assumption (4.7) of the Theorem 4.1.1. As a consequence, we will be able to prove the following statement.

Corollary 4.1.2. *(Exponential and Mittag-Leffler L^∞ moments)*

(a) Suppose $f_0(v) \leq Ce^{-\alpha_0 \langle v \rangle^p}$ for some $\alpha_0 > 0$ and $p < \frac{4}{\nu+2}$. Suppose f is a $e^{\alpha_0 \langle v \rangle^p}$-suitable solution to the Cauchy problem (2.16) *with the cross section* (2.6) *with $0 < \gamma \leq 1$, the angular kernel satisfying* (4.3) *with $\nu \in (0, 1]$ and the initial data $f_0(v)$ which has finite mass, energy and entropy. Then there exist a constant $C_1 > 0$ and $0 < \alpha < \alpha_0$ so that*

$$f(t,v) \leq C_1 e^{-\alpha \langle v \rangle^p}, \qquad \text{for all } \ t \geq 0.$$

(b) Suppose $f_0(v) \leq C\mathcal{E}_{2/p}(\alpha_0^{2/p} \langle v \rangle^2)$ for some $\alpha_0 > 0$ and $p < \frac{4}{\nu+2}$. Suppose f is a $\mathcal{E}_{2/p}(\alpha_0^{2/p} \langle v \rangle^2)$-suitable solution to the Cauchy problem (2.16) *with the cross section* (2.6) *with $0 < \gamma \leq 1$, and the angular kernel* (4.3) *with*

$\nu \in (0,1]$ and the initial data $f_0(v)$ which has finite mass, energy and entropy. Then there exist $C_1 > 0$ and $0 < \alpha < \alpha_0$ so that

$$f(t,v) \leq C_1 \mathcal{E}_{2/p}(\alpha^{2/p}\langle v\rangle^2), \qquad \textit{for all } t \geq 0.$$

4.2 Relevant previous results and tools

Our proof relies on the propagation of the corresponding weighted L^1 bounds. In this section we recall what is known about (weighted) L^∞ bounds. The transition from L^1 to L^∞ type results often employs a classical tool called Carleman representation, which we recall now.

4.2.1 Towards L^∞ bounds: Carleman representation

In previous works [13, 35, 52] on upper L^∞ bounds of solutions to the homogeneous Boltzmann equation, a specific change of variables was used, which is often referred to as Carleman representation. This technique was developed by Carleman [21]. See also [35, 40, 58]. The main idea behind the Carleman representation is to replace variables (v, v_*, σ) by (v, v', w). In this process the integration over the $(d-1)$ dimensional sphere reduces to the integration over a hyperplane that is orthogonal to $v' - v$. In this thesis we will use the version of Carleman representation given below.

Lemma 4.2.1 (Carleman representation, [21,35,40,52,59])**.** *Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} H(v,v')\, f(v'_*)\, B(r,\theta)\, d\sigma dv_* \;=\; \int_{\mathbb{R}^d} H(v,v')\, K_f(v,v')\, dv', \tag{4.9}$$

where the kernel $K_f(v, v')$ is given by

$$K_f(v, v') \; = \; \frac{2^{d-1}}{|v' - v|} \int_{\{w : w \cdot (v' - v) = 0\}} f(v + w) \; B(r, \theta) \; r^{-d+2} \; dw. \qquad (4.10)$$

The new set of variables (v, v', w) satisfies

$$r = \sqrt{|v' - v|^2 + |w|^2}, \;\; \cos\frac{\theta}{2} = \frac{|w|}{r},$$
$$v'_* = v + w, \;\; v_* = v' + w.$$

4.2.2 Weighted L^∞ bounds for the homogeneous Boltzmann equation with the angular cutoff

Once weighted L^1 estimates are developed, the next important question is understanding pointwise behavior of solutions. This has been achieved in the cutoff case for the polynomial weights by Arkeryd [13] and for exponential weights in the work of Gamba, Panferov and Villani [35]. We now provide the statement from [35] on the propagation in time of exponentially weighted L^∞ norms of solutions to the homogeneous Boltzmann equation in the cutoff case.

Theorem (Gamba-Panferov-Villani [35])**.** *Consider the Cauchy problem* (2.16), (2.6)*, for the hard potentials $0 < \gamma \leq 1$ with the angular kernel satisfying $0 \leq b(\cos\theta) \leq c \sin^{-\alpha}\theta$, with $\alpha < d - 1$, which corresponds to a Grad's cutoff. Suppose $f(t, v)$ is the unique solution to this Cauchy problem with initial data satisfying*

$$0 \leq f_0(v) \leq e^{-a_0|v|^2 + c_0}, \qquad \textit{for a.e. } v \in \mathbb{R}^d, \; \textit{for all}\, t \geq 0$$

that conserves the initial mass and energy. Then there exist constants $a > 0$

and $c \in \mathbb{R}$ *so that*

$$f(t,v) \leq e^{-a|v|^2+c}, \qquad \textit{for a.e. } v \in \mathbb{R}^d, \textit{ for all}\, t \geq 0.$$

The key tool for proving the pointwise estimate of [35] is the comparison principle, which was also established in [35], thanks to a monotonicity property of a linear Boltzmann semigroup. A crucial ingredient for a successful application of the comparison principle is an exponentially weighted upper bound of the linear "gain" operator, which was obtained in [35] using Carleman's form of the "gain" term and careful estimates some of which use the propagation of exponentially weighted L^1 norms of the solution.

Although the comparison principle of [35] is stated in the case of a cutoff, the proof implies that it should be expected in a non-cutoff case. However that is not sufficient to obtain the analogue of the point-wise propagation estimate of [35] in a non-cutoff case, since in [35] the application of the comparison principle proceeds via separately estimating the gain and loss terms, the procedure which cannot be carried out in a non-cutoff case. Despite not using the comparison principle[1], our proof of a propagation in time of exponentially decaying point-wise estimates carries a similarity to the idea of [35], in the sense that we too employ the estimates coming from the propagation of exponentially weighted L^1 norms of the solution to obtain weighted L^∞ estimates.

[1]Instead, we modify the contradiction argument from the recent work of Silvestre [52].

4.2.3 Recent L^∞ bounds for the Boltzmann equation

Recently Silvestre [52] obtained certain regularity results for the Boltzmann equation in a non-cutoff case by introducing at the level of the Boltzmann equation techniques inspired by the theory of integro-differential equations. Along the way, Silvestre [52] proved the following pointwise bound for a solution to the Boltzmann equation.

Theorem (Non-weighted pointise bounds, non-cutoff case, [52]). *Suppose $f(t,v)$ is a classical solution to the Boltzmann equation* (2.16) *with finite mass, energy and entropy. Then*

$$\|f(t,v)\|_{L^\infty_v} \leq a + bt^{-d/\nu},$$

for some constants a, b depending only on the initial energy, mass and entropy.

In this thesis we generalize the above estimate, to obtain a propagation in time of *weighted* L^∞ norms of a solution. The proof builds on the known weighted L^1 bounds, and one of the key tools used in that direction is the Carleman representation (Lemma 4.9). The following lemma from [52] provides an estimate that we use on the kernel K_f (see (4.10) for the definition of K_f). This lemma uses the specific structure of the angular kernel as given in (4.3).

Lemma 4.2.2 (Corollary 4.2, [52]). *For the angular kernel that satisfies* (4.3), *the weight function K_f in the Carleman representation* (4.9) *satisfies*

$$K_f(t,v,v') \approx \left(\int_{\{w: w\cdot(v'-v)=0\}} f(v+w)\, |w|^{1+\gamma+\nu}\, dw \right) |v'-v|^{-N-\nu} \tag{4.11}$$

On the other hand, the following lemma from [52] provides a lower bound on the kernel K_f in the Carleman representation on a distinguished set of points that lie on a certain cone. Its proof uses the representation from the above lemma.

Lemma 4.2.3 (Lemma 7.1, [52]). *Suppose f is a non-negative function on $\mathbb{R}^d$ that has finite and strictly positive mass, finite energy and finite entropy. Then, for any $v \in \mathbb{R}^d$, there exists a symmetric subset $A(v)$ of the unit sphere, and there are constants μ, λ, C (that depend on mass, energy and entropy bounds) so that*

(i) $|A(v)| \geq \frac{\mu}{\langle v \rangle}$, *where $|A(v)|$ denotes the $(N-1)$-Hausdorff measure of $A(v)$;*

(ii) For every v' for which the normalized vector $\frac{v'-v}{|v'-v|}$ belongs to the set $A(v)$, we have

$$K_f(v, v') \geq \lambda \langle v \rangle^{1+\gamma+\nu} |v' - v|^{-N-\nu}, \tag{4.12}$$

(iii) for every $\sigma \in A(v)$, $\quad |\sigma \cdot v| \leq C$.

Remark 4.2.1. Given v and the corresponding subset $A(v)$ of the unit sphere determined by the above lemma, we denote by $\Sigma(v)$ the corresponding cone centered at v of all vectors v' for which the normalization $\frac{v'-v}{|v'-v|}$ belongs to the set $A(v)$ i.e.

$$\Sigma(v) := \left\{ v' \in \mathbb{R}^d \, : \, \frac{v'-v}{|v'-v|} \in A(v) \subset \mathcal{S}^{N-1} \right\}.$$

It is for the points $v' \in \Sigma(v)$ that the lower bound in (ii) holds.

The final lemma of this section provides a lower bound of an integral over a cone Σ determined by a vector v and a subset A of the unit sphere. This will be crucial in estimating the negative contribution of the collision operator.

Lemma 4.2.4 (Lemma 7.2, [52])**.** *Assume that the maximum of a function $g(v)$ is achieved at $v = \tilde{v}$ and is equal to $\tilde{m}$. Assume A is a subset of the unit sphere and that $|A| \geq \mu > 0$. Let Σ be the cone centered at v that consists of all vectors $v' \in \mathbb{R}^d$ for which the normalized vector $\frac{v'-v}{|v'-v|}$ belongs to the set A, i.e. $\Sigma := \left\{ v' \in \mathbb{R}^N \, : \, \frac{v'-v}{|v'-v|} \in A \right\}$. Then*

$$\int_{\Sigma} (\tilde{m} - g(v'))\, |\tilde{v} - v'|^{-N-\nu}\, dv' \geq \frac{c\, \tilde{m}^{1+\nu/N}\, \mu^{1+\nu/N}}{\left(\int_{\Sigma} |g(v')| dv' \right)^{\nu/N}}. \tag{4.13}$$

4.3 Proof of Theorem 4.1.1

To prove the propagation in time of weighted L^∞_v norm of solutions to the Boltzmann equation, we modify the contradiction argument of Silvestre used to prove Theorem 4.2.3. Since we too are in the case of a non-cutoff, we cannot use the splitting of the collision operator into the "gain" and "loss" terms. However the standard splitting (see (4.26)) that is often used in non-cutoff cases, and which has been used by Silvestre [52] too, is not adequate for us. We need to further refine the splitting (for details see (4.31)) to be able to obtain *weighted* upper bounds. In particular, the appearance of the term $Q_{1,2}$ in (4.31) is new. To control that term, we need to overcome the singularity of a non-cutoff collision operator, which we do thanks to oscillations present in the weight function. The other substantial difference with respect to [52] is that in our estimates we take the advantage of the known propagation of w-moments, given via (4.7).

Setting up the contradiction argument

Let $\alpha_0 > 0$ and $p \in (0, 2]$ be fixed, and suppose that initial data satisfies

$$\|f_0(v)\ w(v; \alpha_0, p)\|_{L^\infty_v} \leq C < \infty. \tag{4.14}$$

Then for $\alpha = \alpha_0^-$ we have thanks to (4.14)

$$\begin{aligned} \|f_0(v)\ w(v; \alpha, p)\|_{L^1_v} &\leq C \int \frac{w(v; \alpha, p)}{w(v; \alpha_0, p)}\, dv \\ &\leq C \int \frac{1}{\langle v \rangle^{d+}}\, dv \qquad (4.15) \\ &< \infty, \qquad (4.16) \end{aligned}$$

where to obtain (4.15) we used the property (P3). Therefore, assumption (4.7) implies that there exists $\alpha_1 < \alpha_0$ such that

$$\|f(t, v)\ w(v; \alpha_1, p)\|_{L^1_v} < \infty. \tag{4.17}$$

It is convenient to introduce the following notation. For parameters β, p and for any $t \geq 0$, let $m_{\beta,p}(t)$ denote the $w(v; \beta, p)$-weighted L^∞ norm in velocity, i.e.

$$m_{\beta,p}(t) := \|f(t, v)\ w(v; \beta, p)\|_{L^\infty_v}\,. \tag{4.18}$$

In order to prove the theorem, it suffices to find $\alpha_2, a, b > 0$ such that

$$m_{\alpha_2,p}(t) < a + b\, t^{-d/\nu}. \tag{4.19}$$

First, we show that (4.19) is true at $t = 0$ for $\alpha_2 < \alpha_0$ and $a, b > 0$ that will be determined later in the proof. Namely, by the property (P1) that expresses monotonicity of $w(v; \beta, p)$ in β, we have

$$m_{\alpha_2,p}(0) \leq m_{\alpha_0,p}(0) < \infty, \tag{4.20}$$

where the last inequality follows from (4.14). On the other hand, $a + bt^{-N/\nu}$ blows up around $t = 0$. Thus, the inequality (4.19) trivially holds for $t = 0$, and by the continuity of $m_{\alpha_2,p}(t)$ it is satisfied on a time interval of positive measure starting at $t = 0$.

Now, assume that there exists the first time $t_0 > 0$ for which the inequality (4.19) fails. At this time then

$$m_{\alpha_2,p,q}(t_0) = a + bt_0^{-d/\nu}. \tag{4.21}$$

Since f is a $w(v, \alpha_0, p)$-suitable solution, it is also $w(v, \alpha_2, p)$-suitable since $\alpha_2 < \alpha_0$. Therefore, for every time t the norm $L^\infty_{w(v,\alpha_2,p)}$ of $f(t, v)$, i.e. $m_{\alpha_2,p}(t)$, is attained for some velocity v. Let v_0 be such velocity corresponding to time t_0. In other words,

$$m_{\alpha_2,p}(t_0) = f(t, v_0)\, w(v_0, \alpha_2, p) = a + bt_0^{-d/\nu}. \tag{4.22}$$

Hence,

$$\begin{aligned} &f(t, v_0)\, w(v_0, \alpha_2, p) < a + bt^{-d/\nu}, \quad \forall t < t_0, \\ &f(t_0, v_0)\, w(v_0, \alpha_2, p) = a + bt_0^{-d/\nu}. \end{aligned} \tag{4.23}$$

Therefore,

$$\partial_t \left(f(t, v_0)\ w(v_0, \alpha_2, p)\right)_{t=t_0} \geq \partial_t \left(a + bt^{-d/\nu}\right)_{t=t_0}. \tag{4.24}$$

Combining (4.22) and (4.24), we conclude the following lower bound at (t_0, v_0)

$$\partial_t f(t_0, v_0) \geq -\frac{d}{\nu} b^{-\nu/d}\ \frac{1}{w(v_0, \alpha_2, p)}\ \left(m_{\alpha_2,p}(t_0) - a\right)^{1+\frac{N}{\nu}}. \tag{4.25}$$

In the rest of the proof we look for an upper bound on $\partial_t f(t_0, v_0)$ using the Boltzmann equation (2.16). In particular, we estimate the collision operator $Q(f, f)(t_0, v_0)$. The upper bound that we will obtain will contradict (4.25) and will thus conclude our proof.

In the rest of the proof, if parameters of the weight function w are not specified, they are assumed to be α_2 and p.

Splitting of the collision operator

When the Grad's cutoff is not assumed, it is often convenient to split the collision integral into the following two terms, both of which are finite (see [1, 2, 5, 27, 52, 56, 57])

$$Q(f, f) = Q_1(f, f) + Q_2(f, f), \tag{4.26}$$

$$\begin{aligned} Q_1(f, f) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f' - f) f'_* B\ d\sigma dv_* \\ &= \int_{\mathbb{R}^d} (f' - f) K_f(v, v')\ dv', \end{aligned} \tag{4.27}$$

$$Q_2(f, f) = f(v) \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f'_* - f_*) B\ d\sigma dv_*. \tag{4.28}$$

Since we study weighted norms, we introduce new splitting of Q tailored for the building blocks of our calculations, which are functions of the type fw. More precisely, we further split Q_1 into $Q_{1,1}$ and $Q_{1,2}$ according to

$$Q_1 = Q_{1,2} + Q_{1,2},$$

where

$$Q_{1,1}(f,f) \;=\; \frac{1}{w(v)} \int_{\mathbb{R}^d} (f'\; w' - f\; w)\;\; K_f(v,v')\, dv', \tag{4.29}$$

$$Q_{1,2}(f,f) \;=\; \int_{\mathbb{R}^d} f'\; w' \left(\frac{1}{w'} - \frac{1}{w}\right)\; K_f(v,v')\, dv'. \tag{4.30}$$

Hence our overall decomposition of the collision operator is

$$Q(f,f) \;=\; Q_{1,1}(f,f) \;+\; Q_{1,2}(f,f) \;+\; Q_2(f,f). \tag{4.31}$$

This splitting helps us to identify the negative contribution within Q_1 at (t_0, v_0), which is coming from $Q_{1,1}(f,f)(t_0,v_0)$. More precisely, recalling that at time $t = t_0$ the L^∞ norm defining $m_{\alpha_2,p}(t_0)$ is attained at v_0, i.e. $m_{\alpha_2,p}(t_0) = \|f(t_0,v)\; w(v)\|_{L^\infty_v} = f(t_0,v_0)\, w(v_0)$. Therefore,

$$Q_{1,1}(f,f)(t_0,v_0) \;=\; -\frac{1}{w(v_0)} \int_{\mathbb{R}^d} (m_{\alpha_2,p}(t_0) - f(t_0,v')\; w(v'))\;\; K_f(v_0,v')\, dv'. \tag{4.32}$$

Since the integrand is a positive function, $Q_{1,1}(f,f)(t_0,v_0)$ is negative. However this information is not sufficient, and we proceed to obtain a precise upper bound on $Q_{1,1}(f,f)(t_0,v_0)$, as well as on the other two terms. This is what we do below.

Estimating $Q_{1,1}$

As noted above, $Q_{1,1}(f,f)$ is negative at (t_0, v_0). To estimate how negative it is, we reduce the domain of integration to the cone $\Sigma(v_0)$ on which the lower bound (4.12) on K_f is known to hold. This cone was introduced in Lemma 4.2.3 and Remark 4.2.1. This yields

$$Q_{1,1}(t_0, v_0) \leq -C \, \frac{\langle v_0 \rangle^{1+\gamma+\nu}}{w(v_0)} \int_{\Sigma(v_0)} (m_{\alpha_2,p}(t_0) - f(t_0, v') \, w(v')) \; |v' - v_0|^{-d-\nu} \, dv'. \tag{4.33}$$

The above integral, over the cone $\Sigma(v_0)$, is then estimated using Lemma 4.2.4 with $g = fw$ and its maximum value $\tilde{m} = m_{\alpha_2,p}(t_0)$. This implies

$$Q_{1,1}(f,f)(t_0, v_0) \;\leq\; -C \frac{\langle v_0 \rangle^{1+\gamma+\nu}}{w(v_0)} \, (m_{\alpha_2,p}(t_0))^{1+\nu/d} \, \frac{\left(\frac{1}{\langle v_0 \rangle}\right)^{1+\nu/d}}{\left(\int_{\Sigma(v_0)} f' \; w' dv'\right)^{\nu/d}}. \tag{4.34}$$

We proceed the estimate by considering the above integral in two cases, when $|v_0| \leq R$ and when $|v_0| > R$, where the number R is determined in the following way. Recall the statement in Lemma 4.2.3 (iii) according to which for every $\sigma \in A(v_0)$, where $A(v_0)$ is the symmetric subset of the unit sphere that determines the cone $\Sigma(v_0)$, we have $|\sigma \cdot v_0| \leq C$. This means that set $A(v_0)$ lies in a band of the unit sphere of width at most $C/|v_0|$ "around the largest circle on the sphere belonging to the hyperplane that is perpendicular to" v_0. Hence, the larger $|v_0|$ is, the thinner the band is. Therefore, there exists a number R (depending on C), as is noted in [52], such that

$$|v'| > \frac{|v_0|}{2}, \quad \text{whenever } v' \in \Sigma(v_0) \text{ and } |v_0| > R. \tag{4.35}$$

Case 1: $|v_0| \le R$. It immediately follows that

$$\frac{1}{\langle v_0 \rangle} \ge \frac{1}{\langle R \rangle}, \tag{4.36}$$

and consequently

$$\frac{1}{w(v_0)} \ge \frac{1}{w(R)}, \tag{4.37}$$

due to the property (P1) according to which the weight w is strictly positive and radially increasing in v. In addition, by the assumption (4.7) on propagation of weighted L^1 bounds, we have

$$\int_{\Sigma(v_0)} f' \, w' dv' \le \int_{\mathbb{R}^d} f' \, w' dv' \le C, \tag{4.38}$$

where C now also depends on R. Applying estimates (4.36)-(4.38) to (4.34) yields the following estimate on $Q_{1,1}$ whenever $|v_0| \le R$

$$Q_{1,1}(f,f)(t_0, v_0) \;\le\; -C \, \langle v_0 \rangle^{1+\gamma+\nu} \, (m_{\alpha_2,p}(t_0))^{1+\nu/d}. \tag{4.39}$$

Case 2: $|v_0| > R$. Now we need a more refined bound on $\int_{\Sigma(v_0)} f' \, w' dv'$ than the one given by (4.38). To find such a bound, recall from (4.35) that for $|v_0| > R$ and for any $v' \in \Sigma$

$$|v_0| \;<\; 2|v'|. \tag{4.40}$$

For $w(v_0;\alpha_3,p)$, where α_3 will be chosen below, we have

$$\begin{aligned}
\int_\Sigma f' w(v';\alpha_2,p)\,dv' &= \int_\Sigma f'\, w(v';\alpha_2,p)\frac{w(v_0;\alpha_3,p)}{w(v_0;\alpha_3,p)}\,dv' \\
&\le \frac{1}{w(v_0;\alpha_3,p)}\int_\Sigma f'\, w(v';\alpha_2,p)\, w(2v';\alpha_3,p)\,dv' \qquad (4.41)\\
&= \frac{1}{w(v_0;\alpha_3,p)}\int_\Sigma f'\, w(v';\alpha_2+c_2\alpha_3,p)\,dv' \qquad (4.42)\\
&\le C\frac{1}{w(v_0;\alpha_3,p)}, \qquad (4.43)
\end{aligned}$$

where to obtain (4.41) we used monotonicity of w with respect to v

$$w(v_0;\alpha_3,p)\le w(2v';\alpha_3,p), \qquad (4.44)$$

which holds thanks to the property (P1). To obtain (4.42) we used the property (P2). The inequality (4.43) follows from the use of the assumption on the propagation of L^1 weighted bounds (4.17), which can be applied if $\alpha_3>0$ satisfies

$$\alpha_2+c_2\alpha_3<\alpha_1. \qquad (4.45)$$

Now we estimate (4.34) using (4.43)

$$\begin{aligned}
Q_{1,1}(f,f)(t_0,v_0) &\le -C\,\frac{\langle v_0\rangle^{1+\gamma+\nu}}{w(v_0;\alpha_2,p)}\,(m_{\alpha_2,p}(t_0))^{1+\nu/d}\,\frac{\left(\frac{1}{\langle v_0\rangle}\right)^{1+\frac{\nu}{d}}}{\left(\frac{1}{w(v_0;\alpha_3,p)}\right)^{\nu/d}}\\
&= -C\,\langle v_0\rangle^{1+\gamma+\nu}\,(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}\,\langle v_0\rangle^{-1-\frac{\nu}{d}}\,\frac{(w(v_0;\alpha_3,p))^{\nu/d}}{w(v_0;\alpha_2,p)}\\
&\le -C\,\langle v_0\rangle^{1+\gamma+\nu}\,(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}\,\langle v_0\rangle^{-1-\frac{\nu}{d}}\,\langle v_0\rangle^{2} \qquad (4.46)\\
&\le -C\,\langle v_0\rangle^{1+\gamma+\nu}\,(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}, \qquad (4.47)
\end{aligned}$$

where to obtain (4.46) we use the property (P3) according to which

$$\frac{w(v_0;\alpha_3,p)^{\nu/d}}{w(v_0;\alpha_2,p)} \geq C\langle v\rangle^2,$$

provided that

$$\frac{\alpha_3\,\nu}{d} > \alpha_2. \tag{4.48}$$

Now we pause for a moment to choose α_3 to satisfy (4.45) and (4.48). In particular, we choose α_3 such that

$$\frac{\alpha_3\,\nu}{d} = 2\alpha_2,$$

which automatically satisfies (4.48). Then (4.45) implies the condition on α_2

$$\alpha_2 < \frac{\alpha_1}{1+\frac{2\,c_2\,d}{\nu}}. \tag{4.49}$$

For such α_2, the estimates (4.39) and (4.47) imply

$$Q_{1,1}(f,f)(t_0,v_0) \leq -C\,\langle v_0\rangle^{1+\gamma+\nu}\,(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}. \tag{4.50}$$

Estimating $Q_{1,2}$

Recall the definition of $Q_{1,2}$ from (4.29)

$$Q_{1,2}(f,f) \;=\; \int_{\mathbb{R}^d} f'\,w'\,(\frac{1}{w'}-\frac{1}{w})\;K_f(v,v')\,dv'.$$

We start by a simple observation. Since $m_{\alpha_2,p}(t)$ is defined as a supremum of $f(t,v)\;w(v)$ over velocities v, we have $f'w' \leq m_{\alpha_2,p}$ for every $v' \in \mathbb{R}^d$.

Therefore,

$$\begin{aligned}
&Q_{1,2}(f,f)(t,v) \\
&\quad \le\ m_{\alpha_2,p}(t)\ \int_{\mathbb{R}^d} (\frac{1}{w'} - \frac{1}{w})\ K_f(v,v')\ dv' \\
&\quad =\ m_{\alpha_2,p}(t)\ \int_{\mathbb{R}^d} \left(\frac{1}{w(v+z)} - \frac{1}{w(v)}\right)\ K_f(v,v+z)\ dz. \qquad (4.51)
\end{aligned}$$

Since the kernel $K_f(v, v+z)$ has a singularity at $z = 0$, we estimate the above integral inside the unit ball and outside the unit ball separately, using different bounds on $\frac{1}{w(v+z)} - \frac{1}{w(v)}$ in those regions.

Outside the unit ball. Since the singularity of $K_f(v, v + z)$ is at $z = 0$, which is outside the considered region, a coarse bound

$$\left|\frac{1}{w(v+z)} - \frac{1}{w(v)}\right| \le C,$$

which follows from the property (P1). Applying Lemma 4.2.2, followed by a spherical change of coordinates, yields

$$\begin{aligned}
&\int_{|z|>1} \left|\frac{1}{w(v+z)} - \frac{1}{w(v)}\right|\ K_f(v,v')\ dv' \\
&\quad \le\ C\ \int_{|z|>1} \left(\int_{\{w:w\cdot z=0\}} f(v+w)|w|^{1+\gamma+\nu}\ dw\right)\ |z|^{-d-\nu}\ dz \\
&\quad =\ C\ \int_1^\infty \int_{\mathbb{S}^{d-1}} \left(\int_{\{w:w\cdot z=0\}} f(v+w)|w|^{1+\gamma+\nu}\ dw\right)\ \rho^{-d-\nu}\ \rho^{d-1}\ dS(z)\ d\rho \\
&\quad =\ C\ \left(-\rho^{-\nu}\right)_1^\infty\ \int_{\mathbb{S}^{d-1}} \left(\int_{\{w:w\cdot z=0\}} f(v+w)|w|^{1+\gamma+\nu}\ dw\right)\ dS(z) \\
&\quad = C\ \int_{\mathbb{R}^d} f(v+y)\ |y|^{\gamma+\nu}\ dy \qquad (4.52) \\
&\quad \le\ C\langle v\rangle^{\gamma+\nu}, \qquad (4.53)
\end{aligned}$$

where to obtain (4.52) we used the fact that

$$\nu > 0. \tag{4.54}$$

and we applied a classical change of variables as stated in Lemma 4.3.1 below.

Lemma 4.3.1. *Suppose g is any non-negative function. Then*

$$\int_{\mathbb{S}^{d-1}} \int_{\{\omega : \omega\cdot\sigma = 0\}} g(\omega)\, d\omega\, dS(\sigma) \;=\; c_d \int_{\mathbb{R}^d} g(y)\, \frac{dy}{|y|}. \tag{4.55}$$

The inequality (4.53) follows from the change of variables combined with the generation of polynomial moments and conservation of mass.

Inside the unit ball $|z| < 1$. Here we need a better bound on $\left|\frac{1}{w(v+z)} - \frac{1}{w(v)}\right|$ to compensate for the singularity of $K_f(v, v+z)$ at $z = 0$. By the mean-value theorem, we have for some $t \in [0,1]$

$$\begin{aligned}
\left|\frac{1}{w(v+z)} - \frac{1}{w(v)}\right| &= \left|\nabla\left(\frac{1}{w}\right)(tv + (1-t)(v+z))\cdot(v+z-v)\right| \\
&\le C\,(t\langle v\rangle + (1-t)\langle v+z\rangle)\,|z| \qquad (4.56) \\
&\le (\langle v\rangle + |z|)\,|z| \qquad (4.57) \\
&\le 2\langle v\rangle\,|z|,
\end{aligned}$$

where (4.56) follows from the property (P4), while the inequality (4.57) follows from an elementary inequality $\langle v+z\rangle \le \langle v\rangle + |z|$. Therefore, applying again

Lemma 4.2.2 and spherical change of coordinates yields

$$
\begin{aligned}
&\int_{B_1} \left| \frac{1}{w(v+z)} - \frac{1}{w(v)} \right| \; K_f(v,v') \; dv' \\
&\leq \; C \; \langle v \rangle \; \int_{B_1} \left(\int_{\{w: w\cdot z = 0\}} f(v+w)|w|^{1+\gamma+\nu} \; dw \right) \; |z|^{-d-\nu+1} \; dz \\
&= \; C \; \langle v \rangle \; \int_0^1 \int_{\mathbb{S}^{d-1}} \left(\int_{\{w: w\cdot z = 0\}} f(v+w)|w|^{1+\gamma+\nu} \; dw \right) \; \rho^{-d-\nu+1} \; \rho^{d-1} \; dS(z) \; d\rho \\
&= \; C \; \langle v \rangle \; \left(\rho^{1-\nu} \right)_0^1 \; \int_{\mathbb{S}^{d-1}} \left(\int_{\{w: w\cdot z = 0\}} f(v+w)|w|^{1+\gamma+\nu} \; dw \right) \; dS(z) \\
&= \; C \; \langle v \rangle \; \int_{\mathbb{R}^d} f(v+y) \; |y|^{\gamma+\nu} \; dy \\
&\leq C \; \langle v \rangle \; \left(C + C\langle v \rangle^{\gamma+\nu} \right) \\
&\leq C \; \langle v \rangle^{1+\gamma+\nu}.
\end{aligned}
$$

Note that for this calculation to work we need that

$$\nu \leq 1. \tag{4.58}$$

In conclusion, combining the bounds obtained for the inside and outside the ball regions, we get

$$Q_{1,2}(f,f)(t,v) \; \leq \; C \; m(t) \; \langle v \rangle^{1+\gamma+\nu}. \tag{4.59}$$

Estimating Q_2

Recall that Q_2 is defined as

$$Q_2(f,f) \; = \; f(v) \; \int_{\mathbb{R}^d} \int_{\mathbb{S}^{N-1}} (f'_* - f_*) B \; d\sigma dv_*.$$

It is well-known, from the pioneering work on cancelation properties, by Alexandre, Desvillettes, Villani and Wennberg [2], that the above double integral can be represented as a convolution operator. Thus, Q_2 takes the following simplified form

$$Q_2(f,f)(t,v) \;=\; (\tilde{B} * f)(v)\; f(v) \tag{4.60}$$

where

$$\tilde{B}(v) = C|v|^{\gamma}, \tag{4.61}$$

where γ is the potential rate from the collision kernel, and C is a dimensional constant depending on the angular kernel. Because of this simplified representation, one then has the following estimate on Q_2

$$Q_2(f,f)(t,v) \;=\; (\tilde{B} * f)(v)\; f(v) \;\leq\; \begin{cases} C\; m_{\alpha_2,p}(t)\; \langle v\rangle^{\gamma}, & \text{if } \gamma \geq 0 \\ C\; (m_{\alpha_2,p}(t))^{1-\frac{\gamma}{N}}, & \text{if } \gamma < 0. \end{cases} \tag{4.62}$$

Conclusion

In summary, the following are the estimates (4.50), (4.59), (4.62) of all three parts (4.29) of the collision operator

$$\begin{aligned} Q_{1,1}(f,f)(t_0,v_0) \;&\leq\; -C\; (m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}\; \langle v_0\rangle^{1+\gamma+\nu}, \\ Q_{1.2}(f,f)(t_0,v_0) \;&\leq\; C\; m_{\alpha_2,p}(t_0)\langle v_0\rangle^{1+\gamma+\nu}, \\ Q_2(f,f)(t_0,v_0) \;&\leq\; C\; m_{\alpha_2,p}(t_0)\; \langle v_0\rangle^{\gamma}. \end{aligned}$$

Combining the three estimates yields

$$
\begin{aligned}
Q(f,f)(t_0,v_0) \;&\leq\; -c\,(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}\,\langle v_0\rangle^{1+\gamma+\nu} \;+\; C\,m_{\alpha_2,p}(t_0)\,\langle v_0\rangle^{1+\gamma+\nu} \\
&=\; \left(-c\,(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}} + C\,m_{\alpha_2,p}(t_0)\right)\langle v_0\rangle^{1+\gamma+\nu} \\
&\leq\; -\frac{c}{2}(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}\,\langle v_0\rangle^{1+\gamma+\nu} \qquad (4.63)\\
&\leq\; -\frac{c}{2}(m_{\alpha_2,p}(t_0))^{1+\frac{\nu}{d}}, \qquad (4.64)
\end{aligned}
$$

where the inequality (4.63) holds provided that

$$\left(\frac{2C}{c}\right)^{d/\nu} \leq m_{\alpha_2,p}(t_0) = a + bt_0^{-d/\nu}. \qquad (4.65)$$

So, we choose a to be

$$a := \left(\frac{2C}{c}\right)^{d/\nu}. \qquad (4.66)$$

Now, let us recall (4.25)

$$Q(f,f)(t_0,v_0) = \partial_t f(t_0,v_0) \;\geq\; -\frac{d}{\nu}b^{-\nu/d}\,(m_{\alpha_2,p}(t_0)-a)^{1+\frac{\nu}{d}}. \qquad (4.67)$$

Hence, if we choose b so that

$$\frac{c}{2} = \frac{d}{\nu}b^{-\nu/d}, \qquad (4.68)$$

we get the contradiction with the upper bound (4.64). This completes the proof of the theorem.

4.4 Examples of weight functions and the proof of Corollary 4.1.2

We now provide two examples of functions that satisfy properties (P1)-(P4) and to which Theorem 4.1.1 can be applied, as will be proven bellow.

Example 1.

$$w_1(v;\alpha,p) = e^{\alpha\langle v\rangle^p}.$$

Now we proceed to check that w_1 indeed satisfies (P1)-(P4). It is easy to see that $w_1(v;\alpha,p)$ is strictly positive, radially increasing in v, and increasing in α. Therefore it satisfies property (P1).

Next, note that for any $\alpha_1, \alpha_2, p > 0$ we have

$$\begin{aligned} w_1(v;\alpha_1,p)\; w(2v;\alpha_2,p) &= e^{\alpha_1\langle v\rangle^p}\; e^{\alpha_2\langle 2v\rangle^p} \\ &\leq e^{(\alpha_1+2^p\alpha_2)\langle v\rangle^p} \\ &= w_1(v;\ \alpha_1+2^p\alpha_2,\ p), \end{aligned}$$

thus w_1 satisfies condition (P2) as well.

To check that condition (P3) holds, let $\delta\in[0,1]$, and let $\alpha_1,\alpha_2,p>0$ and $k\geq 0$. If $\delta\alpha_1<\alpha_2$, then

$$\begin{aligned} \frac{w_1(v;\alpha_1,p)^\delta}{w_1(v;\alpha_2,p)} &= \frac{e^{\delta\alpha_1\langle v\rangle^p}}{e^{\alpha_2\langle v\rangle^p}} \\ &= e^{(\delta\alpha_1-\alpha_2)\langle v\rangle^p} \\ &\leq CD\,\langle v\rangle^k, \end{aligned}$$

where C is a constant that depends on parameters $k,\delta,\alpha_1,\alpha_2,p$. The last inequality holds because $\delta\alpha_1-\alpha_2<0$, so the exponential $e^{(\delta\alpha_1-\alpha_2)\langle v\rangle^p}$ decays faster than any polynomial.

Similarly, if $\delta\alpha_1 > \alpha_2$, then

$$\begin{aligned}\frac{w_1(v;\alpha_1,p)^\delta}{w_1(v;\alpha_2,p)} &= e^{(\delta\alpha_1-\alpha_2)\langle v\rangle^p} \\ &\geq D\,\langle v\rangle^k,\end{aligned}$$

where D is a constant that depends on parameters $k, \delta, \alpha_1, \alpha_2, p$. The last inequality holds because $\delta\alpha_1 - \alpha_2 > 0$, so the exponential $e^{(\delta\alpha_1-\alpha_2)\langle v\rangle^p}$ grows faster than any polynomial. In conclusion, w_1 satisfies condition (P3).

Finally, it is easy to check that for any $\alpha, p > 0$ we have

$$\begin{aligned}\left|\nabla_v\left(\frac{1}{w_1(v;\alpha,p)}\right)\right| &= \left|\nabla_v\left(e^{-\alpha\langle v\rangle^p}\right)\right| \\ &\leq |v|\,\left(\alpha p\,\langle v\rangle^{p-2}\,e^{-\alpha\langle v\rangle^p}\right) \\ &\leq \langle v\rangle\,\left(q+\alpha p\,\langle v\rangle^{p-2}\frac{1}{\alpha\langle v\rangle^p}\right) \\ &\leq (q+p)\langle v\rangle.\end{aligned}$$

Therefore w_1 satisfies property (P4).

Example 2. Second example are Mittag-Leffler functions

$$w_2(v;\alpha,p) = \mathcal{E}_{2/p}(\alpha^{2/p}\langle v\rangle^2).$$

For simplicity we now verify that $w_2(v;\alpha,p)$, i.e. a Mittag-Leffler function, satisfies (P1)-(P4), because those functions are used in Corollary 4.1.2.

Recall (3.4) that

$$ce^{\alpha\langle v\rangle^p} \leq \mathcal{E}_{2/p}(\alpha^{2/p}\langle v\rangle^2) \leq Ce^{\alpha\langle v\rangle^p}.$$

Using this equivalence relation and properties of classical exponential functions proved in Example 1, it is easy to check that a Mittag-Leffler function $w_2(v;\alpha,p)$ satisfies first three properties (P1)-(P3). It remains to show that it satisfies condition (P4) as well.

$$\begin{aligned}
\left|\nabla_v\left(\frac{1}{\mathcal{E}_{2/p}(\alpha^{2/p}\langle v\rangle^2)}\right)\right| &= \left|\nabla_v\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)^{-1}\right| \\
&\leq \left(\sum_{k=1}^{\infty}\frac{2k\alpha^{2k/p}\ \langle v\rangle^{2k-1}}{\Gamma(\frac{2k}{p}+1)}\right)\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)^{-2} \\
&\leq p\left(\sum_{k=1}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k-1}}{\Gamma(\frac{2k-2}{p}+1)}\right)\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)^{-2},
\end{aligned}$$

where in the last inequality we used

$$\frac{2k}{\Gamma(\frac{2k}{p}+1)}=\frac{p}{\Gamma(\frac{2k}{p})}\leq\frac{p}{\Gamma(\frac{2k-2}{p}+1)}.$$

Therefore, by simple algebraic manipulations, we get

$$\begin{aligned}
\left|\nabla_v\left(\frac{1}{\mathcal{E}_{2/p}(\alpha^{2/p}\langle v\rangle^2)}\right)\right| &\leq p\ \alpha^{2/p}\ \langle v\rangle\left(\sum_{k=1}^{\infty}\frac{\alpha^{(2k-2)/p}\ \langle v\rangle^{2k-2}}{\Gamma(\frac{2k-2}{p}+1)}\right)\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)^{-2} \\
&= p\ \alpha^{2/p}\ \langle v\rangle\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)^{-2} \\
&= p\ \alpha^{2/p}\ \langle v\rangle\left(\sum_{k=0}^{\infty}\frac{\alpha^{2k/p}\ \langle v\rangle^{2k}}{\Gamma(\frac{2k}{p}+1)}\right)^{-1} \\
&\leq p\ \alpha^{2/p}\ \langle v\rangle;
\end{aligned}$$

hence the property (P4) holds for the Mittag-Leffler function $w_2(v,\alpha,p)$.

We are now in a position to prove Corollary 4.1.2.

Proof of Corollary 4.1.2. We provide details of the proof of (a). Part (b) can be proved in an analogous way. First we observe that

$$Ce^{\alpha\langle v\rangle^p} = Cw_1(v;\alpha,p),$$

where w_1 is the function introduced in Example 1. Therefore we know that $Ce^{\alpha\langle v\rangle^p}$ satisfies (P1)-(P4). On the other hand, by Theorem 3.1.1, the propagation condition (4.8) of Theorem 4.1.1 is satisfied. The claim follows from an application of Theorem 4.1.1.

□

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