Neural Network Algorithms for Stochastic Optimal Control

SIAM OP26 – MS38 “Learning-informed Optimization and Control”

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Stochastic Optimal Control: Setup and Applications

Setup state $\mathbf{z} \in \mathbb{R}^d$, control $\mathbf{u} \in \mathbb{R}^m$:

$$\Phi(t, \mathbf{x}) = \min_{\mathbf{u}} \mathbb{E}\left[g(\mathbf{z}(T)) + \int_t^T L(s, \mathbf{z}, \mathbf{u})\, ds \right] \quad \text{s.t.} \quad d\mathbf{z}(s) = f(s, \mathbf{z}, \mathbf{u})\, ds + \sigma\, dW, \quad \mathbf{z}(t) = \mathbf{x}$$

- drift f and noise $\sigma\, dW$ govern the dynamics
- running cost L along the path, terminal cost g at $s = T$

Rare Event Analysis

- f: system dynamics
- L: trajectory penalty
- g: event indicator

e.g. chaotic Lorenz, extreme weather, Burgers–Huxley

Bayesian Inverse Problems

- f: score / flow prior
- L: control penalty
- g: neg. log-likelihood

e.g. inverse scattering, structural health, inverse design

Stochastic PDE Control

- f: discretized SPDE
- L: tracking + control
- g: terminal error

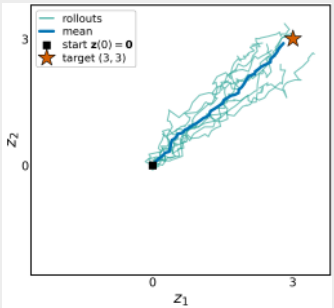
e.g. stochastic reaction-diffusion, FitzHugh–Nagumo

common challenge: high-dim, complex dynamics, reliable training

Two Running Examples

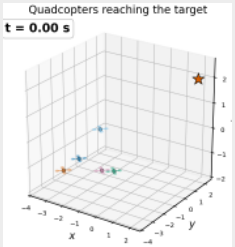
Ex 1: 100-D Shifted Target

- $d\mathbf{z} = 2\mathbf{u}\,ds + \sigma\,dW$, $\mathbf{z}(0) = \mathbf{0}$ Li et al. 2024
- $L = \|\mathbf{u}\|^2$; $g(\mathbf{x}) = \ln\left(\frac{1}{2}(1 + \|\mathbf{x} - \mathbf{x}_{\text{ref}}\|^2)\right)$
- $\mathbf{x}_{\text{ref}} = (3, \ldots, 3) \in \mathbb{R}^{100}$; high-$d$, target far from initial distribution



Ex 2: 12-D Quadcopter

- state (d=12): position, Euler angles Θ, linear velocity $\mathbf{v}$ & angular velocity $\boldsymbol{\omega}$
- drift $f = \left(\mathbf{v},\ \boldsymbol{\omega},\ \frac{F}{m}\mathbf{n}(\Theta) - g_0\mathbf{e}_3,\ \boldsymbol{\tau}\right)$: 1 thrust F + 3 torques $\boldsymbol{\tau}$
- $\mathbf{n}(\Theta) = \left(s_\psi s_\varphi + c_\psi s_\theta c_\varphi,\ -c_\psi s_\varphi + s_\psi s_\theta c_\varphi,\ c_\theta c_\varphi\right)$: attitude rotates the thrust (nonlinear)



The Hamilton–Jacobi–Bellman (HJB) Equation

The value function $\Phi(t, \mathbf{x})$ is the minimal expected cost-to-go from state $\mathbf{x}$ at time t.

Quick facts from optimal control theory (write $\nabla\Phi$ for the spatial gradient $\nabla_{\mathbf{x}}\Phi$)

1. feedback form (Pontryagin Maximum Principle, PMP) relates optimal control and value function

$$\mathbf{u}^*(s) \in \arg\min_{\mathbf{u}} \mathcal{H}(s, \mathbf{z}(s), \mathbf{p}(s), \mathbf{u}), \qquad \mathbf{p}(s) := \nabla\Phi(s, \mathbf{z}(s))$$

   with Hamiltonian $\mathcal{H}(s, \mathbf{z}, \mathbf{p}, \mathbf{u}) = \mathbf{p}^\top f(s, \mathbf{z}, \mathbf{u}) + L(s, \mathbf{z}, \mathbf{u})$; the costate $\mathbf{p}$ *is* the value gradient $\nabla\Phi$ evaluated along $\mathbf{z}(s)$

2. value function Φ satisfies HJB:

$$\partial_s\Phi(s, \mathbf{x}) + \tfrac{\sigma^2}{2}\Delta\Phi(s, \mathbf{x}) + \min_{\mathbf{u}} \mathcal{H}(s, \mathbf{x}, \nabla\Phi, \mathbf{u}) = 0$$

$$\Phi(T, \mathbf{x}) = g(\mathbf{x})$$

costate $\mathbf{p} = \nabla\Phi$ links control and value — every method below approximates it

Neural Network Solvers for High-Dimensional SOC

- **PINN-based HJB solvers** Sirignano and Spiliopoulos 2018; Raissi 2018
  learn Φ_θ on a fixed sampling distribution; loss $=$ HJB residual
- **Forward–Backward SDE (FB-SDE) / Deep BSDE** Han et al. 2018
  parametrize Φ_θ and $\sigma^\top \nabla \Phi_\theta$ along an uncontrolled forward rollout; loss $=$ backward-SDE (BSDE) terminal residual (Feynman–Kac)
- **Neural ODE/SDE (NeuralSOC)** Onken et al. 2022; Li et al. 2024
  learn Φ_θ; sample via PMP feedback; loss $=$ HJB residual $+$ terminal
- **Diffusion-Optimization (matching)** Domingo-Enrich et al. 2024; Domingo-Enrich et al. 2025; Blessing et al. 2025
  learn $\mathbf{u}_\theta$ directly; sample under current policy; loss $=$ matching / log-ratio

Forward–Backward SDE (FB-SDE) / Deep BSDE

Feynman–Kac semilinear representation Han et al. 2018 along an uncontrolled forward rollout $d\mathbf{z} = f(s, \mathbf{z}, \mathbf{0})\, ds + \sigma\, dW$, $\mathbf{z}(0) \sim \rho_0$, the value function satisfies the backward identity:

$$\Phi(s, \mathbf{z}(s)) \;=\; g(\mathbf{z}(T)) \;+\; \int_s^T \min_{\mathbf{u}} \big\{ (\nabla\Phi)^\top f(r, \mathbf{z}, \mathbf{u}) + L(r, \mathbf{z}, \mathbf{u}) \big\}\, dr \;-\; \int_s^T \big(\sigma^\top \nabla\Phi\big)^\top dW$$

equality holds pathwise (a.s. along the rollout) — Itô's lemma applied to $\Phi(s, \mathbf{z}(s))$

What is learned scalar $\Phi_{\boldsymbol{\theta}}(0, \mathbf{x})$ plus a network $\mathbf{v}_{\boldsymbol{\theta}}(s, \cdot) \approx \sigma^\top \nabla\Phi_{\boldsymbol{\theta}}(s, \cdot)$ at each Euler–Maruyama step $s_0, \ldots, s_N$ (so $\nabla\Phi_{\boldsymbol{\theta}} = \mathbf{v}_{\boldsymbol{\theta}}/\sigma$)

Training problem accumulate the value forward via the BSDE increment, then match $\widehat{\Phi}_N$ to the terminal g:

$$\widehat{\Phi}_{n+1} = \widehat{\Phi}_n - \min_{\mathbf{u}} \big\{ (\nabla\Phi_{\boldsymbol{\theta}})^\top f + L \big\} \Delta s + \mathbf{v}_{\boldsymbol{\theta}}(s_n, \mathbf{z}_n)^\top \Delta W_n, \quad \min_{\boldsymbol{\theta}} \mathbb{E} \big\| \widehat{\Phi}_N - g(\mathbf{z}(T)) \big\|^2$$

How it learns $\widehat{\Phi}_N$ depends on all the gradient networks $\mathbf{v}_{\boldsymbol{\theta}}(s_0, \cdot), \ldots, \mathbf{v}_{\boldsymbol{\theta}}(s_{N-1}, \cdot)$; the policy-free path is the convenience (mesh-free) *and* the later limitation (cannot probe a target it never visits)

mesh-free, scales to $d \approx 100$ — but only where the uncontrolled rollout visits

A 100-D Semilinear HJB Benchmark Han et al. 2018

Setup $d = 100$, $T = 1$, $\sigma = \sqrt{2}$

- $d\mathbf{z} = 2\mathbf{u}\,ds + \sigma\,dW$, $\mathbf{z}(0) = \mathbf{0}$
- $L = \|\mathbf{u}\|^2$
- terminal $g(\mathbf{x}) = \ln\big((1 + \|\mathbf{x}\|^2)/2\big)$

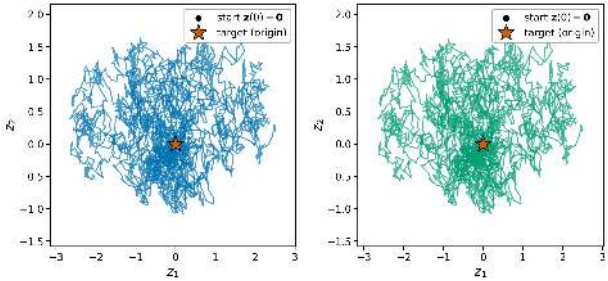
HJB
$$\partial_s \Phi + \Delta\Phi - \|\nabla\Phi\|^2 = 0$$

Reference Feynman–Kac, closed form

$$\Phi(s, \mathbf{x}) = -\ln \mathbb{E}\Big[\exp\big(-g(\mathbf{x} + \sqrt{2}\, W_{T-s})\big)\Big]$$

Baselines at $d = 100$ Deep BSDE Han et al. 2018 matches the reference via uncontrolled forward rollouts; DGM / PINN residual solvers Sirignano and Spiliopoulos 2018 via sampled collocation

2D projection of 100-D trajectories (NeuralSOC)



uncontrolled (left): $d\mathbf{z} = \sigma\,dW$

PMP controlled (right): $d\mathbf{z} = -\nabla\Phi_\theta\,ds + \sigma\,dW$

learned control tiny: mean $\|\mathbf{u}\| \approx 0.14$; rollout gap $< 1.2\%$ of the $\|\mathbf{z}_T\| \approx 14.8$ noise spread

the target sits at the origin where Brownian motion already goes — Deep BSDE / PINN learn near-zero control, so controlled and uncontrolled rollouts coincide

NeuralSOC: A Continuous-time Learning Problem

Three ingredients Onken et al. 2022; Li et al. 2024 extension of the method of characteristics for HJB (cf. Onken et al. deterministic NeuralOC):

- **PMP-driven sampling**: roll out under the feedback control $\mathbf{u}_\theta$
- **HJB** residual along the sampled trajectory penalizes the value function
- **discretize-then-optimize (DTO)** through the neural SDE by reverse-mode AD

Objective given dynamics f and costs L, g, learn Φ_θ (feedback $\mathbf{u}_\theta = \arg\min_{\mathbf{u}} \mathcal{H}$):

$$\min_{\boldsymbol{\theta}} \mathbb{E}_{\mathbf{x}\sim\rho_0}\big[\,\alpha\big(\ell(T) + g(\mathbf{z}(T))\big) + \beta_1\, c_{\mathrm{HJ}}(T) + \beta_2\,\big|\Phi_{\boldsymbol{\theta}}(T,\mathbf{z}(T)) - g(\mathbf{z}(T))\big|\,\big]$$

$$\text{s.t. } d\begin{pmatrix}\mathbf{z}\\ \ell\\ c_{\mathrm{HJ}}\end{pmatrix} = \begin{pmatrix} f(\mathbf{z},\mathbf{u}_{\boldsymbol{\theta}})\\ L(s,\mathbf{z},\mathbf{u}_{\boldsymbol{\theta}})\\ \big|\partial_s\Phi_{\boldsymbol{\theta}} + \mathcal{H}^*(s,\mathbf{z},\nabla\Phi_{\boldsymbol{\theta}}) + \frac{\sigma^2}{2}\Delta\Phi_{\boldsymbol{\theta}}\big|\end{pmatrix} ds + \begin{pmatrix}\sigma\, dW\\ 0\\ 0\end{pmatrix},$$

$\mathbf{z}(0) = \mathbf{x}$, $\ell(0) = c_{\mathrm{HJ}}(0) = 0$

PMP feedback Both examples are control-affine with $L = \frac{1}{2}\|\mathbf{u}\|^2$, so the argmin is explicit: $\mathbf{u}_\theta = -B(\mathbf{z})^\top \nabla\Phi_\theta$ with $B(\mathbf{z}) = \partial_{\mathbf{u}} f$. Ex 1 ($f{=}2\mathbf{u}$): $\mathbf{u}_\theta = -\nabla\Phi_\theta$; quadcopter: closed-form thrust + torque channels.

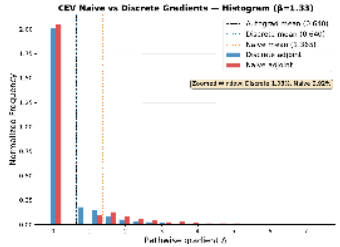
NeuralSOC = HJB residual + PMP-driven forward roll-out

Differentiating Through SDEs

Setup Leburu et al. 2026 gradient of $J(\mathbf{x}_0) = \mathbb{E}[\Phi(\mathbf{z}_T)]$ for an Itô SDE — the building block behind NeuralSOC's DTO adjoint.

Two tales **DTO**: AD through Euler–Maruyama (Itô) / Heun (Stratonovich) steps; **OTD**: continuous adjoint SDE, then discretize.

CEV model, $\beta = 1.33$



discrete adjoint $= 0.640$ (correct); naive $= 1.365$ (biased)

Leburu–Nurbekyan–R. 2026 Leburu et al. 2026 (tutorial leveraging Kidger 2022)

- DTO is *exact* for the discrete objective in both Itô and Stratonovich
- naive OTD on Itô is biased when $\partial_{\mathbf{x}} g$ depends on $\mathbf{x}$ — fix: Itô→Stratonovich (Øksendal)

DTO via AD is exact — in both Itô and Stratonovich

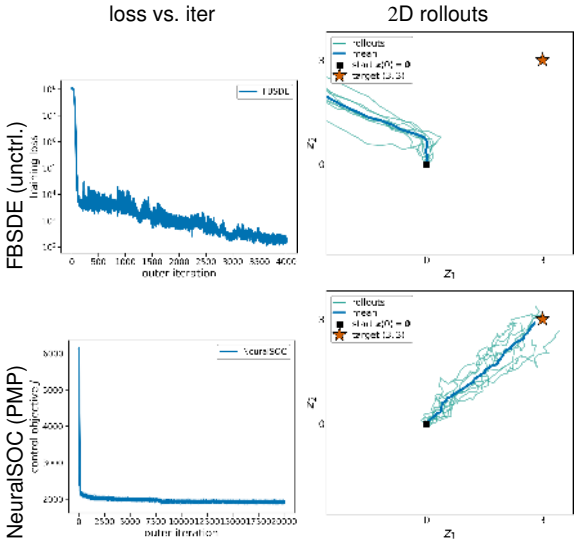
100-D Stress Test: Shifted Target

Same benchmark, two changes

$d = 100$, $T = 1$

- $d\mathbf{z} = 2\mathbf{u}\,ds + \sigma\,dW$, $\mathbf{z}(0) = \mathbf{0}$, $L = \|\mathbf{u}\|^2$
- $g(\mathbf{x}) = \ln\big((1 + \|\mathbf{x} - \mathbf{x}_{\text{ref}}\|^2)/2\big)$, $\mathbf{x}_{\text{ref}} = (3, \ldots, 3)$
  target moved far off the origin
- noise $\sigma = 2\sqrt{2}/5$: lowered so the shift dominates the Brownian spread (else $\sigma\sqrt{T}\sqrt{d} \approx 14$ washes it out)

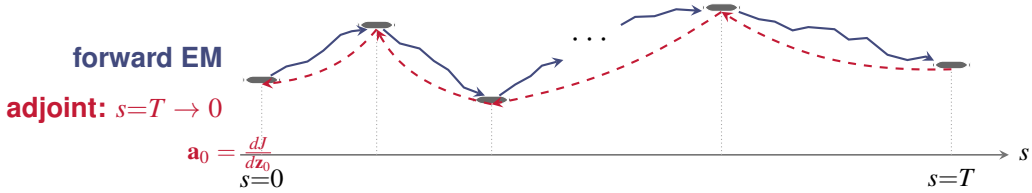
Why it is hard Brownian sampling never reaches $\mathbf{x}_{\text{ref}}$, so the BSDE loss cannot inform Φ along the optimal path Li et al. 2024.



uncontrolled FBSDE misses; PMP sampling reaches ⋆ (same MLP architecture)

loss decay ≠ success: FBSDE never reaches $\mathbf{x}_{\text{ref}}$; PMP sampling does

Computational Cost of NeuralSOC



Discrete pathwise adjoint differentiate $J = g(\mathbf{z}_N)$ w.r.t. $\mathbf{z}_0$ (drop $\mathbf{u}$; ΔW_n independent of $\mathbf{z}_0$):

$$\mathbf{z}_{n+1} = \mathbf{z}_n + f(\mathbf{z}_n)\,\Delta s + \sigma\,\Delta W_n, \qquad \mathbf{a}_N = \nabla g(\mathbf{z}_N), \qquad \mathbf{a}_n = \big(I + \Delta s\,\partial_{\mathbf{z}} f(\mathbf{z}_n)\big)^\top \mathbf{a}_{n+1}$$

$$\frac{dJ}{d\mathbf{z}_0} = \mathbf{a}_0 = M_0^\top M_1^\top \cdots M_{N-1}^\top \nabla g(\mathbf{z}_N), \qquad M_n := I + \Delta s\,\partial_{\mathbf{z}} f(\mathbf{z}_n)$$

- one backward sweep reuses the stored forward states — cost $\approx$ one forward pass, memory $O(N\,d_{\text{act}})$

classic DTO: store the forward rollout, sweep the pathwise adjoint backward

From Full Adjoint to Lean Adjoint

Setting adjoint method on NeuralSOC's controlled SDE $d\mathbf{z} = f(\mathbf{z}, \mathbf{u}_\theta)\, ds + \sigma\, dW$; which terms can we drop?

Full closed-loop adjoint (BSDE)

$$-d\mathbf{p} = \big[\partial_{\mathbf{z}} f^\top \mathbf{p} + \partial_{\mathbf{z}} L \;+\; \underbrace{\partial_{\mathbf{z}} \mathbf{u}_{\boldsymbol{\theta}}^\top\, \partial_{\mathbf{u}} \mathcal{H}}_{= D(s,\mathbf{z})}\big]\, ds - \mathbf{q}\, dW, \quad \mathbf{p}(T) = \nabla g(\mathbf{z}(T))$$

with PMP residual $\partial_{\mathbf{u}} \mathcal{H} = \partial_{\mathbf{u}} f^\top \mathbf{p} + \partial_{\mathbf{u}} L$.

Drop D $\partial_{\mathbf{z}} \mathbf{u}_{\boldsymbol{\theta}}$ is the policy input-Jacobian at every EM step. Dropping it leaves a plain backward sweep, with lean costate $\tilde{\mathbf{p}}$ and lean target $\tilde{\mathbf{a}}$:

$$-d\tilde{\mathbf{p}} = \big[\partial_{\mathbf{z}} f^\top \tilde{\mathbf{p}} + \partial_{\mathbf{z}} L\big]\, ds - \mathbf{q}\, dW, \;\; \tilde{\mathbf{p}}(T) = \nabla g(\mathbf{z}(T)), \;\; \tilde{\mathbf{a}} = -\sigma^\top \tilde{\mathbf{p}}$$

At the optimum, lean = full

- PMP: $\partial_{\mathbf{u}} \mathcal{H}^\star \equiv 0 \Rightarrow D(\mathbf{z}^\star) \equiv 0$; $\mathbf{u}^\star$ critical point of $\mathcal{L}_{\mathrm{AM}}$ Domingo-Enrich et al. 2025; off-policy $D \neq 0$, vanishes as $\mathbf{u}_{\boldsymbol{\theta}} \to \mathbf{u}^\star$

lean is biased off-policy; the bias vanishes as $\mathbf{u}_\theta \to \mathbf{u}^\star$

Adjoint Matching: The Algorithm

Adjoint Matching Domingo-Enrich et al. 2025; Domingo-Enrich and Han 2026
(one outer iteration; repeat to convergence)

(i) **Forward simulate** under frozen $\bar{\mathbf{u}}_{\boldsymbol{\theta}} = \mathrm{sg}(\mathbf{u}_{\boldsymbol{\theta}})$: $d\mathbf{z} = f(\mathbf{z}, \bar{\mathbf{u}}_{\boldsymbol{\theta}})\, ds + \sigma\, dW$, $\mathbf{z}(0) \sim \rho_0$

(ii) **Lean adjoint BSDE** backward: $-d\tilde{\mathbf{p}} = \big[\partial_{\mathbf{z}} f^\top \tilde{\mathbf{p}} + \partial_{\mathbf{z}} L\big] ds - \mathbf{q}\, dW$, $\tilde{\mathbf{p}}(T) = \nabla g$, $\tilde{\mathbf{a}} = -\sigma^\top \tilde{\mathbf{p}}$

(iii) **Regress** $\mathbf{u}_{\boldsymbol{\theta}}$ onto $\tilde{\mathbf{a}}$: $\min_{\boldsymbol{\theta}}\ \mathbb{E}\int_0^T \frac{1}{2}\, \|\mathbf{u}_{\boldsymbol{\theta}}(s, \mathbf{z}(s)) - \tilde{\mathbf{a}}(s)\|^2\, ds$

Why this loss? when the regression residual *is* the PMP residual ($L = \frac{1}{2}\|\mathbf{u}\|^2$, $\partial_{\mathbf{u}} f = \sigma$), its gradient *is* the policy gradient:

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}_{\mathrm{AM}} = \mathbb{E}\int_0^T (\partial_{\boldsymbol{\theta}} \mathbf{u}_{\boldsymbol{\theta}})^\top \big(\mathbf{u}_{\boldsymbol{\theta}} - \tilde{\mathbf{a}}\big)\, ds = \mathbb{E}\int_0^T (\partial_{\boldsymbol{\theta}} \mathbf{u}_{\boldsymbol{\theta}})^\top \underbrace{\big(\partial_{\mathbf{u}} L + \sigma^\top \tilde{\mathbf{p}}\big)}_{=\partial_{\mathbf{u}} \mathcal{H}}\, ds = \nabla_{\boldsymbol{\theta}} J[\mathbf{u}_{\boldsymbol{\theta}}]$$

convex regression, **no backprop through the SDE**; **weight-free** = no importance / Girsanov reweighting (unlike SOCM log-ratio losses), a plain L_2 fit onto the Gauss–Newton / PMP-residual direction; minimized with Adam.

the regression gradient *is* the policy gradient — no backprop through the SDE

Lean AM on the 100-D Shifted Target ($\partial_{\mathbf{z}} f \equiv 0$)

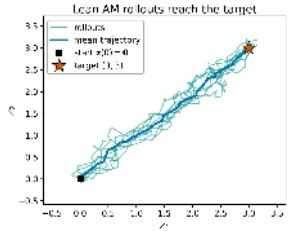
Controlled comparison (our code)

- Ex 1: $f = 2\mathbf{u} \Rightarrow \partial_{\mathbf{z}} f \equiv 0$ — dropping D costs nothing structurally
- NeuralSOC: value Φ-net (OT-Flow, $20{,}184$ p.) *or* MLP control net ($628{,}836$ p.); lean AM uses the same MLP
- matched 2×10^4 iters, batch 64 (representative run) — final J: lean AM $884 \approx$ NeuralSOC-MLP $908 <$ Φ-net 1956

Why this works $\partial_{\mathbf{z}} f \equiv 0$: lean AM and NeuralSOC on the same MLP converge together — the lean drop is free.

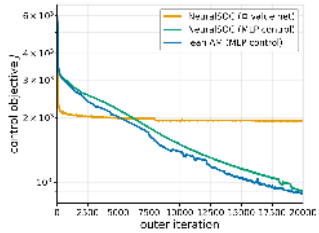
lean AM matches NeuralSOC where $\partial_{\mathbf{z}} f \equiv 0$ — matching's home regime

Lean AM trajectories



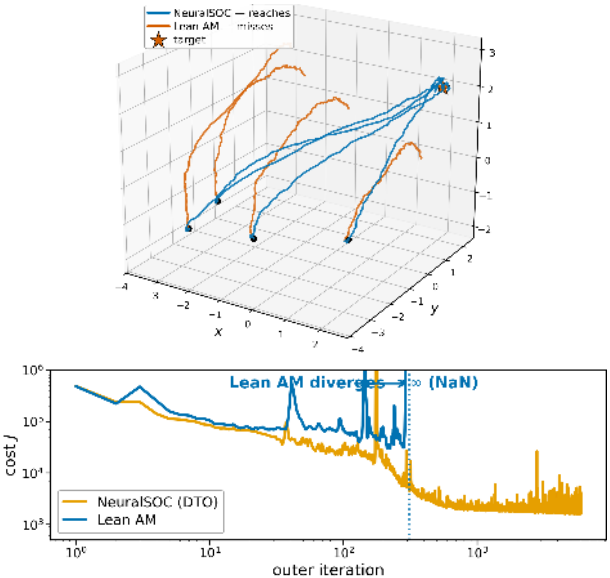
2D projection, start $\mathbf{0}$, target $\star$ at $(3,3)$

Control objective vs. outer iter



all converge (E=0); the two MLP variants track each other

Lean AM on the 12-D Quadcopter (expansive $\partial_{\mathbf{z}} f$)



Experiment control-affine drift f (see Two Examples), wind $\sigma{=}0.2$; attitude tilts $\mathbf{n}(\Theta) \Rightarrow \partial_{\mathbf{z}} f$ expansive Onken et al. 2022

- architecture controlled: both use an MLP policy — NeuralSOC (MLP control, DTO adjoint) *vs.* lean AM (lean adjoint); same T, noise, seeds

Interpretation

- NeuralSOC reaches across $\sigma \in \{0, 0.1, 0.3, 1.0\}$; lean AM $\to \infty$/NaN
- closed-loop $E > 0$; bias amplifies $\sim e^E$ — the lean-vs-full adjoint is the cause

DTO converges, lean AM diverges — same problem

The Lean-Drop Bias and the Expansion Integral E

Setting linearize around $(\mathbf{z}^\star, \mathbf{u}^\star)$ ($\sigma_W{=}0$, mean perturbation), $B := \partial_{\mathbf{u}} f$; perturb $\mathbf{u}_{\boldsymbol{\theta}} = \mathbf{u}^\star + \delta\mathbf{u}$ and track one outer iteration $\delta\mathbf{u} \mapsto \delta\mathbf{u}^+$.

1. Forward variational ODE along $\mathbf{z}^\star$:

$$\tfrac{d}{ds}\delta\mathbf{z} = \partial_{\mathbf{z}} f\, \delta\mathbf{z} + B\, \delta\mathbf{u}, \qquad \delta\mathbf{z}(0) = 0$$

2. Adjoint linearization linearizing the lean adjoint about $\tilde{\mathbf{p}}^\star$: lean generator $\partial_{\mathbf{z}} f^\top$ plus a curvature source ($\partial_{\mathbf{z}} L$, $\partial_{\mathbf{z}} f^\top \mathbf{p}$ differentiated):

$$-\tfrac{d}{ds}\delta\tilde{\mathbf{p}} = \partial_{\mathbf{z}} f^\top \delta\tilde{\mathbf{p}} + \underbrace{(\partial_{\mathbf{zz}} \mathcal{H}^\star)\, \delta\mathbf{z}}_{\text{curvature source}}, \quad \delta\tilde{\mathbf{p}}(T) = \nabla^2 g\, \delta\mathbf{z}(T)$$

3. Regression update $\mathbf{u}_{\boldsymbol{\theta}}^+$ regresses onto $\mathbf{a} = -B^\top \tilde{\mathbf{p}}$ at fixed $\mathbf{z} \Rightarrow \delta\mathbf{u}^+ \propto -B^\top \delta\tilde{\mathbf{p}}$.

Composing $\delta\mathbf{u} \to \delta\mathbf{z} \to \delta\tilde{\mathbf{p}} \to \delta\mathbf{u}^+$ is a linear map whose gain is the state-transition matrix (matrizant) of $\partial_{\mathbf{z}} f$ along $\mathbf{z}^\star$:

$$\|\delta\mathbf{u}^+\| \;\le\; \exp(E)\, \|\delta\mathbf{u}\|, \qquad \boxed{E \;=\; \int_0^T \partial_{\mathbf{z}} f\big(\mathbf{z}^\star(s), \mathbf{u}^\star(s)\big)\, ds}$$

$E > 0$: lean iteration diverges before the on-policy bias can vanish

Why Common Fixes Don't Help in This Example

Common diagnosis E depends on $\partial_{\mathbf{z}} f$ *on the optimal trajectory* — it is a property of the problem, not of the optimizer.

- **Noise** σ leaves $\partial_{\mathbf{z}} f$ on $\mathbf{z}^\star$ unchanged $\Rightarrow$ E unchanged; on the quadcopter $\hat{E} > 0$ for every σ we tried.
- **Trust region / regularizers** select *which* $\mathbf{z}^\star, \mathbf{u}^\star$ the iteration converges to — but not the sign of E along it.
- **More inner iterations** drive $\mathbf{u}_{\boldsymbol{\theta}}$ harder onto the *wrong* target (dropped-D adjoint); amplify error.

What is needed better matching algorithms with convergence theory *beyond* LQG: augmented-Hamiltonian / extended-MSA Li et al. 2017 and trust-region matching Blessing et al. 2025.

expansion is a problem property, not an optimizer one — better matching algorithms are needed

Implicit Controls: The Lean Idea, Jacobian-Free

When the feedback is implicit if L is non-quadratic or f is *not* control-affine, $\mathbf{u}^\star = \arg\min_{\mathbf{u}} \mathcal{H}$ has no closed form — $\mathbf{u}$ is pinned only by $\partial_{\mathbf{u}}\mathcal{H} = 0$.

Jacobian-Free Backpropagation Gelphman et al. 2025 keep the value net $\Phi_{\boldsymbol{\theta}}$; make $\mathbf{u}_{\boldsymbol{\theta}}$ the fixed point $\mathbf{u} = T(\mathbf{u})$ of Hamiltonian gradient ascent, and train end-to-end while *dropping* the inverse-Jacobian $(I - \partial_{\mathbf{u}} T)^{-1} \to I$ — still a descent direction under contractivity (Thm 1).

The connection

- same drop-the-Jacobian idea as the **lean adjoint** (which drops the policy Jacobian D) — here it buys *implicit* controls
- $\mathcal{O}(m^2)$ not $\mathcal{O}(m^3)$; reaches 80-D state / 40-D control where plain AD runs out of memory
- *deterministic* so far — the stochastic / matching extension is open

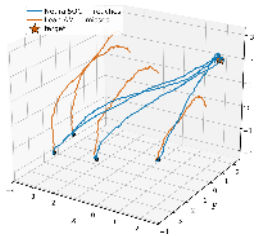
the Jacobian-free idea also unlocks implicit, high-dim controls — stochastic is next

Σ: Neural SOC — where matching wins, where it doesn't

- **NeuralSOC**: learn $\Phi_{\boldsymbol{\theta}}$; HJB residual + PMP sampling; DTO adjoint
- **Adjoint Matching**: drop $D \to$ **lean** adjoint; **weight-free** regression; gradient = policy gradient at $\mathbf{u}^\star$
- **Limits**: *empirically*, $E = \int_0^T \partial_{\mathbf{z}} f \, ds \leq 0$ is *sufficient* (not necessary) to contract; the expansive quadcopter ($E > 0$) diverges

What we learned

- *in our experiments*, matching works when f is contractive ($E \leq 0$) — the regime it was built for (incl. gen-AI fine-tuning)
- lean AM = continuous-time MSA Domingo-Enrich and Han 2026 (deterministic limit; deep-learning ext. Li et al. 2017)



better algorithms are needed to get the advantages of both

Joint with K. Keegan, R. Leburu, X. Li,

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Questions?

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