Emory Math 788, Stacks
TuTh 11:30 - 12:45

All assignments
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1 Functors and Yoneda's lemma

1. Prove a slightly more general version of Yoneda's Lemma – let C be a category, $X \in C$ an object, h_X the functor $h_X(T) = Hom_C(T, X)$, and $F\colon C^{op} \to \mathbf{Sets}$ a functor. Then $Hom(h_X, F) \cong F(X)$.

2. Play the game "find the representing object" whenever you get the chance. Determine if the following functors are representable. If they are, find the representing object.

   (a) The functor $\mathrm{Top}^{\mathrm{op}} \to \mathbf{Sets}$ taking a topological space X to the set of open subsets of X.

   (b) The functor $\mathrm{Top}^{\mathrm{op}} \to \mathbf{Sets}$ taking a topological space X to the set of closed subsets of X.

   (c) The functor $\mathrm{Top}^{\mathrm{op}} \to \mathbf{Sets}$ taking a topological space X to the open subsets of X whose complement is also open.

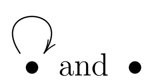
   (d) The functor $\mathbf{HausTop}^{\mathrm{op}} \to \mathbf{Sets}$ taking a topological space X to the set of open subsets of X. (**HausTop** is the category of Hausdorff topological spaces.)

   (e) The functor $\mathbb{A}^n - \{(0, \ldots, 0)\}\colon (\mathbf{Sch}^{\mathrm{op}}) \to (\mathbf{Sets})$ taking a scheme T to $\{(f_1, \ldots, f_n) \in \mathcal{O}_T(T)^n | \text{the } f_i \text{ do not all simultaneously vanish at the origin}\}$.

   (f) The functor $(\mathbb{A}^n - \{(0, \ldots, 0)\})/\mathbb{G}_m\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}$ taking a scheme T to $(\mathbb{A}^n - \{(0, \ldots, 0)\})(T)/\sim$, where $\sim$ is the equivalence relation $(f_1, \ldots, f_n) \sim (f_1', \ldots, f_n')$ if there is a unit $u \in \mathcal{O}_T(T)$ such that $f_i' = uf_i$ for each i.

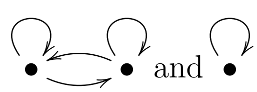
3. Give 3 examples of equivalences of categories that are not isomorphisms of categories.

4. Are the categories



   equivalent?

5. Are the categories



   equivalent?

2 Limits, colimits, group objects

1. Let C be a category and let $h\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Sets})$ be the Yoneda embedding. Show that for any arrows $X \to Y$ and $Z \to Y$ in C, there is a natural isomorphism

$$h_{X \times_Y Z} \to h_X \times_{h_Y} h_Z$$

of functors, where $h_X \times_{h_Y} h_Z$ is the functor

$$h_X \times_{h_Y} h_Z \colon W \mapsto h_X(W) \times_{h_Y(W)} h_Z(W).$$

2. Let G be an object of a category C. Show that the functor of points

$$h_G \colon C^{\mathrm{op}} \to \mathbf{Sets}$$

factors through the forgetful functor from groups

$$\begin{array}{ccc} & & \mathbf{Groups} \\ & \nearrow & \downarrow \\ h_G\colon C^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \end{array}$$

if and only if G is a group object of C.

3. Let $\star$ be one of $\operatorname{Spec} k$ (with k a field) or $\operatorname{Spec} \mathbb{Z}$. Let

$$G = \coprod_{g \in \mathbb{Z}/2\mathbb{Z}} \star$$

be the group object corresponding to $\mathbb{Z}/2\mathbb{Z}$. Work out explictily the group structure. In other words, work out the maps in terms of rings, and show that G represents the sheafification of the functor

$$\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Groups},\ W \mapsto \mathbb{Z}/2\mathbb{Z}$$

4. (a) Let $n \geq 1$ be an integer and let

$$\mathrm{GL}_n \colon (\mathbf{Sch})^{\mathrm{op}} \to \mathbf{Sets}$$

be the functor sending a scheme Y to the set $\mathrm{GL}_n(\Gamma(Y, \mathcal{O}_Y))$. Prove that GL_n is a representable functor.

   (b) Let X represent the functor GL_n. Prove that the group structures on the sets $\mathrm{GL}_n(\Gamma(Y, \mathcal{O}_Y))$ induce the structure of a group scheme on X. (I.e. use the previous exercise, noting that there is one detail to check.)

5. Give an example of a category C, a subcategory C', and a diagram $D\colon I \to C'$ such that the limit (or colimit, your choice) in C' is not the limit in C.

6. Use the functor of points to define a map $\mathbb{A}^1 \to \mathbb{A}^1$ given by the formula $z \mapsto z^2$. Compare this with how one would define such a map with locally ringed spaces.

7. **Monomorphisms**.

   (a) Show that a morphism $X \to Y$ is a monomorphism if and only if for every $T \in C$, the map of sets $X(T) \to Y(T)$ is injective.

   (b) Show that a map of schemes which is injective topologically may not be a monomorphism.

   (c) Show that a map of schemes which is surjective topologically may not be an epimorphism.

8. **Limits and colimits of sets**. Let $D \colon I \to Set$ be a diagram. Show that

   (a) $\varprojlim D = \{(x_i) \in \prod_{i \in I} D(i) \text{ s.t. } \forall i, j \in I, \forall \phi \in \mathrm{Hom}(i,j), D(\phi)(x_i) = x_j\}$.

   (b) $\varinjlim D = \coprod D(i)/\sim$, where $\sim$ is given by $\forall i, j \in I, \forall \phi \in \mathrm{Mor}(i,j), x_i \sim D(\phi)(x_i)$.

9. Consider the diagram

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ Y & \longrightarrow & Z & \xrightarrow{h} & W \end{array}$$

   Suppose that h is a monomorphism. Show that $X \times_Z Y \to X \times_W Y$ is an isomorphism.

3 Étale morphisms and sheaves

1. Think of lots of examples and non examples of étale morphisms, and work out the details as explicitly as you can.

2. Show that the sheaf axiom of Hartshorne is equivalent to our sheaf axiom.

3. Let C be a category and let $D\colon I \to \mathrm{Fun}(C^{op}, \mathbf{Sets})$ be a diagram.

   (a) Show that $\varprojlim D$ is the functor $X \mapsto \varprojlim D(i)(X)$.

   (b) Show that $\varinjlim D$ is the functor $X \mapsto \varinjlim D(i)(X)$.

4. Let C be a category and let $D\colon I \to \widetilde{\mathbf{Sch}}$ be a diagram, where $\widetilde{\mathbf{Sch}}$ is the subcategory of sheaves in $\mathrm{Fun}(C^{op}, \mathbf{Sets})$.

   (a) Show that $\varprojlim D$ (in the category of sheaves) is the functor $X \mapsto \varprojlim D(i)(X)$.

   (b) Show that $\varinjlim D$ (in the category of sheaves) is the *sheafication* of the functor $X \mapsto \varinjlim D(i)(X)$.

5. Let F be a sheaf, F' be a presheaf, and $f\colon F' \to F$ be an injection (as presheaves). Show that the sheafication of F' is isomorphic to F if and only if every section of F is locally in the image of f.

4 Sheaves and topoi

1. Let T be a topos. Define a topology τ on T by declaring a morphism $F' \to F$ to be a covering if it is a surjection of sheaves. (This is called the canonical topology.)

   (a) Show that the canonical topology is a topology.

   (b) Show that the associated topos is equivalent to T.

   (c) Show that the canonical topology is the largest topology preserving the topos; i.e., prove that if $p\colon X' \to X$ is a morphism and if every sheaf F in T satisfies the sheaf axiom with respect to p, then p is a surjection of sheaves in T.

2. Define a site C to be *subcanonical* if for every object $X \in C$, h_X is a sheaf. (So, for instance, the etale site of a scheme is subcanonical.) Give an example of a site C which is not subcanonical.

3. Let $X' \to X$ be an étale surjection of affine schemes and let Y be a scheme. Show that the 'first sheaf axiom' is satisfied, i.e., that the map

$$h_Y(X) \to h_Y(X')$$

   is injective. (Assume that we already know this for Y affine.)

4. Let $F\colon C \to D$ and $G\colon D \to C$ be a pair of functors. We say that F is *left adjoint* to G if there is an isomorphism of (bi)-functors

$$\operatorname{Hom}(F(-), -) \cong \operatorname{Hom}(-, G(-)).$$

   Let C' be a category and let C be a subcategory. Denote by i the inclusion $C \to C'$, and suppose that i has a left adjoint $a\colon C' \to C$. (For instance, a could be sheafification.) Let $D\colon I \to C$ be a diagram.

   Prove that $a(\varinjlim i \circ D) = \varinjlim D$

5 Comma Category and adjunction

1. Let C be a site, and let X be an object in C. Recall that the comma category C/X inherits the structure of a site. Assume that C is subcanonical (which means that for every $X \in C$, h_X is a sheaf).

   (a) Show that there is an equivalence of categories between $Sh(C/X)$ and $Sh(C)/h_X$.

   (b) Show that $j^*\colon Sh(C) \to Sh(C)/h_X$, given by $F \mapsto (F \times h_X \xrightarrow{p_2} h_X)$ commutes with finite limits and has a right adjoint j_*. (Describe j_* explicitly.)

6 2-categories

1. Let C and D be categories and calculate very explicitly the 2-limit of the diagram

$$C \rightrightarrows\!\!\!\rightarrow D \tag{6.0.1}$$

2. Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of categories is a monomorphism (i.e., fully faithful) iff the diagonal is an equivalence.

3. Let C be a site and let $X' \to X$ be a covering in C. Show that the category $Sh(X' \to X)$ is equivalent to the 2-limit of the diagram

$$\widetilde{X'} \rightrightarrows \widetilde{X''} \rightrightarrows\!\!\!\rightarrow \widetilde{X'''} \tag{6.0.2}$$

4. Let $D \to C$ be a fibred category. Show that the maps $D(V) \to D(U)$ defined in class are functors, and that, for a pair of maps $U \to V \to W$, the composition of the functors $D(W) \to D(V) \to D(U)$ is isomorphic to $D(W) \to D(U)$.

5. Prove the 2-Yoneda lemma.

6. Let $\mathcal{X} \to$ **Sch** be a fibered category. Show that if the fibers are setoids, then $\mathcal{X}$ is equivalent to $\mathbf{Sch}_{/F}$ for some functor F. Show that in this case F is a sheaf iff $\mathcal{X} \to$ **Sch** is a stack.

7 Additional problems to proofread and incorporate

1. Let C be a category. F be a functor. Show that the diagonal is representable iff every map $X \to F$, with $X \in C$, is representable.
2. Show that h_X and "$X(R)$ are isomorphic functors.
3. Show that the sheaf axiom of Hartshorne is equivalent to our sheaf axiom using coproducts.
4. Show that the "locally isomorphic Zariski topology" and the usual Zariski topology give the same topos.
5. Disjoin unions vs open sets.
6. Verify that p^{-1} and p_* of a sheaf is a sheaf.
7. Let F be a presheaf and let $p\colon X' \to X$ and $q\colon Y' \to Y$ be two morphisms such F satisfies the sheaf axiom with respect to every base change of p and q. Prove that F satisfies the sheaf with respect to $p \times q\colon X' \times Y' \to X \to Y$.
8. Adjoint functor is fully faithful if and only if the unit (or counit) is an isomorphism. (Hint: Yoneda's lemma.)
9. **Diagonal**.
   (a) Prove that that the diagonal is an isomorphism if and only if f is etale.
10. A functor on X_{zar} with an open cover by schemes is a scheme.
11. Show, explicitly, that the map $[G/G]^{ps} \to \star$ is an equivalence of categories.
12. Stackify the stack $B_{\mathbb{G}_m}$ by hand.
13. Let $R \to X \times X$ be an equivalence relation. Show that the diagonal $\Delta\colon X \to X \times X$..
14. Let C be a site. Let $T \in C$, and let $X, Y \in C/T$. Define a functor $\underline{\mathrm{Hom}}(X, Y)$ by

$$T' \mapsto \mathrm{Hom}_{T'}(X \times_T T', Y \times_T T')$$

   (a) Show that $\underline{\mathrm{Hom}}(X, Y)$ is a sheaf.
   (b) Let $C = \mathrm{Aff}$ be the category of affine schemes with the Zariski topology and let $X = Y = \mathbb{A}^1$. Show that $\underline{\mathrm{Hom}}(\mathbb{A}^1, \mathbb{A}^1)$ is not representable by an affine scheme.
   (c) Let $C = \mathbf{Sch}$ with the Zariski topology and let $X = Y = \mathbb{A}^1$. Show that $\underline{\mathrm{Hom}}(\mathbb{A}^1, \mathbb{A}^1)$ is not representable by a scheme.
   (d) Let $C = \mathbf{Sch}$ with the étale topology and let $X = Y = \mathbb{A}^1$. Show that $\underline{\mathrm{Hom}}(\mathbb{A}^1, \mathbb{A}^1)$ is not representable by an algebraic space.