MATH 250 HANDOUT 12 - COMPOSITIONS AND INJECTIVITY/SURJECTIVITY

(1) Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function $f(x) = \frac{1}{1+x^2}$ and let $g\colon \mathbf{R} \to \mathbf{R}$ be the function $g(x) = e^x$.

  (a) What is $g \circ f(0)$?
  (b) What is $f \circ g(0)$?
  (c) Give a formula for $f \circ g$ and $g \circ f$.

(2) Let $f\colon \mathbf{R} \to \mathbf{Z}$ be the function $f(x) = \lfloor x \rfloor$ (i.e., round x down to the nearest integer) and let $g\colon \mathbf{Z} \to \mathbf{Z}$ be the function $g(n) =$ 'the number of distinct prime factors of n'. (So $g(0) = g(1) = 0$, $g(4) = 1$, $g(6) = 2$))

  (a) What is $g \circ f(\pi)$?
  (b) What is $g \circ f(91.1023124)$?
  (c) Is $g \circ f$ injective? Surjective?

(3) Let $f\colon \mathbf{Z} \to P(\mathbf{Z})$ be the function $f(n) = \{n\}$ and let $g\colon P(\mathbf{Z}) \to P(\mathbf{Z})$ be the function $g(S) = S \cap \{1\}$.

  (a) What is $g \circ f(0)$?
  (b) What is $g \circ f(1)$?
  (c) Give a formula for $g \circ f$.

(4) Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove or disprove each of the following:
  (a) If f and g are injections, then $g \circ f$ is an injection.
  (b) If f and g are surjections, then $g \circ f$ is a surjection.
  (c) If f and g are bijections, then $g \circ f$ is a bijection.
  (d) If $g \circ f$ is an injection, then f and g are injections.
  (e) If $g \circ f$ is a surjection, then f and g are surjections.
  (f) If $g \circ f$ is a bijection, then f and g are bijections.
  (g) If $g \circ f$ is an injection, then f is an injection.
  (h) (HW) If $g \circ f$ is an injection, then g is an injection.
  (i) (HW) If $g \circ f$ is a surjection, then f is a surjection.
  (j) (HW) If $g \circ f$ is a surjection, then g is a surjection.
  (k) If $g \circ f$ is a bijection, then f is a bijection.
  (l) If $g \circ f$ is a bijection, then g is a bijection.
  (m) If $g \circ f$ is an injection and g is a bijection, then f is an injection.

(5) Let $f\colon A \to B$ be a function. Let $X, Y \subset A$ and let $W, V \subseteq B$. Each of the following statements are false as stated. Which become true if we assume that f is injective or surjective? In each case (f is injective, or f is surjective), prove your assertion or give a counterexample.
  (a) $X \subseteq Y \Leftarrow f(X) \subseteq f(Y)$.
  (b) (HW)$f(X \cap Y) \subseteq f(X) \cap f(Y)$.
  (c) $f(X) - f(Y) \subseteq f(X - Y)$.
  (d) $X \subseteq f^{-1}(f(X))$.
  (e) $W \subseteq f(f^{-1}(W))$.
  (f) $V \subseteq W \Leftarrow f^{-1}(V) \subseteq f^{-1}(W)$.