MATH 250 HANDOUT 1 - LOGIC

A **statement** is a sentence for which 'true or false' is meaningful.

1. Which of these are **statements**?

(1) Today it is raining.
(2) What is your name?
(3) Every student in this class is a math major.
(4) $2+2=5$.
(5) $x+1>0$.
(6) $x^2+1>0$.
(7) If it is raining, then I will wear my raincoat.
(8) Give me that.
(9) This sentence is false.
(10) If x is a real number, then $x^2>0$.

2. Which of these are true?

(1) (T or F) Every student in this class is a math major and a human being.
(2) (T or F) Every student in this class is a math major or a human being.
(3) (T or F) $2+2=5$ or $1>0$.
(4) (T or F) If x is a real number, then $x^2 \geq 0$.
(5) (T or F) If x is a complex number, then $x^2 \geq 0$.

3. Write the negations of the following.

(1) $2+2=5$
(2) $1>0$.
(3) $2+2=5$ or $1>0$.
(4) Every student in this class is a math major.
(5) Every student in this class is a math major or a human being.
(6) If x is a real number, then $x^2>0$.

4. Prove the following using truth tables.

(1) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$,
(2) $(P \vee Q) \vee R = P \vee (Q \vee R)$. (We thus write $P \vee Q \vee R$ for both.)
(3) $\neg(P \vee Q) = \neg P \wedge \neg Q$,
(4) $\neg(P \wedge Q) =$ (make a guess similar to problem 3),
(5) $\neg(\neg P) = P$.

5. In exercise 6, you may use the following variants of exercise 4.

(1) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$,
(2) $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$. (We thus write $P \wedge Q \wedge R$ for both.)
(3) $P \vee Q = Q \vee P$.
(4) $P \wedge Q = Q \wedge P$.

6. Prove or disprove the following *without* using truth tables.

(1) $\neg(P \wedge \neg Q) = \neg P \vee Q$.
(2) $P \vee ((Q \wedge R) \wedge S) = (P \wedge Q) \vee (P \wedge R) \vee (P \wedge S)$.
(3) $P \vee (Q \wedge R) \wedge S) = (P \vee Q) \wedge (P \vee R) \wedge (P \vee S)$.

7. Write the negations of the following implications.

(1) If n is even, then n^2 is even.
(2) If $1 = 0$, then $2 + 2 = 5$.
(3) If there is free coffee, then DZB will drink it
(4) If $1 = 0$ and $2 + 2 = 5$, then the sky is blue and kittens are popular on youtube
(5) If x and y are real numbers such that $xy = 0$, then $x = 0$ or $y = 0$.

8. Which of these are true?

(1) (T or F) For all $x \in \mathbf{Z}$, x is divisible by 2.
(2) (T or F) There exists an $x \in \mathbf{Z}$ such that x is divisible by 2.
(3) (T or F) For all $x \in \mathbf{R}$, if $x \neq 0$, then there exists a $y \in \mathbf{R}$ such that $xy = 1$.
(4) (T or F) For all $x \in \mathbf{R}$, there exists a $y \in \mathbf{R}$ such that $xy = 1$.

9. Write the negations of the following.

(1) For all $x \in \mathbf{Z}$, x is divisible by 2.
(2) There exists an $x \in \mathbf{Z}$ such that x is divisible by 2.
(3) $\neg(\forall x, P(x))$,
(4) $\neg(\exists x$ s.t. $Q(x))$
(5) $\forall x, (P(x) \wedge Q(x))$.
(6) If $\exists x \in \mathbf{R}$ such that $2x = 1$, then for all y, $y^2 < 0$.
(7) For all $x \in \mathbf{R}$, there exists a $y \in \mathbf{R}$ such that $xy = 1$.

10. Write the converse and contrapositive of the statements from problem 7.

Here are some basic identities.

(1) $P \wedge Q = Q \wedge P$
(2) $P \vee Q = Q \vee P$
(3) $(P \wedge Q) \wedge R = P \wedge (Q \wedge R) = P \wedge Q \wedge R$
(4) $(P \vee Q) \vee R = P \vee (Q \vee R) = P \vee Q \vee R$

Here are some useful identities.

(1) $\neg(P \wedge Q) = \neg P \vee \neg Q$
(2) $\neg(P \vee Q) = \neg P \wedge \neg Q$
(3) $\neg(\neg P) = P$
(4) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
(5) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
(6) $\neg(P \Rightarrow Q) = P \wedge \neg Q$
(7) $\neg(\forall x, P(x)) = \exists x$ such that $\neg P(x)$
(8) $\neg(\exists x$ such that $P(x)) = \forall x, \neg P(x)$

We can combine these to negate more complicated statements

(1) $\neg(P \Rightarrow (Q \vee R)) =$
$P \wedge \neg(Q \vee R)) =$
$P \wedge \neg Q \wedge \neg R$

(2) If $1 = 0$ and $2 + 2 = 5$, then the sky is blue and kittens are cute
If (P and Q) then (R and T)

Its negation:
(P and Q) and not (R and T)
($1 = 0$ and $2 + 2 = 5$) and (the sky is not blue or kittens are not cute)

(3) $\neg Q \Rightarrow \neg P$
$\neg(\neg Q \Rightarrow \neg P)$
$\neg Q \wedge \neg(\neg P)$
$\neg Q \wedge P$

This last example is called the **contrapositive**, and is a useful proof technique! (Try it on your homework.)

$$(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P) \text{ because they have the same negation.}$$