MATH 250 HANDOUT 3 - PROOF BY CONTRADICTION

- (1) Prove that if x + y > 5, then x > 2 or y > 3.
- (2) Let $0 < \alpha < 1$. Prove that $\sqrt{\alpha} > \alpha$.
- (3) Prove that there are no integer solutions to the equation $x^2 = 4y + 2$
- (4) Prove that there are no positive integer solutions to the equation $x^2 y^2 = 10$.
- (5) Prove that there is no smallest positive real number.
- (6) Let b_1, b_2, b_3, b_4 be positive integers such that

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} = 1.$$

Prove that at least one of the b_k 's is even. Hint: clear the denominators.

- (7) Show that if a is rational and b is irrational, then a + b is irrational.
- (8) Prove that $\sqrt{3}$ is irrational.
- (9) Prove that if $r^3 + r + 1 = 0$ then r is irrational.
- (10) Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even. (Harder problem: show that a or b must be even.)
- (11) If a, b, c are odd integers, prove that $ax^2 + bx + c = 0$ does not have a solution x such that x is a rational number.
- (12) Prove that if $3 \mid (a^2 + b^2)$, then $3 \mid a$ and $3 \mid b$. Hint: If $3 \not\mid a$ and $3 \not\mid b$, what are the possible remainders of a, b, a^2 , and b^2 upon division by 3?
- (13) Prove that $\log_{10} 7$ is irrational.
- (14) Let $b \in \mathbb{Z}_{>1}$. Prove that $\log_b 3/\log_b 2$ is irrational.
- (15) Prove that $\sqrt[5]{5}$ is irrational.
- (16) Prove that the equation

$$(x^2 - y^2)(x^2 - 4y^2) = 7$$

has no solutions with $x, y \in \mathbf{Z}$.

- (17) Prove that there are infinitely many primes of the form 6n + 1 or there are infinitely many primes of the form 6n + 5.
- (18) Prove that there are infinitely many primes of the form 6n + 5.
- (19) Try to prove that there are infinitely many primes of the form 6n + 1. What goes wrong in the argument from the previous problem?
- (20) Prove that if $n \ge 2$, then $\sqrt[n]{n}$ is irrational. Hint: use that if n > 2, then $2^n > n$.