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A locally anisotropic fluid-structure interaction remeshing strategy for thin structures with application to a hinged rigid leaflet

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SUMMARY

An immersed finite element Fluid-Structure Interaction (FSI) algorithm with an anisotropic remeshing strategy for thin rigid structures is presented in two-dimensions. The algorithm consists in remeshing only the fluid elements that are cut by the solid such that they fit the solid geometry. This approach allows to keep the initially provided fluid mesh vertexes fixed during the whole simulation. Furthermore, essential constraints between the fluid and the solid may be directly enforced in the finite element spaces and elements allowing the stress to be discontinuous across the structure may be employed. However, remeshed elements may be strongly anisotropic while the adopted interpolation schemes, always stable on isotropic meshes, may be unstable on anisotropic ones. Therefore a proper element choice needs to be done, which is one of the focuses of the present work. Regarding the FSI algorithm, the fluid and the solid are strongly coupled within a time advancing implicit Euler scheme.

In the numerical tests, we focus on inf-sup stability of several mixed finite elements and we numerically show that some elements might not be suitable for such an FSI algorithm, in particular the mixed elements having a discontinuous pressure. Using the proposed methodology, we carefully address the motion of a hinged rigid leaflet, a problem already discussed in the literature but for which we propose a detailed analysis. Copyright © 0000 John Wiley & Sons, Ltd.

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1. INTRODUCTION

A classical approach for the numerical simulation of fluid-structure interaction is the Arbitrary Lagrangian Eulerian (ALE) method (see, e.g., [1] or [2]). It is well known that this method may not be adequate when the structure undergoes large and fast deformations. In this case, alternatives are provided by the so-called “immersed” approaches, also known under the name of Immersed Boundary Method (see, e.g., [3] or [4]), Fictitious Domain (see, e.g., [5]), embedded/unfitted (see, e.g., [6]), etc. In these methods, on the contrary to the ALE method, the fluid mesh is given *a-priori* and independently of the location of the structure.

Many immersed approaches are known for lacking of accuracy with respect to the ALE method (see, e.g., [7] and references therein). The loss of accuracy is in general due to the non conformity of the fluid/solid meshes and/or an inaccurate enforcement of the coupling constraints. Accordingly,

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many researches in the last decade have focused on developing accurate immersed approaches and today we may distinguish two types of accurate strategies: iterative (see, e.g., [8] and [9]) and direct strategies. Among the latter, we mention the Immersed Interface Method (see, e.g., [10]), the eXtended Finite Element Method (XFEM) (see, e.g., [11]), and a local refinement strategy (see, e.g., [12] or [13]). The present work deals with the latter mentioned strategy, based on local refinement.

In particular, the presented method shares similarities with the XFEM, which consists in constructing a finite element basis able to accurately resolve the singularities introduced by the structure. A major difficulty of the XFEM is to enforce the fluid/solid interface constraints. This issue has recently been an important field of research (see, e.g., [14] or [15] and references therein). Differently, the local refinement approach consists in locally refining the initial fluid mesh such that it conforms with the solid mesh. The method avoids a complex implementation and, thanks to conformity, the fluid-structure interface constraints may be enforced directly in the finite element spaces. In [13] a smoothing strategy is used to maintain a relative isotropy of the refined mesh. One of the drawbacks of this approach is that the more the isotropy of the refined mesh is guaranteed, the more the refined mesh is modified with respect to the original mesh. On the contrary, in [12] the original mesh is used without changing its topology (i.e., only elements cut by the immersed structure are modified). As in [12], the present strategy employs anisotropic elements and, therefore, we talk of a *locally anisotropic remeshing* approach.

Since our problem consists of an incompressible fluid and we deal with it via mixed finite elements, we have to guarantee their stability on anisotropic meshes. The combination of mixed finite elements and anisotropic meshes may in fact lead to two major issues: an ill-conditioned linear system (issue possibly already present in the standard finite element method with anisotropic elements) and a possible lack of inf-sup stability even for elements that are inf-sup stable on isotropic meshes. We point out that in [12] a streamline upwind Petrov Galerkin scheme with low order elements is used and it helps circumvent the inf-sup condition on distorted meshes (see, e.g., [16] or [17]). In the present work we focus on solving the inf-sup stability issue with higher order elements. In addition, we also discuss the linear system ill-conditioning.

The present work takes inspiration from [12], where: i) only the elements crossed by the solid are remeshed so to fit with the immersed boundary; ii) lower order elements are used to ensure that no additional degrees of freedom result from the remeshing strategy; and iii) the nodes lying on the immersed boundary are not considered as degrees of freedom, precluding strongly coupled strategies. Instead, we want to have the freedom of using high order elements, so to avoid stabilization of the inf-sup condition, and of using strongly coupled strategies; we thus do not eliminate the nodes on the internal boundary. In [18] it is investigated the use of the $\mathbf{P}_2/P_1$ element for a 2D steady incompressible Stokes problem using the locally anisotropic remeshing strategy and it was shown that $\mathbf{P}_2/P_1$ may be not inf-sup stable at corner triangles with Dirichlet boundary conditions applied on two edges, but that adding a bubble to the velocity space stabilizes $\mathbf{P}_2/P_1$ with the present remeshing strategy. Furthermore, the gradient of the velocity and the pressure of the fluid are discontinuous across the structure, and thus it may be convenient to use mixed finite elements with discontinuous pressures, as used in [13], and for this reason we study also: $\mathbf{P}_2/P_0$ and $\mathbf{P}_2^+/P_1^d$, respectively. However, as we shall see these two elements have inf-sup related issues on anisotropic meshes and thus we will end up using the mixed finite elements studied in [18], namely, $\mathbf{P}_2/P_1$ and $\mathbf{P}_2^+/P_1$ but in such a way that the pressure is discontinuous across the solid.

Finally, in terms of the fluid-solid interaction scheme, we use a strongly coupled algorithm with a time advancing implicit Euler scheme such that added mass effects are avoided (see, e.g., [19]).

The outline of the article is as follows. In Section 2 we introduce the continuous models for the fluid and the structure as well as for the FSI coupling. In Section 3 we present the time discretization as well as the Picard scheme for the Navier-Stokes equations used for the fluid model. In Section 4 we present the locally anisotropic remeshing strategy, the considered finite element spaces, the interpolation techniques for mapping the solutions between the initial mesh and the remeshed one, and the discrete problem. In Section 5 we present the algebraic formulation of the problem with details on how we enforce the coupling constraints. In Section 6 we present the numerical tests. We finally draw our conclusions in Section 7.

2. CONTINUOUS PROBLEM

In this section we present the Navier-Stokes equations used to model the fluid, we describe the initial and boundary conditions employed, as well as the model for the rigid leaflet and its initial conditions.

2.1. The fluid problem

For the fluid, a standard Newtonian model with constant density is adopted. Therefore, the set of equations governing the motion of the fluid is given by the classical incompressible Navier-Stokes equations:

$$\begin{cases} \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \operatorname{div}(\mu \nabla^s \mathbf{u}) + \nabla p - \mathbf{f} = \mathbf{0}, \\ \operatorname{div}(\mathbf{u}) = 0, \end{cases} \quad (1)$$

where ρ_f designates the density, μ the dynamic viscosity, $\mathbf{u}$ the velocity, p the pressure and $\nabla^s \mathbf{u}$ is defined as $\nabla^s \mathbf{u} = \nabla \mathbf{u} + (\nabla \mathbf{u})^T$.

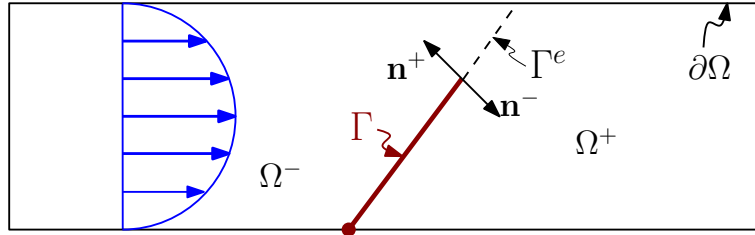


Figure 1. The fluid domain is denoted by Ω . It includes the leaflet denoted by Γ which is subjected to a flow (in blue). The leaflet rotates around $\mathbf{R}$ and it divides the domain Ω in two parts: Ω^- (upstream) and Ω^+ (downstream). The dividing frontier is defined by $\Gamma \cup \Gamma^e$, where Γ^e is an extension such that Ω may be conveniently divided.

2.1.1. Boundary and initial conditions We consider Eq. (1) in a fixed domain $\Omega = \Omega^- \cup \Omega^+ \cup (\Gamma \cup \Gamma^e)$ (see Fig. 1). The boundary of Ω is defined by $\partial\Omega = \Sigma_D \cup \Sigma_N$ such that $\Sigma_D \cap \Sigma_N = \emptyset$. We close Eq. (1) by adding the following boundary and initial conditions:

$$\begin{cases} \mathbf{u} = \mathbf{b}_D & \text{on } \Sigma_D, \\ -p\mathbf{n} + \mu(\nabla^s \mathbf{u})\mathbf{n} = \mathbf{b}_N & \text{on } \Sigma_N, \end{cases} \quad (2)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_i(\mathbf{x}) \quad \text{on } \Omega, \quad (3)$$

where $\mathbf{n}$ designates the outward normal of $\partial\Omega$. We assume that Σ_N is not empty such that the pressure is uniquely defined.

2.2. The solid problem

For the solid we choose a hinged rigid leaflet of length L with negligible width, only one rotational degree of freedom, and a rotational spring of stiffness $\kappa \geq 0$ (see Fig. 2).

For describing the motion of the leaflet we use a polar coordinate system with pole $\mathbf{R} \in \mathbb{R}^2$, the point of rotation of the leaflet. We denote by θ and r the angular and radial coordinates, respectively.

The leaflet is initially at θ_0 and has a constant mass distribution ρ_s . The leaflet equation is thus given by

$$I \frac{d^2 \theta}{dt^2} + \kappa (\theta - \theta_0) = \tau, \quad (4)$$

where τ is the torque exerted on the bar, that is

$$\tau(t) = \int_{\Gamma} r f_s(r, t) dr, \quad (5)$$

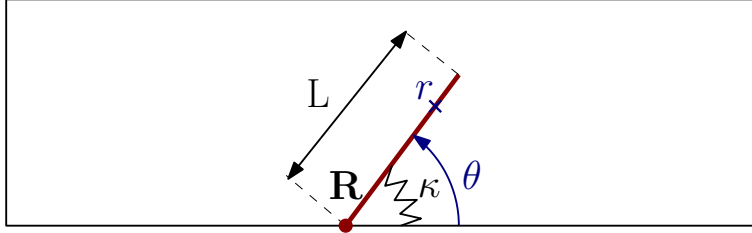


Figure 2. An example of a hinged rigid leaflet with a rotational spring attached.

where f_s is the net balance of the stress acting on the two sides of the bar (see Fig. 2), and I is the moment of inertia given by

$$I = \int_{\Gamma} \rho_s r^2 dr = \frac{\rho_s L^3}{3}.$$

2.3. Fluid-structure interaction coupling

The coupling between the fluid and the solid is exerted by the fulfillment of the kinematic constraint

$$\mathbf{u} = r \frac{d\theta}{dt} \mathbf{n}^+, \quad (6)$$

and the conservation of momentum

$$f_s(\theta, t) = P^- - P^+,$$

where the right hand side is the drop of pressure across the solid. Defining

$$\begin{cases} P^+ \mathbf{n}^+ &= -p^+ \mathbf{n}^+ + \mu(\nabla^s \mathbf{u}^+) \mathbf{n}^+, \\ P^- \mathbf{n}^- &= -p^- \mathbf{n}^- + \mu(\nabla^s \mathbf{u}^-) \mathbf{n}^-, \end{cases}$$

we obtain that

$$f_s(\theta, t) = \llbracket p \mathbf{n}^+ - \mu(\nabla^s \mathbf{u}) \mathbf{n}^+ \rrbracket_{\Gamma} \cdot \mathbf{n}^+ \quad (7)$$

where $\llbracket \cdot \rrbracket_{\Gamma}$ denotes the jump across Γ , i.e.,

$$\llbracket p \mathbf{n}^+ - \mu(\nabla^s \mathbf{u}) \mathbf{n}^+ \rrbracket_{\Gamma} = (p|_{\Gamma^+} - p|_{\Gamma^-}) \mathbf{n}^+ - \mu(\nabla^s (\mathbf{u}|_{\Gamma^+} - \mathbf{u}|_{\Gamma^-})) \mathbf{n}^+,$$

with $v|_{\Gamma}$ the trace of v on Γ .

3. TIME DISCRETIZATION AND LINEARIZATION

For the fluid we use the first order backward Euler scheme to discretize the velocity time derivative, that is

$$\left. \frac{\partial \mathbf{u}}{\partial t} \right|_{t^{n+1}} \approx \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}, \quad (8)$$

where δt is the time step. We also choose the velocity at the previous time step as the convective velocity, according to the classical Picard approach such that the convective term in Eq. (1) is given by

$$(\mathbf{u} \cdot \nabla \mathbf{u})_{t^{n+1}} \approx \mathbf{u}^n \cdot \nabla \mathbf{u}^{n+1}, \quad (9)$$

leading to a first-order scheme in time.

For the solid we employ a centered scheme for the acceleration, that is

$$\left. \frac{d^2 \theta}{dt^2} \right|_{t^{n+1}} \approx \frac{\theta^{n+1} - 2\theta^n + \theta^{n-1}}{\delta t^2}, \quad (10)$$

and for the velocity we use a backward Euler scheme, that is

$$\left. \frac{d\theta}{dt} \right|_{t^{n+1}} \approx \frac{\theta^{n+1} - \theta^n}{\delta t}. \quad (11)$$

3.1. Strong formulation of the coupled problem

Using Eqs. (8)-(11) in Eqs. (1)-(7), the time-discretized problem in strong form reads:

Problem 2: Given $\mathbf{f}^{n+1}$, $\mathbf{b}_D^{n+1}$, $\mathbf{b}_N^{n+1}$, $\mathbf{u}^n$, θ^n , and θ^{n-1} , find $\mathbf{u} = \mathbf{u}^{n+1}$, $p = p^{n+1}$, $\theta = \theta^{n+1}$ such that

$$\left\{ \begin{array}{ll} \rho_f \left(\frac{\mathbf{u}}{\delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u} \right) - \operatorname{div}(\mu \nabla^s \mathbf{u}) + \nabla p & \text{in } \Omega \setminus \Gamma^n \\ = \mathbf{f}^{n+1} + \rho_f \frac{\mathbf{u}^n}{\delta t} & \end{array} \right. \quad (12a)$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega \quad (12b)$$

$$\mathbf{u} = \mathbf{b}_D^{n+1} \quad \text{on } \Sigma_D \quad (12c)$$

$$-p\mathbf{n} + \mu(\nabla^s \mathbf{u})\mathbf{n} = \mathbf{b}_N^{n+1} \quad \text{on } \Sigma_N \quad (12d)$$

$$\mathbf{u} - r \frac{\theta}{\delta t} \mathbf{n}^+ = -r \frac{\theta^n}{\delta t} \mathbf{n}^+ \quad \text{on } \Gamma^n \quad (12e)$$

$$\tau^{n+1} = \int_{\Gamma^n} r \llbracket p\mathbf{n}^+ - \mu(\nabla^s \mathbf{u})\mathbf{n}^+ \rrbracket \cdot \mathbf{n}^+ dr \quad (12f)$$

$$I \frac{\theta}{\delta t^2} + \kappa\theta - \tau^{n+1} = \kappa\theta_0 + 2I \frac{\theta^n}{\delta t^2} - I \frac{\theta^{n-1}}{\delta t^2}. \quad (12g)$$

Notice that we impose the coupling between the fluid and the solid on Γ at time t_n , denoted by Γ^n . Indeed, the position of the leaflet at time t_{n+1} is an unknown of the problem and enforcing the coupling on Γ^n is coherent with our choice for the time stepping, i.e., a first-order scheme in time.

3.2. Weak formulation of the coupled problem

We consider the classical Sobolev spaces; accordingly $L^2(\Omega)$ denotes the space of square summable functions in Ω , $H^1(\Omega)$ is the space of functions that are in $L^2(\Omega)$ with first derivatives in $L^2(\Omega)$ as well. The space $H_\Sigma^1(\Omega)$ denotes the space of functions in $H^1(\Omega)$ with vanishing traces on Σ_D and $H_{\Sigma, \Gamma}^{1,n}(\Omega)$ is the space of functions in $H^1(\Omega)$ with vanishing trace on $\Sigma_D \cup \Gamma^n$.

For simplicity and without loss of generality, let assume that $\mathbf{b}_D = \mathbf{0}$ (see Eq. (2)), and thus only the essential interfacial constraints are imposed on Γ^n such that we have the lifting of the interfacial constraints defined as $\mathbf{g}^{n+1} \in [H_\Sigma^1(\Omega)]^2$ and $\mathbf{u}_0^{n+1} \in [H_{\Sigma, \Gamma}^{1,n}(\Omega)]^2$ has a null trace on Γ^n with

$$\mathbf{u}^{n+1} = \mathbf{u}_0^{n+1} + \mathbf{g}^{n+1}, \quad (13)$$

where $\mathbf{g}_{|\Gamma^n}^{n+1} = (r(\theta^{n+1} - \theta^n)/\delta t)\mathbf{n}^+$ (see Eq. (12e)).

Then our weak formulation for Problem 2 reads:

Problem 3: Given $\mathbf{f}^{n+1}$, $\mathbf{b}_N^{n+1}$, $\mathbf{u}^n$, θ^n , θ^{n-1} , find $\mathbf{u}_0 = \mathbf{u}_0^{n+1} \in [H_{\Sigma, \Gamma}^{1,n}(\Omega)]^2$, $p = p^{n+1} \in L^2(\Omega)$, and $\theta = \theta^{n+1} \in \mathbb{R}$, such that $\forall (\mathbf{v}, q, \gamma) \in [H_{\Sigma, \Gamma}^{1,n}(\Omega)]^2 \times L^2(\Omega) \times \mathbb{R}$ we have

$$\left\{ \begin{aligned} & \int_{\Omega} \rho_f \left(\frac{\mathbf{u}_0}{\delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u}_0 \right) \cdot \mathbf{v} + \int_{\Omega} \mu \nabla^s \mathbf{u}_0 : \nabla \mathbf{v} - \int_{\Omega} p \operatorname{div}(\mathbf{v}) \\ & + \int_{\Omega} \rho_f \left(\frac{\mathbf{g}}{\delta t} \mathbf{u}^n \cdot \nabla \mathbf{g} \right) \cdot \mathbf{v} + \int_{\Omega} \mu \nabla^s \mathbf{g} : \nabla \mathbf{v} \end{aligned} \right. \quad (14a)$$

$$\left\{ \begin{aligned} & = \int_{\Omega} \mathbf{f}^{n+1} \cdot \mathbf{v} + \int_{\Omega} \rho_f \frac{\mathbf{u}^n}{\delta t} \cdot \mathbf{v} + \int_{\Sigma_N} \mathbf{b}_N^{n+1} \cdot \mathbf{v} \\ & - \int_{\Omega} q \operatorname{div}(\mathbf{u}_0) - \int_{\Omega} q \operatorname{div}(\mathbf{g}) = 0 \end{aligned} \right. \quad (14b)$$

$$\tau^{n+1} = \int_{\Gamma^n} r (\llbracket p \mathbf{n}^+ - \mu (\nabla^s \mathbf{g}) \mathbf{n}^+ \rrbracket) \cdot \mathbf{n}^+ \, dr \quad (14c)$$

$$\mathbf{g}|_{\Gamma^n} = (r(\theta - \theta^n)/\delta t) \mathbf{n}^+, \quad (14d)$$

$$\left(I \frac{\theta}{\delta t^2} + \kappa \theta - \tau^{n+1} \right) \gamma = \left(\kappa \theta_0 + 2I \frac{\theta^n}{\delta t^2} - I \frac{\theta^{n-1}}{\delta t^2} \right) \gamma. \quad (14e)$$

4. LOCALLY ANISOTROPIC REMESHING STRATEGY AND FINITE ELEMENTS

This section is devoted to the construction of a partition $\mathcal{T}^n$ of Ω based on an initial partition $\mathcal{T}$ of Ω , which is given independently of the position of the leaflet, as well as of the associated finite element spaces. The construction of $\mathcal{T}^n$ is built upon a locally anisotropic remeshing. A more detailed presentation of the locally anisotropic remeshing strategy is given in [18], but we give here a short summary with additional considerations for the tip of the leaflet. Furthermore, for the problem under consideration, the partition $\mathcal{T}^n$ is built in such a way that it is fairly easy to:

1. Enforce the kinematic constraint between the fluid and the solid;
2. Construct finite element spaces with discontinuous pressures and velocity derivatives across the solid.

Moreover, since we use high order finite elements for the velocity field and a local remeshing we explain how we interpolate the values of the velocity field computed at previous time steps on several nodes of the remeshed elements. We then give a discrete formulation of Problem 3. Moreover, since some nodal values of the velocity field, on the initial mesh, are not directly known from computation at time t^{n+1} , we have to use an interpolation strategy. Finally, we summarize in the form of an algorithm all previously discussed steps. Next section, Section 5, will be then devoted to the algebraic formulation of the problem.

4.1. Locally anisotropic remeshing

We start by an isotropic triangulation $\mathcal{T}$ of Ω given independently of the position of the leaflet at time t^n (see, e.g., [20] for a definition of an isotropic mesh named there a shape regular mesh and [21] for a discussion on shape assumptions for triangular meshes). The employed strategy is to remesh $\mathcal{T}$ solely on elements crossed by the solid. As often requested in the literature (see, e.g., [22]), we assume that Γ^n may intersect at most only twice the edges of a triangle and that it crosses two distinctive edges (see the *full-cut* in Fig. 3). Also, as pointed out in [22], given any solid geometry, there always exists a sufficiently fine fluid mesh such that the assumptions previously discussed are satisfied. Furthermore, we also accept the tip of the leaflet to cut a single triangle edge (see the *tip-cut* on Fig. 3).

Now that we have the intersection points of Γ^n with the edges of the elements of $\mathcal{T}$ we can remesh. We recall that by hypothesis Γ^n crosses a triangle boundary of $\mathcal{T}$ on two points for a full-cut, each

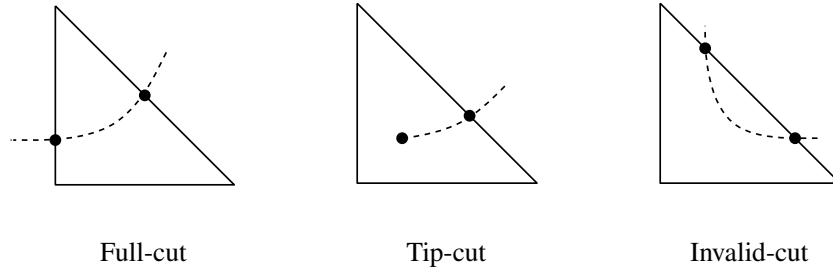
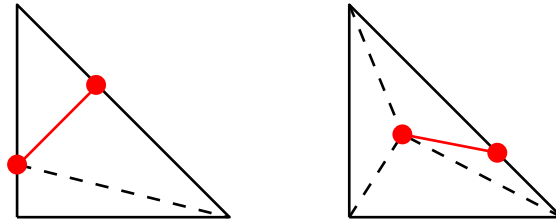


Figure 3. Mesh assumptions with respect to Γ^n (dashed). Full and tip cuts are the only two admissible ones.

point belonging to a distinct edge, or only once for a tip-cut. For each element of $\mathcal{T}$ crossed by Γ^n we consider a Delaunay triangulation of the set of points composed by the vertices of the triangle and the two intersection points for a full-cut, or the single intersection point and the tip of the leaflet for a tip-cut (see Fig. 4). We call this remeshing a *sub-triangulation*. We point out that a Delaunay triangulation maximizes the minimal angles (see, e.g., [23]).

We now can construct a new partition denominated $\mathcal{T}^n$ that consists of all triangles of $\mathcal{T}$ that are not crossed by Γ^n and the triangles of the sub-triangulation of all the elements cut by Γ^n . We define the remeshed partition by $\Omega_h^n = \bigcup_k T_k$ where T_k is a triangle of $\mathcal{T}^n$. As opposed to $\mathcal{T}$, notice that $\mathcal{T}^n$ is not in general isotropic. It has been noted in [24] that an isotropic partition is not a necessary assumption for the optimal convergence of the finite element method. However, the distortion of the mesh affects the conditioning of the system (see, e.g., [25]). In Section 6 we will discuss these topics in more details.



(a) Full-cut sub-triangulation subdivided into 3 subtriangles. (b) Tip-cut sub-triangulation subdivided into 4 subtriangles

Figure 4. Spatial discretization: full and tip triangle subdivisions.

It is interesting to observe that using the partition $\mathcal{T}^n$ we can easily enforce the kinematic constraints on Γ^n since we have nodes of $\mathcal{T}^n$ lying on Γ^n . Furthermore, across Γ^n the stress may jump, that means that we have a discontinuous pressure and a discontinuity in the derivatives of the velocity that have to be properly taken into account. With $\mathcal{T}^n$ a pressure basis that allows a jump across Γ_h^n is promptly constructed, while the jump in the derivatives of the velocity is automatically managed by standard nodal C^0 -Lagrange basis.

4.2. Choice of the finite element spaces

Anisotropic partitions may impact adversely inf-sup stability of mixed finite elements. In this section we introduce the various mixed finite elements used in the numerical tests and we discuss their inf-sup stability in the context of the present method. More details are given in the numerical tests section.

We consider two pressure choices:

1. Discontinuous pressure that intrinsically allows jumps in the pressure across element edges. Indeed, the pressure is in L^2 and thus discontinuous elements are allowed;
2. “Continuous” pressure that are continuous everywhere in the fluid domain, except across the immersed boundary. Obviously, such an element is harder to implement and it requires additional degrees of freedom along the structure.

Accordingly to the previous discussion, in the present work we consider the following mixed approximation schemes:

- a. $\mathbf{P}_2/P_0$: continuous piecewise quadratic velocity and piecewise constant pressure:

$$\begin{cases} \mathbf{V}_{h,\Gamma}^{n+1} &= \{\mathbf{v} : \mathbf{v}|_T \in (\mathcal{P}_2)^2, \forall T \in \mathcal{T}^n\} \cap [H_{\Sigma,\Gamma}^{1,n}(\Omega)]^2, \\ Q_h^{n+1} &= \{q : q|_T \in \mathcal{P}_0, \forall T \in \mathcal{T}^n\} \cap L^2(\Omega). \end{cases}$$

Here $\mathcal{P}_k$ denotes the space of polynomials of order k .

- b. $\mathbf{P}_2^+/P_1^d$: continuous piecewise quadratic velocity with a cubic bubble and discontinuous piecewise linear pressure:

$$\begin{cases} \mathbf{V}_{h,\Gamma}^{n+1} &= \{\mathbf{v} : \mathbf{v}|_T = \mathbf{v}_{|T}^2 + \mathbf{v}_{|T}^+; \mathbf{v}_{|T}^1 \in (\mathcal{P}_2)^2, \\ &\quad \mathbf{v}_{|T}^+ \in (\mathcal{P}_3)^2, \mathbf{v}_{|\partial T}^+ = \mathbf{0}, \forall T \in \mathcal{T}^n\} \cap [H_{\Sigma,\Gamma}^{1,n}(\Omega)]^2 \\ Q_h^{n+1} &= \{q : q|_T \in \mathcal{P}_1, \forall T \in \mathcal{T}^n\} \cap L^2(\Omega), \end{cases}$$

- c. $\mathbf{P}_2/P_1$: (Hood-Taylor) continuous piecewise quadratic velocity and continuous piecewise linear pressure but discontinuous across the structure:

$$\begin{cases} \mathbf{V}_{h,\Gamma}^{n+1} &= \{\mathbf{v} : \mathbf{v}|_T \in (\mathcal{P}_2)^2, \forall T \in \mathcal{T}^n\} \cap [H_{\Gamma,\Sigma}^{1,n}(\Omega)]^2 \\ Q_h^{n+1} &= \{q : q|_T \in \mathcal{P}_1, \forall T \in \mathcal{T}^n\} \cap \{H^1(\Omega \setminus \Gamma^n) \cap L^2(\Omega)\}. \end{cases}$$

- d. $\mathbf{P}_2^+/P_1$: continuous piecewise quadratic velocity with a cubic bubble and continuous piecewise linear pressure but discontinuous across the structure:

$$\begin{cases} \mathbf{V}_{h,\Gamma}^{n+1} &= \{\mathbf{v} : \mathbf{v}|_T = \mathbf{v}_{|T}^2 + \mathbf{v}_{|T}^+; \mathbf{v}_{|T}^1 \in (\mathcal{P}_2)^2, \\ &\quad \mathbf{v}_{|T}^+ \in (\mathcal{P}_3)^2, \mathbf{v}_{|\partial T}^+ = \mathbf{0}, \forall T \in \mathcal{T}^n\} \cap [H_{\Sigma,\Gamma}^{1,n}(\Omega)]^2 \\ Q_h^{n+1} &= \{q : q|_T \in \mathcal{P}_1, \forall T \in \mathcal{T}^n\} \cap \{H^1(\Omega \setminus \Gamma^n) \cap L^2(\Omega)\}. \end{cases}$$

For a general and detailed presentation of the previously presented finite element spaces and an extensive presentation of the theory on mixed finite elements see [26].

One of the major issues for mixed elements on anisotropic mesh is that the inf-sup constant, indicated in the following with β , may degenerate to zero for highly anisotropic elements. The effects of a very low inf-sup constant are twofold. Firstly, the velocity and the pressure fields may have a very large bound, of $\mathcal{O}(\beta^{-1})$ and $\mathcal{O}(\beta^{-2})$, respectively. Secondly, the conditioning of the system is affected since dependence of the condition number of the Schur complement for the steady incompressible Stokes problem on β is $\mathcal{O}(\beta^{-2})$ (see, e.g., [26] or [20]). We point out that the conditioning of the Schur complement is also affected by the distortion of the elements even for inf-sup stable elements on anisotropic meshes (see Proposition 4.47 in [27]) since the conditioning of the pressure mass matrix usually worsens as the elements are more distorted.

We now provide the various results on distorted meshes regarding the mixed finite elements used in this work.

- a $\mathbf{P}_2/P_0$: We know from [28] that $\mathbf{P}_2/P_0$ is stable on a large class of distorted meshes. Despite these results, using a similar argument as in [29], it may be shown that $\mathbf{P}_2/P_0$ can be unstable with anisotropic elements and not only in corners (see [30]).
- b $\mathbf{P}_2^+/P_1^d$: To the best knowledge of the authors there are no results for this element. Nevertheless, we show that $\mathbf{P}_2^+/P_1^d$ (which is used in [13]) is highly unstable on anisotropic elements, (see also [30]).
- c $\mathbf{P}_2/P_1$: It has been shown numerically in [31] that $\mathbf{P}_2/P_1$ may fail on distorted meshes while $\mathbf{P}_2^+/P_1$ passes all proposed tests. In a different study (see [29]) it has been shown numerically that, for a similar method as proposed in the present document but for the steady 2D incompressible Stokes problem, $\mathbf{P}_2/P_1$ may fail on corner triangles for which Dirichlet boundary conditions are applied on the two boundary edges. Actually such a situation is unlikely to occur for our FSI problem, except perhaps for refined elements near the point of rotation of the leaflet. Nevertheless, no spurious modes are seen in our numerical tests with this element.
- d $\mathbf{P}_2^+/P_1$: On the contrary to $\mathbf{P}_2/P_1$ and as suggested from the results in [31], the $\mathbf{P}_2^+/P_1$ element passes all tests (see also [29]). For that reason, $\mathbf{P}_2^+/P_1$ is our element of choice. However, no formal proof of the stability of $\mathbf{P}_2^+/P_1$ on anisotropic meshes is known to the authors.

We point out that, when a mixed element is inf-sup unstable, the effect of the spurious modes are only local and on very small elements, which may have a limited impact on the motion of the leaflet, as obtained in the numerical tests. The rigidity of the leaflet may also play an important role in this. However, among all tested mixed elements in this work, the best and safest is surely $\mathbf{P}_2^+/P_1$.

Remark 1

Regarding the use of $\mathbf{P}_2^+/P_1$, in the current implementation we add the bubble over the whole mesh. It is however interesting to observe that a more computationally efficient approach would be to add the bubble only on the distorted elements. Furthermore, it is not necessary to integrate exactly all polynomial degrees higher than those present using $\mathbf{P}_2/P_1$ (i.e., terms involving the bubble shape functions), making it a competitive alternative to $\mathbf{P}_2/P_1$ (see [29]).

4.3. Previous time step velocity interpolation

In Problem 3, two velocity terms at time t^n are required, but since we remesh elements crossed by the structure some nodal values are unknown with respect to the velocity solution over the initial mesh $\mathcal{T}$ (see Fig. 5). We therefore need to interpolate.

In order to compute the missing values we build the quadratic interpolator Π over $\mathcal{T}$, i.e., over the initial mesh. Let $\mathbf{V}_h$ be the finite dimensional subspace of $[H^1(\Omega)]^2$ built over $\mathcal{T}$ using piecewise quadratic Lagrange shape functions (each shape function is denoted by $\mathbf{N}_i$) such that $\mathbf{V}_h = \text{span}(\mathbf{N})$. More precisely, we assume that we know $\hat{\mathbf{w}}^n$, i.e., the velocity nodal values at time t^n over $\mathcal{T}$, and thus the interpolation $\Pi \mathbf{w}^n$ is defined by

$$\Pi \mathbf{w}^n(\mathbf{x}) = \sum_i \hat{\mathbf{w}}_i^n \mathbf{N}_i(\mathbf{x}). \quad (15)$$

Remark 2

The use of the interpolant introduces an error. In particular, $\Pi \mathbf{w}^n(\mathbf{x}^n)$ is not divergence free in general. Such an issue is well known. However, as pointed out in [13], the error introduced is found to be small. We point out that the issue could be avoided for the convective term resorting to a full implicit strategy by using $\mathbf{u}_h^{n+1} \cdot \nabla \mathbf{u}_h^{n+1}$ instead of $\Pi \mathbf{w}^n(\mathbf{x}^{n+1}) \cdot \nabla \mathbf{u}_h^{n+1}$. However, such a strategy has not been tested here.

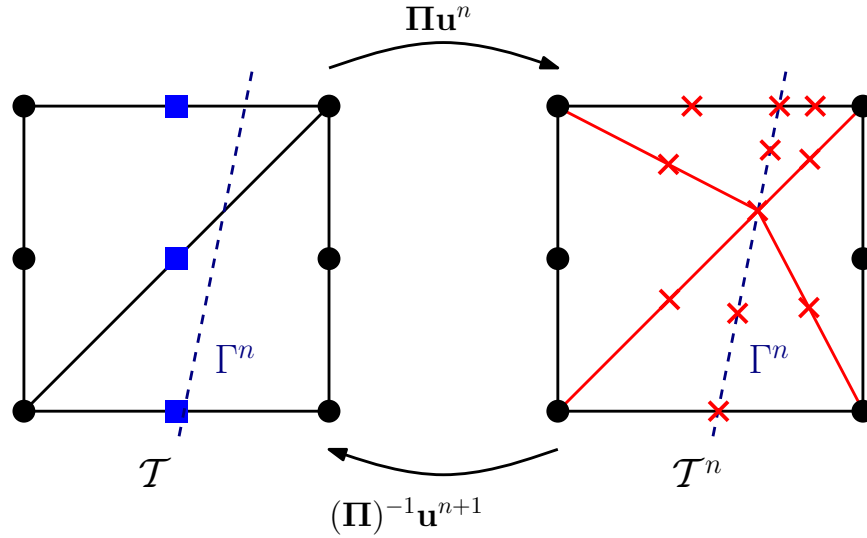


Figure 5. The solution $\mathbf{u}^n$ is required for computing $\mathbf{u}^{n+1}$ but some nodal values are unknown on $\mathcal{T}^n$. They are depicted by red crosses. Hence an interpolation operator Π is employed assuming that we know $\mathbf{u}^n$ on $\mathcal{T}$ (discussed in Sec. 4.3). In the same manner from the mesh $\mathcal{T}$ some nodal values of the solution $\mathbf{u}^{n+1}$ are unknown (depicted with a blue square), and thus an interpolation using the operator $(\Pi)^{-1}$ is required (see Sec. 4.5) in order to obtain all nodal values on $\mathcal{T}$ of $\mathbf{u}^{n+1}$.

4.4. Discrete problem

Now using the elements defined in Sec. 4.2, we build finite dimensional spaces $\mathbf{V}_{h,\Gamma}^{n+1}$ and $\mathbf{Q}_h^{n+1}$ such that $\mathbf{V}_{h,\Gamma}^{n+1} \subset [H_{\Sigma,\Gamma}^{1,n}(\Omega)]^2$ and $\mathbf{Q}_h^{n+1} \subset L^2(\Omega)$. It follows that the discretized problem is given by:

Problem 4: Given $\mathbf{f}^{n+1}$, $\mathbf{b}_N^{n+1}$, $\mathbf{u}_h^n$, θ^n , θ^{n-1} , find $\mathbf{u}_{0,h} = \mathbf{u}_{0,h}^{n+1} \in \mathbf{V}_{h,\Gamma}^{n+1}$, $p_h = p_h^{n+1} \in \mathbf{Q}_h^{n+1}$, and $\theta = \theta^{n+1} \in \mathbb{R}$, such that $\forall (\mathbf{v}_h, q_h, \gamma) \in \mathbf{V}_{h,\Gamma}^{n+1} \times \mathbf{Q}_h^{n+1} \times \mathbb{R}$ we have

$$\left\{ \begin{aligned} & \int_{\Omega_h^n} \rho_f (\mathbf{u}_{0,h}/\delta t + \Pi \mathbf{w}^n \cdot \nabla \mathbf{u}_{0,h}) \cdot \mathbf{v}_h + \int_{\Omega_h^n} \mu \nabla^s \mathbf{u}_{0,h} : \nabla \mathbf{v}_h - \int_{\Omega_h^n} p_h \operatorname{div}(\mathbf{v}_h) \\ & + \int_{\Omega_h^n} \rho_f (\mathbf{g}_h/\delta t + \Pi \mathbf{w}^n \cdot \nabla \mathbf{g}_h) \cdot \mathbf{v}_h + \int_{\Omega_h^n} \mu \nabla^s \mathbf{g}_h : \nabla \mathbf{v}_h \\ & = \int_{\Omega_h^n} \mathbf{f}^{n+1} \cdot \mathbf{v}_h + \int_{\Omega_h^n} \rho_f \Pi \mathbf{w}^n / \delta t \cdot \mathbf{v}_h + \int_{\Sigma_N} \mathbf{b}_N^{n+1} \cdot \mathbf{v}_h \end{aligned} \right. \quad (16a)$$

$$\left\{ \begin{aligned} & - \int_{\Omega_h^n} q_h \operatorname{div}(\mathbf{u}_{0,h}) - \int_{\Omega_h^n} q_h \operatorname{div}(\mathbf{g}_h) = 0 \end{aligned} \right. \quad (16b)$$

$$\tau^{n+1} = \int_{\Gamma^n} r (\llbracket p \mathbf{n}^+ - \mu(\nabla^s \mathbf{g}) \mathbf{n}^+ \rrbracket) \cdot \mathbf{n}^+ dr \quad (16c)$$

$$\mathbf{g}_h|_{\Gamma^n} = (r(\theta - \theta^n)/\delta t) \mathbf{n}^+ \quad \forall r \in [0, L] \quad (16d)$$

$$\left(I \frac{\theta}{\delta t^2} + \kappa \theta - \tau^{n+1} \right) \gamma = \left(\kappa \theta_0 + 2I \frac{\theta^n}{\delta t^2} - I \frac{\theta^{n-1}}{\delta t^2} \right) \gamma \quad (16e)$$

4.5. Velocity “re”-interpolation for the initial mesh

When using a linear interpolant, all nodal values required to construct the interpolant are directly available from all computed time steps (which makes low order elements quite competitive), while with a higher order interpolant such as $\mathbf{P}_2$, this is no longer true (see Fig. 5). Indeed, for a $\mathbf{P}_2$ interpolant the mid edge nodes of the elements cut by the solid are not known directly from the

computation at time t^{n+1} . Therefore, we need to interpolate them using an interpolant built upon $\mathcal{T}^n$. We point out that the “re”-interpolation strategy requires to interpolate on distorted elements. However, the Lagrange- L^2 interpolant is bounded for arbitrary anisotropic meshes (see, e.g., [32]). The present implementation uses a quadratic interpolant and not a linear one.

4.6. Fluid-Structure interaction algorithm

To conclude this section, we summarize the various operations presented earlier, to give an overview of processes required to perform a simulation:

Algorithm 1:

Data: Given $\mathbf{u}_i, \Gamma^0, \mathcal{T}$, loads, and BCs
for Time step: t^n **do**
 Step 1: Build $\mathcal{T}^{n+1}$ from Γ^n and $\mathcal{T}$ (see Sec. 4)
 Step 2: Solve Problem 4 (see Sec. 4.4 and 4.3)
 Step 3: Store nodal velocity results for a $\mathbf{P}_2$ interpolant built upon $\mathcal{T}$ (see Sec. 4.5)
 Step 4: Update leaflet position
 Step 5: Next time step: $t^n \leftarrow t^{n+1}$
end

5. ALGEBRAIC FORMULATION

Considering the finite element spaces described in Sec. 4.2, we can construct (see Eq. (13))

$$\mathbf{u}_h = \sum_{j=1}^{n_u} \mathbf{N}_j \hat{\mathbf{u}}_j + \sum_{j=n_u+1}^{n_u+n_\Gamma} \mathbf{N}_j \hat{\mathbf{g}}_j, \quad (17)$$

where $\hat{\mathbf{u}}_j$ are the fluid velocity nodal values in the fluid domain, i.e., nodes in $\Omega_h^n \setminus \Gamma^n$, and $\hat{\mathbf{g}}_j$ are the fluid velocity nodal values on the leaflet, i.e., nodes lying on Γ^n . Furthermore, in Eq. (17) we have $\hat{\mathbf{u}} \in \mathbb{R}^{n_u}$ and $\hat{\mathbf{g}} \in \mathbb{R}^{n_\Gamma}$ (where n_u and n_Γ are the number of velocity degrees of freedom in $\Omega_h^n \setminus \Gamma^n$ and on Γ^n , respectively) and we define $\mathbf{V}_{h,0}^{n+1} = \text{span}(\mathbf{N}^n)$ such that $\mathbf{V}_{h,0}^{n+1} \subset [H_0^1(\Omega)]^2$ and thus $\mathbf{u}_h \in \mathbf{V}_{h,0}^{n+1}$; while

$$p_h = \sum_{j=1}^m M_j^n \hat{p}_j \quad (18)$$

with $\hat{p} \in \mathbb{R}^m$ (where m is the number of degrees of freedom for the pressure) and $Q_h^{n+1} = \text{span}(\mathbf{M}^n)$ implying that $p_h \in Q_h^{n+1}$. We point out that from here on we do not explicitly state time dependence of $\mathbf{N}^n$ and $\mathbf{M}^n$ for ease of notation since there are no references in the following to the shape function defined on $\mathcal{T}$.

Now, substituting Eq. (17) and Eq. (18) in Eq. (16), with a suitable construction of $\mathbf{N}$ and using the Bubnov-Galerkin method we can obtain the following algebraic representation of the coupled problem where all arrays are introduced in detail in the following three sections.

$$\begin{bmatrix} \mathbf{A} & \mathbf{A}_l & \mathbf{D}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{0} & \mathbf{l}_\theta \\ \mathbf{D} & \mathbf{D}_l & \mathbf{0} & \mathbf{0} \\ \mathbf{l}_u & \mathbf{l}_l & \mathbf{l}_p & s \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{g}} \\ \hat{p} \\ \theta \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{b}}_u \\ \hat{\mathbf{b}}_g \\ \hat{0} \\ \lambda \end{Bmatrix}. \quad (19)$$

Note that, in the following, we associate the velocity nodal values in the fluid domain by the index set $\mathcal{I}_u = \{1, 2, \dots, n_u\}$, and the velocity nodal values on the leaflet by the index set $\mathcal{I}_g = \{n_u + 1, n_u + 2, \dots, n_u + n_\Gamma\}$, while $\mathcal{I}_p = \{1, 2, \dots, m\}$ is the set of pressure degrees of freedom.

5.1. The algebraic fluid part

We have defined in Eq. (19):

$$\mathbf{A} = \mathbf{M}/\delta t + \mathbf{O} + \mathbf{K}$$

with

$$\begin{cases} \mathbf{M}_{|ij} &= \int_{\Omega_h^n} \rho_f \mathbf{N}_j \cdot \mathbf{N}_i \\ \mathbf{K}_{|ij} &= \int_{\Omega_h^n} \mu \nabla^s \mathbf{N}_j : \nabla \mathbf{N}_i \\ \mathbf{O}_{|ij} &= \int_{\Omega_h^n} \rho_f (\mathbf{\Pi} \mathbf{w}^n \cdot \nabla \mathbf{N}_j) \cdot \mathbf{N}_i \end{cases} \quad \forall (i, j) \in \mathcal{I}_u \times \mathcal{I}_u. \quad (20)$$

Analogously, we have defined

$$\mathbf{A}_l = \mathbf{M}_l/\delta t + \mathbf{O}_l + \mathbf{K}_l$$

with the matrices defined as in Eq. (20) but for all $(i, j) \in \mathcal{I}_u \times \mathcal{I}_g$. Then we have

$$\begin{cases} \mathbf{D}_{|ij} = - \int_{\Omega_h^n} M_i \operatorname{div}(\mathbf{N}_j) & \forall (i, j) \in \mathcal{I}_p \times \mathcal{I}_u \\ \mathbf{D}_{l|ij} = - \int_{\Omega_h^n} M_i \operatorname{div}(\mathbf{N}_j) & \forall (i, j) \in \mathcal{I}_p \times \mathcal{I}_g \end{cases},$$

and the right hand side is given by

$$\hat{\mathbf{b}}_{u|i} = \int_{\Omega_h^n} (\mathbf{f}^{n+1} + \mathbf{\Pi} \mathbf{w}^n / \delta t) \cdot \mathbf{N}_i \quad \forall i \in \mathcal{I}_u. \quad (21)$$

5.2. The algebraic coupling part

We first deal with the kinematic constraint (i.e., Eq. (16d)) which is represented by the matrix $\mathbf{C}$ and the vectors $\hat{\mathbf{b}}_g$ and $\mathbf{l}_\theta$.

Let $\mathbf{x}_i$ be the set of points of intersection of the fluid mesh $\mathcal{T}$ with Γ^n and the mid points between two intersection points (since we use a Lagrange $\mathcal{P}_2$ basis, see Fig. 5 and associated discussion on construction of the quadratic interpolant), and let $\mathbf{R}$ be the axis of rotation, then

$$\hat{r}_i = \sqrt{(\mathbf{x}_i - \mathbf{R})^T (\mathbf{x}_i - \mathbf{R})} \quad (22)$$

is the discrete counterpart of the radial coordinate r of the leaflet. Then the discrete counterpart of Eq. (16d) is given by

$$\mathbf{C}_{|ij} = \delta_{ij} \quad \forall (i, j) \in \mathcal{I}_g,$$

and $\mathbf{l}_\theta = \{\hat{l}_\theta^x, \hat{l}_\theta^y\}^T$ (where upper-scripts x and y denote the x and y components, respectively) with

$$\begin{cases} \hat{l}_{\theta|i}^x &= -(\hat{r}_i / \delta t) n_x^+ \\ \hat{l}_{\theta|i}^y &= -(\hat{r}_i / \delta t) n_y^+ \end{cases},$$

where $\mathbf{n}^+ = \{n_x^+, n_y^+\}^T$ (see Fig. 1). The right hand side is given by $\hat{b}_g = \{\hat{b}_g^x, \hat{b}_g^y\}^T$ with

$$\begin{cases} \hat{b}_{g|i}^x &= -(\hat{r}_i \theta^n / \delta t) n_x^+ \\ \hat{b}_{g|i}^y &= -(\hat{r}_i \theta^n / \delta t) n_y^+ \end{cases}.$$

To construct the discrete counterpart of the torque acting on the solid (see Eq. (5) or Eq. (16c)) we use a residual approach implemented as follows.

Following [33], we prolong Γ_h^n into $\Gamma_h^{n,e}$ (see Fig. 1, where $\Gamma_h^{n,e}$ is the discrete counter part of Γ^e) such that it divides Ω_h^n into two domains: $\Omega_h^{n,-}$ and $\Omega_h^{n,+}$ with $\Omega_h^n = \Omega_h^{n,-} \cup \Omega_h^{n,+} \cup (\Gamma_h^n \cup \Gamma_h^{n,e})$. Because we use a conforming method we can multiply the first equation of the system of equations (12) by functions $\mathbf{N}_i$ with support on Γ_h^n . Then, by integrating by parts on each subdomain $\Omega_h^{n,-}$ and $\Omega_h^{n,+}$ and finally summing both equations we obtain that

$$\begin{aligned} & \int_{\Omega_h^n} \rho_f (\mathbf{u}_h^{n+1}/\delta t + \Pi \mathbf{w}^n \cdot \nabla \mathbf{u}_h^{n+1}) \cdot \mathbf{N}_i + \int_{\Omega_h^n} \mu \nabla^s \mathbf{u}_h^{n+1} : \nabla \mathbf{N}_i - \int_{\Omega_h^n} \operatorname{div}(\mathbf{N}_i) p_h^{n+1} \\ & - \int_{\Omega_h^n} (\mathbf{f}^{n+1} + \rho_f \Pi \mathbf{w}^n / \delta t) \cdot \mathbf{N}_i = \int_{\Gamma^n \cup \Gamma_h^{n,e}} \llbracket p_h^{n+1} \mathbf{n}^+ - \mu(\nabla^s \mathbf{u}_h^{n+1}) \mathbf{n}^+ \rrbracket \cdot \mathbf{N}_i. \end{aligned} \quad (23)$$

Assuming the stress to be continuous over $\Gamma_h^{n,e}$, that is $\llbracket p_h^{n+1} \mathbf{n}^+ - \mu(\nabla^s \mathbf{u}_h^{n+1}) \mathbf{n}^+ \rrbracket_{\Gamma_h^{n,e}} = \mathbf{0}$, and using Eq. (23) in the velocity and pressure definition of Eqs. (17) and (18) with the notation of Eq. (19), we deduce that the nodal hydrodynamic force on the leaflet is given by

$$\hat{\mathbf{f}}_f = \bar{\mathbf{A}}_u \hat{\mathbf{u}} + \bar{\mathbf{A}}_l \hat{\mathbf{g}} + \bar{\mathbf{D}}_l^T \hat{p} - \hat{\mathbf{b}}_l \quad (24)$$

where the matrices are given as in Eq. (20) but for all $(i, j) \in \mathcal{I}_g \times \mathcal{I}_u$, $(i, j) \in \mathcal{I}_g \times \mathcal{I}_g$, $(i, j) \in \mathcal{I}_g \times \mathcal{I}_p$, respectively, and $\hat{\mathbf{b}}_l$ is given as in Eq. (21) but for all $i \in \mathcal{I}_g$.

Now, using Eq. (7), we obtain the nodal coupling load:

$$\hat{f}_s = \left(\bar{\mathbf{A}}_u \hat{\mathbf{u}} + \bar{\mathbf{A}}_l \hat{\mathbf{g}} + \bar{\mathbf{D}}_l^T \hat{p} - \hat{\mathbf{b}}_l \right) \cdot \mathbf{n}^+. \quad (25)$$

There is a slight abuse of notation in Eq. (25), where the symbol $\cdot$ designates the scalar product with respect to the x and y components of the vector $\bar{\mathbf{A}}_u \hat{\mathbf{u}} + \bar{\mathbf{A}}_l \hat{\mathbf{g}} + \bar{\mathbf{D}}_l^T \hat{p} - \hat{\mathbf{b}}_l$.

In Eq. 25 writing the radial coordinate of the leaflet (i.e., r) defined in Section 2.2 (see Fig. 2) as an isoparametric representation based on the nodal values $\hat{r}_i$ (see Eq. (22)) and the finite element basis $\mathbf{N}$ (where we only take the scalar part), we have

$$r = \sum_{i \in \mathcal{I}_g} \hat{r}_i N_i. \quad (26)$$

Then, using Eq. (26) in Eq. (5) we obtain

$$\tau = \sum_{i \in \mathcal{I}_g} \hat{r}_i \int_{\Gamma} f_s N_i, \quad (27)$$

with

$$\int_{\Gamma} f_s N_i = \hat{f}_{s|i}. \quad (28)$$

Now, by using Eq. (25) with Eq. (28) in Eq. (27), we conclude that

$$\tau^{n+1} = \hat{r}^T [(\bar{\mathbf{A}}_u \hat{\mathbf{u}} + \bar{\mathbf{A}}_l \hat{\mathbf{g}} + \bar{\mathbf{D}}_l^T \hat{p} - \hat{\mathbf{b}}_l) \cdot \mathbf{n}^+]. \quad (29)$$

Using Eq. (29) with Eq. (16e) and the matrices as defined in Eq. (19), we deduce that

$$\begin{cases} \mathbf{l}_u &= -\hat{r}^T (\bar{\mathbf{A}}_u \cdot \mathbf{n}^+) \\ \mathbf{l}_g &= -\hat{r}^T (\bar{\mathbf{A}}_l \cdot \mathbf{n}^+) \\ \mathbf{l}_p &= -\hat{r}^T (\bar{\mathbf{D}}_l^T \cdot \mathbf{n}^+) \end{cases}.$$

5.3. The algebraic solid part

Substituting Eq. (16e) in Eq. (19) we obtain

$$s = I/\delta t^2 + \kappa,$$

and, for the right hand side using the notation in Eq. (29), we have

$$\lambda = -\hat{r}^T (\hat{\mathbf{b}}_l \cdot \mathbf{n}^+) + I/\delta t^2 (2\theta^n - \theta^{n-1}) + \kappa\theta_0.$$

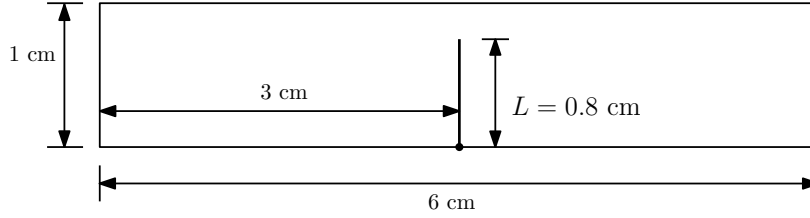


Figure 6. Test 1: Geometric parameters.

6. NUMERICAL EXPERIMENTS

In this section we perform four numerical tests:

- *Test 1: validation.* We consider a test performed in References [33] and [34]. This test originally aimed at studying some effects on the flow of blood in the case of aortic valve stenoses. We use the $\mathbf{P}_2^+/P_1$ for stability reasons as explained in Section 4.2.
- *Test 2: massless leaflet without a rotational spring and effects of the triangles anisotropy on stability of the mixed elements.* The leaflet is massless and there is no rotational spring attached. We compare the results with the approximated asymptotic solution provided in [35]. We perform the test with all four finite element schemes and we discuss the effects of the distortion of the elements on the inf-sup constant, as well as on the conditioning of the linear system.
- *Test 3: massive leaflet without a rotational spring.* We take into account the inertial effects by setting various values of the moment of inertia of the leaflet and without a rotational spring. Because we now focus on the motion of the leaflet this test is only performed with the $\mathbf{P}_2^+/P_1$ element.
- *Test 4: massive leaflet with a rotational spring.* We take into account the inertial effects and the effects of a rotational spring by analyzing its motion. For the same reason as with Test 3, we perform the test only with the $\mathbf{P}_2^+/P_1$ element.

It is interesting to observe that in general only few benchmarks exist in the literature for fluid-structure interaction problems with a rigid leaflet. For instance, we recall [33] with the velocity-pressure Navier-Stokes formulation, and [36] and [37] using the streamfunction-vorticity formulation of the Navier-Stokes equations. Therefore, it is difficult to perform a comparison, since different formulations come with different boundary conditions, and thus a different behavior of the flow. However, we think that even a qualitative comparison is important to assess reliability of the results in absence of analytical solutions.

We use in these tests the CSG unit system.

6.1. Test 1: validation

The present test is mutated from test case 1 of [33]. The test mimics aortic valve stenosis. For this reason, the leaflet in this test is artificially blocked at 90° and for 3 cases, at: 10° , 20° , 45° . In this test we do not get in details in the clinical conclusions obtained on the distortion of the flow from the aortic valve stenosis. Rather, we focus on comparing the results obtained with respect to those in [33] to “validate” our code.

The mesh is defined on the rectangular domain $[-3 \text{ cm}, 3 \text{ cm}] \times [0, 1 \text{ cm}]$. The leaflet has a length of 0.8 cm with an initial angle of $\pi/2$ and $\{0, 0\}^T$ is its point of rotation. The geometry of the problem is also described in Fig. 6

The moment of inertia of the leaflet is $I = 0.51 \text{ g} \cdot \text{cm}^2$. The fluid density and viscosity are $\rho_f = 1 \text{ g} \cdot \text{cm}^{-3}$ and $\mu = 0.03 \text{ g} \cdot \text{s}^{-1}$, respectively. The simulation is performed from 0 s to 4 s

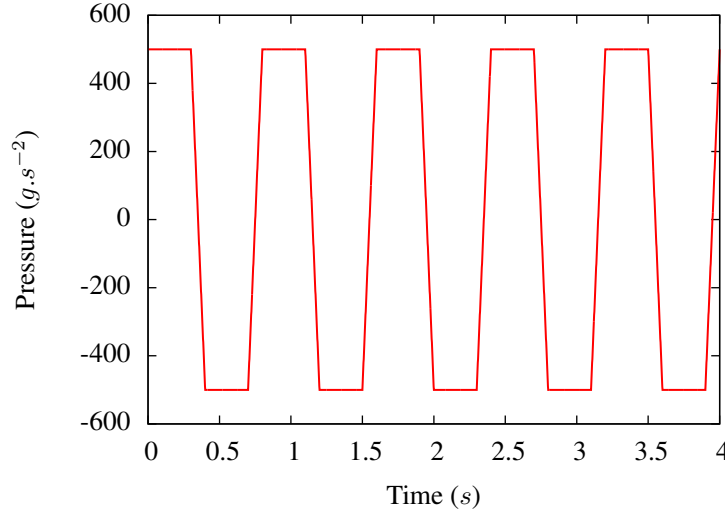


Figure 7. Test 1: Pressure inflow condition defined in Eq. 30.

with $\delta t = 4/500$ s. The problem is pressure-driven, where the boundary condition on $x = 3$ cm is a “do-nothing” condition (that is $p\mathbf{n} - \mu(\nabla\mathbf{u})\mathbf{n} = \mathbf{0}$), and the pressure on $x = -3$ cm is the periodic function of period 0.8 s defined in Eq. (30):

$$p_{in}(t) = \begin{cases} 500 \text{ g} \cdot \text{s}^{-2} & 0 \leq t < 0.3, \\ 5000(0.7 - 2t) \text{ g} \cdot \text{s}^{-2} & 0.3 \leq t < 0.4, \\ -500 \text{ g} \cdot \text{s}^{-2} & 0.4 \leq t < 0.7, \\ 5000(2t - 1.5) \text{ g} \cdot \text{s}^{-2} & 0.7 \leq t < 0.8. \end{cases} \quad (30)$$

On the wall, i.e., at $y = \{0, 1\}$ cm, a no-slip condition is enforced (i.e. $\mathbf{u} = \mathbf{0}$).

The fluid mesh is a 139×57 “union jack” discretization of the domain. We point out that the high number of elements is not driven by the fluid-structure problem but rather by stability issues due to Neumann boundary conditions at the inflows. In addition, it is well known that using $\nabla^s \mathbf{u}$ with do-nothing boundary conditions induces some undesired spurious effects (see, e.g., [38] or [39]) so we retain $\nabla \mathbf{u}$ for this test, i.e., the Cauchy tensor for the viscous term.

The results are provided in Fig. 8, where a good qualitative behavior as compared with [33] is clearly observed.

6.2. Test 2: massless leaflet without a rotational spring and effects of the triangles anisotropy on the mixed elements

In this problem the leaflet has no mass and there is no spring attached to it, and thus Eq. (4) reduces to

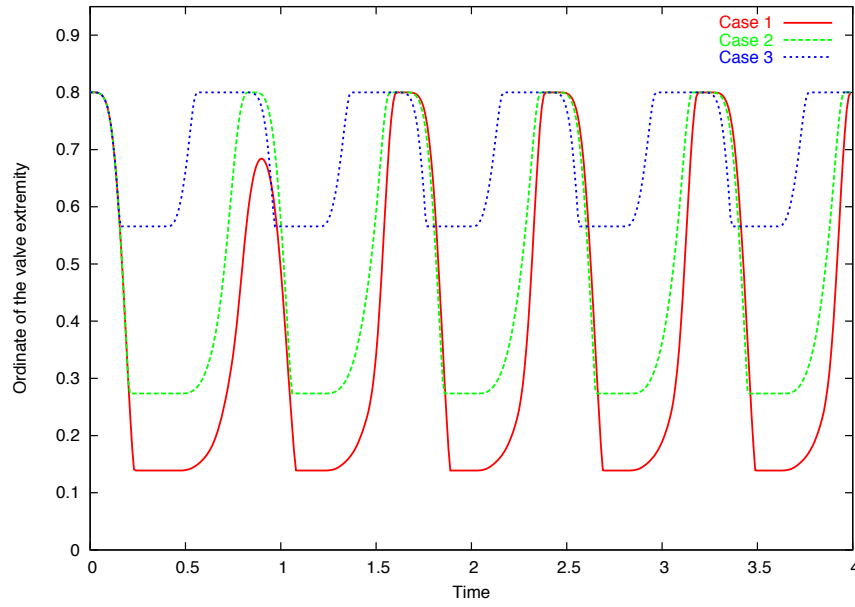
$$\tau(t) = 0.$$

This problem has been studied in [35] where an asymptotic analysis of the valve opening without vortex shedding was presented. An approximation of the asymptotic solution of the motion of the rigid leaflet problem, assuming that no vortices are generated behind the leaflet, is given by (see [35]):

$$\frac{d\theta}{dt} = \frac{2u_x(t) \sin(\theta)}{(\sin(\theta) - 2)}, \quad (31)$$

with $u_x(t) = (1 - \cos(\pi t/T))/2$.

The computational domain is the rectangle $[-1 \text{ cm}, 6 \text{ cm}] \times [0, 1 \text{ cm}]$. The length of the leaflet is $L = 0.999$ cm. The geometry of the problem is also presented in Fig. 9.



(a) Results extracted from [34].

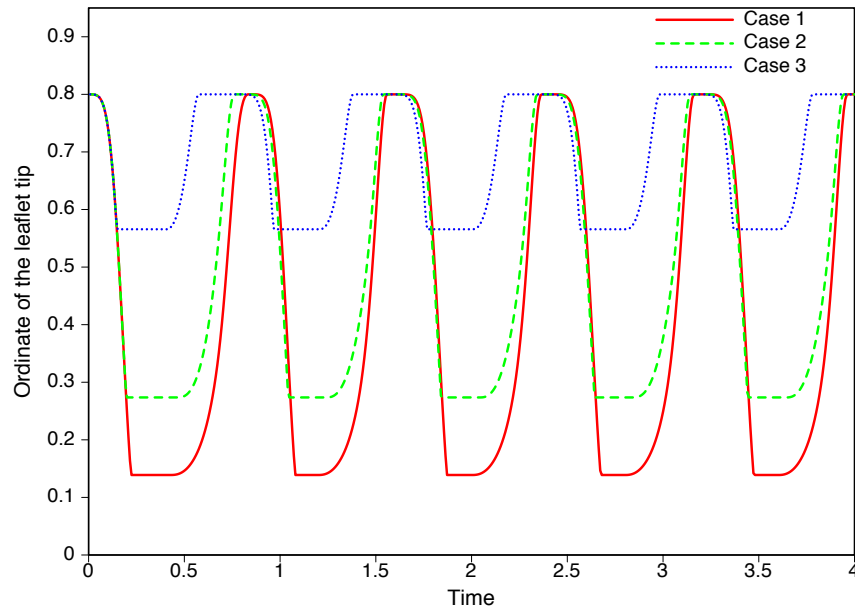
(b) Results from the present method using $\mathbf{P}_2^+/P_1$.

Figure 8. Comparison of the ordinate of the leaflet tip between [34] (see also [33]) and the present method. Case 1, Case 2, and Case 3, denote the motion of leaflet such that $\theta \in [10^\circ, 90^\circ]$, $\theta \in [20^\circ, 90^\circ]$, and $\theta \in [45^\circ, 90^\circ]$, respectively.

At the inflow $x = -1$ cm, the velocity is given by Eq. (32).

$$\mathbf{u}(x, y, t) = \{(1 - \cos(\pi t/T))/2, 0\}^T. \quad (32)$$

The no-slip boundary condition on $y = 0$ is applied. The do-nothing (or “stress-free”) boundary condition is applied on $x = 6$ cm (that is $p\mathbf{n} - \mu(\nabla^s \mathbf{u})\mathbf{n} = \mathbf{0}$). A symmetry boundary condition is imposed on $y = 1$ cm, i.e., only normal velocity components are set to zero and tangential ones are set free (see, e.g., [40]). The initial condition is $\mathbf{u}_i = \mathbf{0}$. The time period is set to $T = 10$ s. The

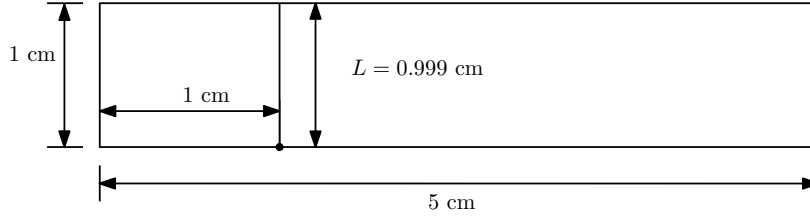


Figure 9. Test 2: Geometric parameters.

viscosity is set to $\mu = 0.001 \text{ g} \cdot \text{cm}^{-2}$. We use a 127×19 discretization of the fluid domain, the time step is set to $\delta t = 10/128 \text{ s}$ and the simulation is performed from 0 to 10 s.

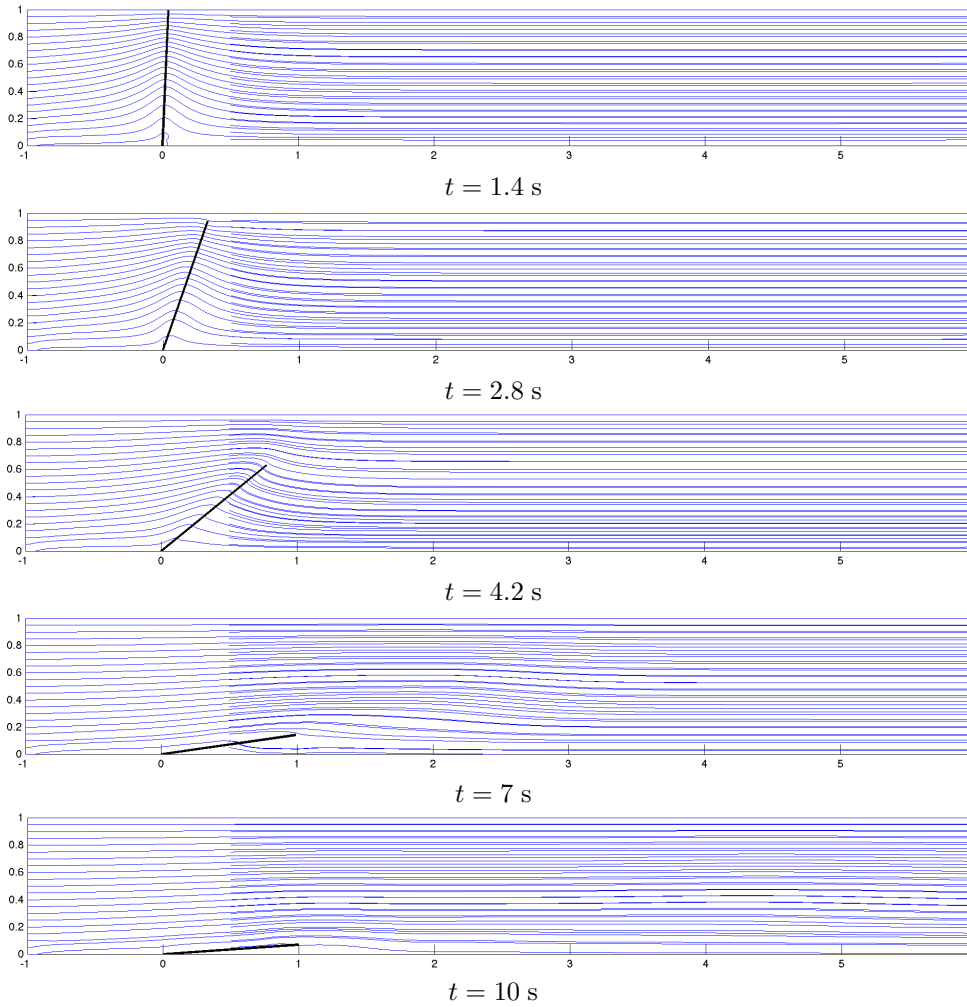


Figure 10. Test 2: Streamlines snapshots, for a massless leaflet without a rotational spring attached and using $\mathbf{P}_2^+/P_1$. It can be observed that no vortex shedding is present.

On Fig. 10 we can observe that there are no vortices shedding. On Fig. 11, we compare the solution obtained with the four finite element schemes and the approximated asymptotic solution in Eq. (31) and we observe that the motion of the leaflet is quite in accordance with the asymptotic analysis. We deduce that the solution proposed in [35] may be suitable for our problem. We recall

that Eq. (31) is only an approximation of the asymptotic solution. We can see that all four finite element schemes provide a similar displacement of the leaflet.

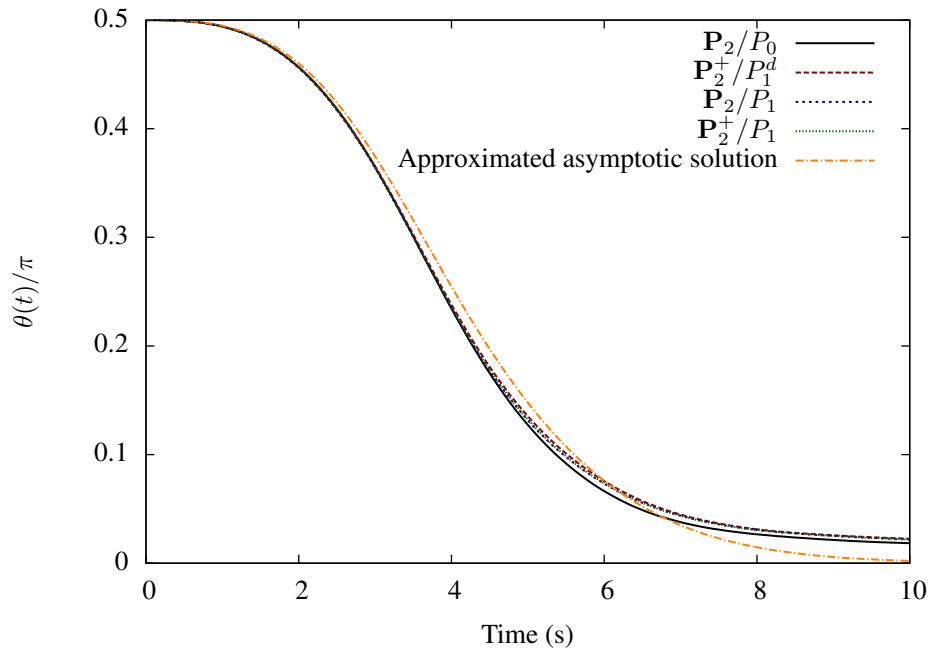


Figure 11. Test 2: Confrontation of the leaflet motion for the various elements with respect to the solution of Eq. (31).

As we pointed out in Sec. 4 and as seen in Fig. 12 the mesh is not isotropic, leading to possible issues with the inf-sup stability and to condition number issues. In Fig. 13 we report the condition number of the linear system (see Eq. (19)) for the four finite element schemes, whose value is affected by the inf-sup constant β . We recall that for the steady incompressible Stokes problem, the conditioning of its Schur complement scales as β^{-2} . Since β is affected by the distortion of the elements, the condition number of the linear system will also be affected. Inf-sup unstable finite elements are therefore expected to show a much worse conditioning than stable elements. We also recall that the conditioning of the Schur complement is also affected by the conditioning of the pressure mass matrix (see Proposition 4.47 in [27]). In particular:

1. For the $\mathbf{P}_2/P_0$ element we see in Fig. 13 two peaks indicating ill-conditioning, of one order of magnitude higher than the highest peaks using $\mathbf{P}_2/P_1$ and $\mathbf{P}_2^+/P_1$. Indeed, we can observe in Fig. 15, which represents the pressure field at the time of the first peak, spurious modes (a zoom on the culprit is showed in Fig. 16).
2. For the $\mathbf{P}_2^+/P_1^d$ element, the associated linear system is very ill-conditioned with respect to the other elements, indicating that the inf-sup constant (and/or the conditioning of the associated pressure mass matrix) is much more sensitive to mesh distortion. In Fig. 15 we can observe some spurious modes on the pressure field at the time of one of the peaks present in Fig. 13. A zoom on the spurious modes is presented in Fig. 16.
3. For the $\mathbf{P}_2/P_1$ element, as pointed out in Sec. 4.2, the inf-sup constant may be very small on small elements in recessed corners with Dirichlet boundary conditions enforced on two boundary edges but such a situation is very unlikely to occur here, leading to a stable scheme. Indeed, no spurious modes are visible in Fig. 15.
4. The $\mathbf{P}_2^+/P_1$ element is stable, as discussed in [29] and no spurious modes are visible in Fig. 15. The conditioning is of the same order as with $\mathbf{P}_2/P_1$.

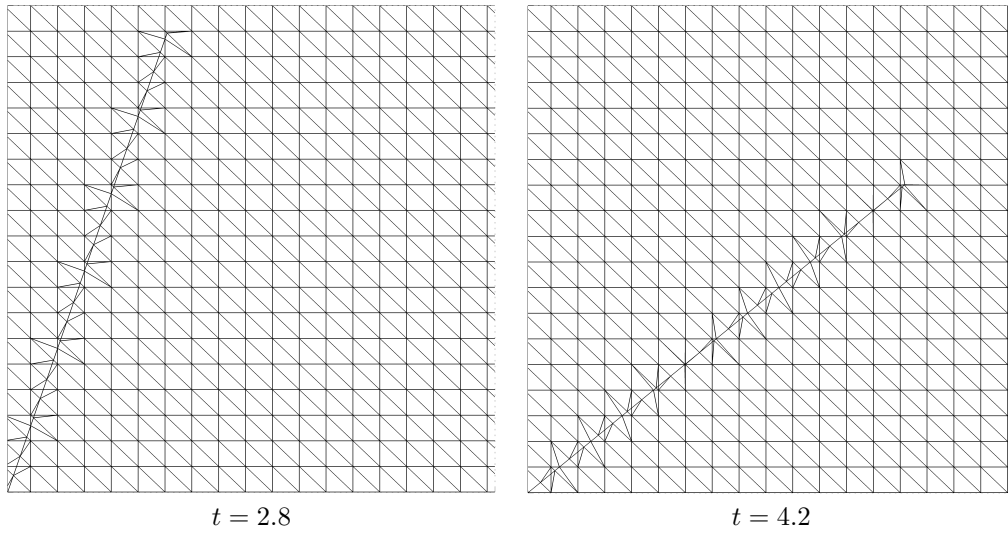
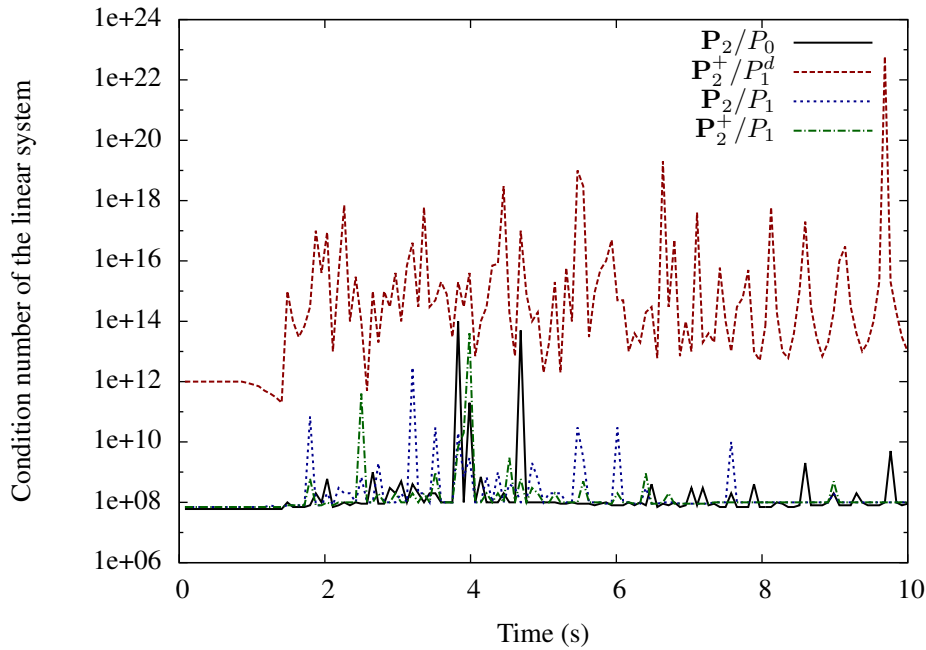
Figure 12. Test 2: Distortion of the mesh with the $\mathbf{P}_2^+/P_1$ element.

Figure 13. Test 2: Condition number of the linear system for all elements.

The stable schemes $\mathbf{P}_2/P_1$ and $\mathbf{P}_2^+/P_1$ show a much better conditioning than the inf-sup unstable schemes pointing out the importance of having inf-sup stable elements on distorted meshes.

Remark 3

The method employed in [13] uses the $\mathbf{P}_2^+/P_1^d$ element with a similar approach for the local refinement as performed here but maintaining good element ratios by a smoothing procedure. However, our results show that extending the method presented in [13] to very stretched elements is not straightforward since there are inf-sup stability issues with $\mathbf{P}_2^+/P_1^d$ on anisotropic meshes. We thus show the necessity of the smoothing procedure with $\mathbf{P}_2^+/P_1^d$.

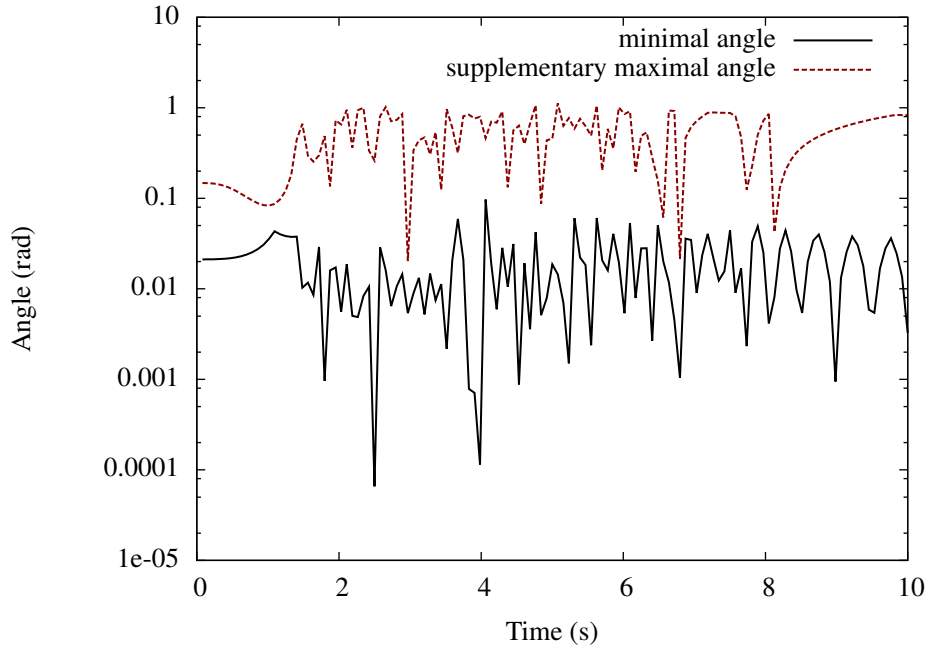


Figure 14. Test 2: Minimal and supplementary maximal angles when using $\mathbf{P}_2^+/P_1$.

In, e.g., [24], [41], and [21], it was showed that the Lagrange H^1 interpolant requires a maximal angle condition (i.e., the maximum angle of every triangle must be strictly below π) to be satisfied such that it is bounded. This condition is much less restrictive than the minimal angle condition (i.e., required for a mesh to be isotropic) discussed in, e.g., [42]. However, we do not prove that the remeshed elements satisfies the maximal angle condition, mainly due to concerns with the refinement of the tip of the leaflet. Nevertheless, we point that in [43] it was showed that even if the maximal angle condition is not satisfied, the finite element method may converge, though the Lagrange H_1 interpolant does not. However, we wish to compare the minimal and maximal angles obtained with the present method. For this reason, in Fig. 14, we confront the minimal and the supplementary maximal angles of the remeshed triangles of the mesh during the simulation for Test 2 using the $\mathbf{P}_2^+/P_1$ element. It clearly appears that the largest angle has a much larger bound away from π than the smallest angle away from 0. That difference is of an order of magnitude. The largest difference between the minimal and supplementary maximal angles is of two orders of magnitude.

In Fig. 13 we compare the conditioning of the linear system associated to $\mathbf{P}_2^+/P_1$ and the minimal and maximal angles and we observe that only very small minimal angles have an impact on the conditioning of the system. Indeed, we can observe that the two highest condition numbers correspond to two very small angles, at time $t = 2.5$ s and $t = 3.98$ s.

6.3. Test 3: inertial effects without a rotational spring

In this problem, inspired by [37], we study the sole effect of inertia. The domain under consideration is $[-3 \text{ cm}, 3 \text{ cm}] \times [0, 4 \text{ cm}]$. The length of the leaflet is $L = 1 \text{ cm}$. The geometry of the problem is also described in Fig. 17.

A no-slip boundary condition on $y = 0 \text{ cm}$ and a symmetric boundary condition on $y = 4 \text{ cm}$ are applied, while the inflow boundary condition is imposed on $x = \pm 3 \text{ cm}$ and it is given by

$$\mathbf{u}(x, y, t) = \{\sin(2\pi t), 0\}^T. \quad (33)$$

The fluid density is set to $\rho_f = 1.0 \text{ g} \cdot \text{cm}^{-3}$ and the viscosity to $\mu = 0.005 \text{ g} \cdot \text{s}^{-1}$. The initial condition is set by $\mathbf{u}_i = \mathbf{0}$. The mesh size is 113×63 and the time step is set at $\delta t = 0.001 \text{ s}$ with

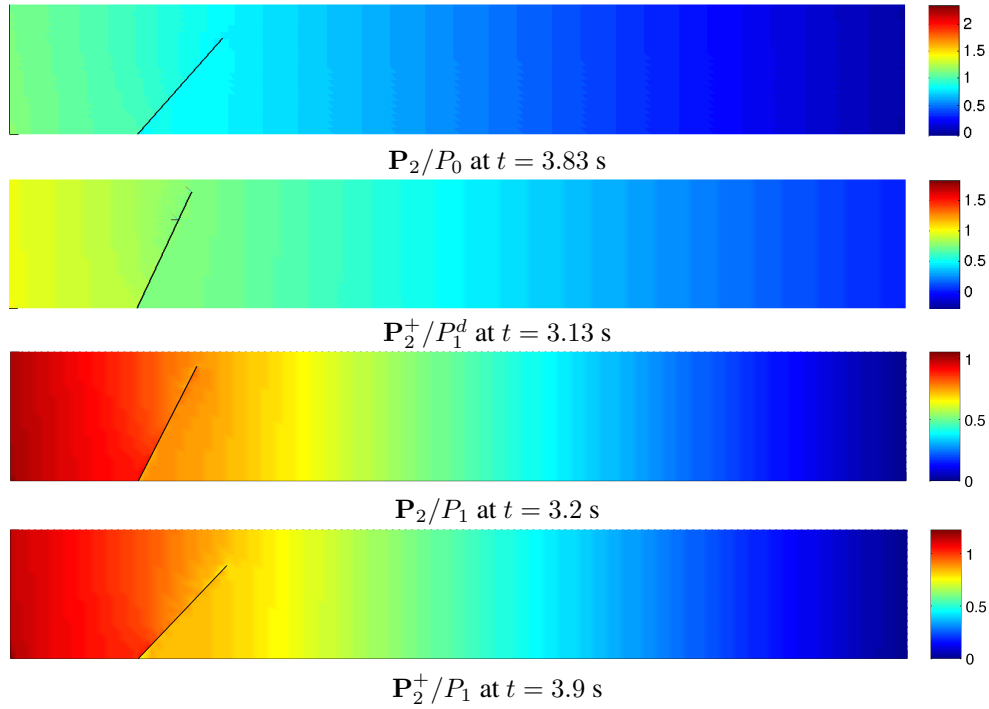


Figure 15. Test 2: Pressure field of the elements, at times corresponding to ill-conditioned linear systems. It shows the inf-sup stability issue for the $\mathbf{P}_2/P_0$ and $\mathbf{P}_2^+/P_1^d$ elements. (For observation of the spurious modes, colors are necessary. The reader is referred to the web version of this article.)

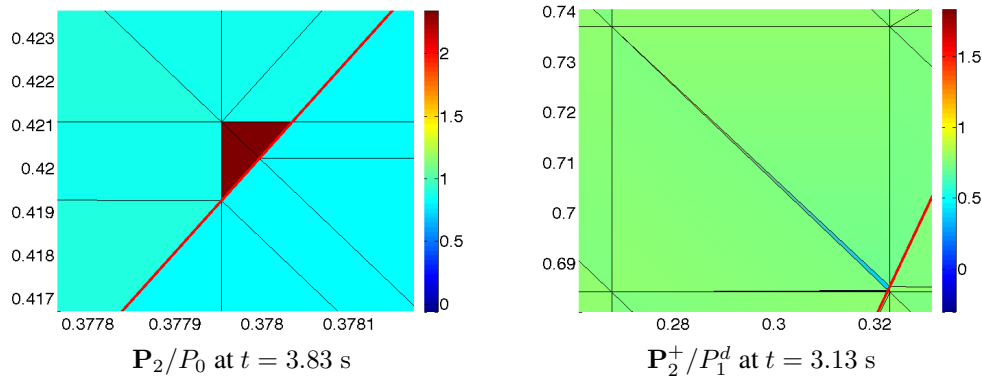


Figure 16. Test 2: Zoom on an effect of element distortion: presence of spurious modes due to inf-sup instabilities. The leaflet is depicted in red. (For observation of the spurious modes, colors are necessary. The reader is referred to the web version of this article.)

a time range from 0 to 5 s. Various values for the mass of the leaflet are considered, as we vary the moment of inertia of the leaflet (see Eq. (4)) where $I = 0.1, 0.5, 1, 2, 10 \text{ g} \cdot \text{cm}^2$.

We can observe from Fig. 18 that it is more difficult to set in motion the leaflet as it is heavier. We may also see that the average angle of the leaflet over a period is not constant in time, in particular when the leaflet is light.

6.4. Test 4: inertial effects with a rotational spring

In this problem, we propose to test the complete system, with different values for κ , the stiffness of the spring. We choose $\kappa = 0, 1, 5, 10 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$.

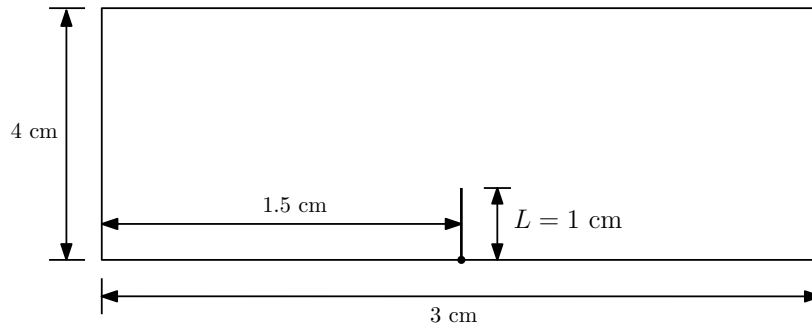


Figure 17. Test 3: Geometric parameters

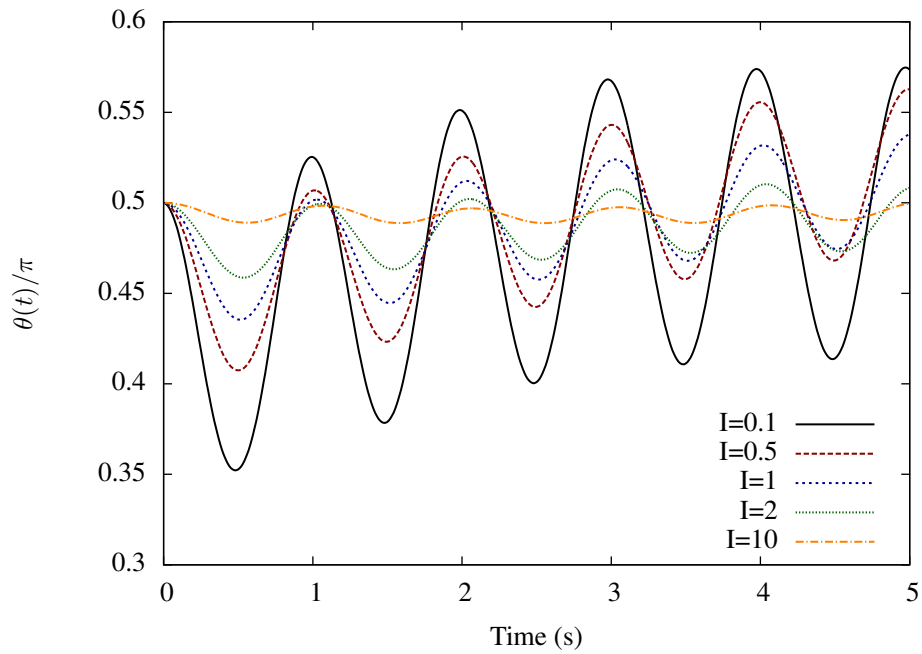
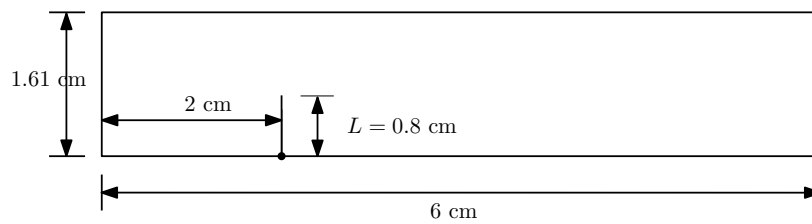
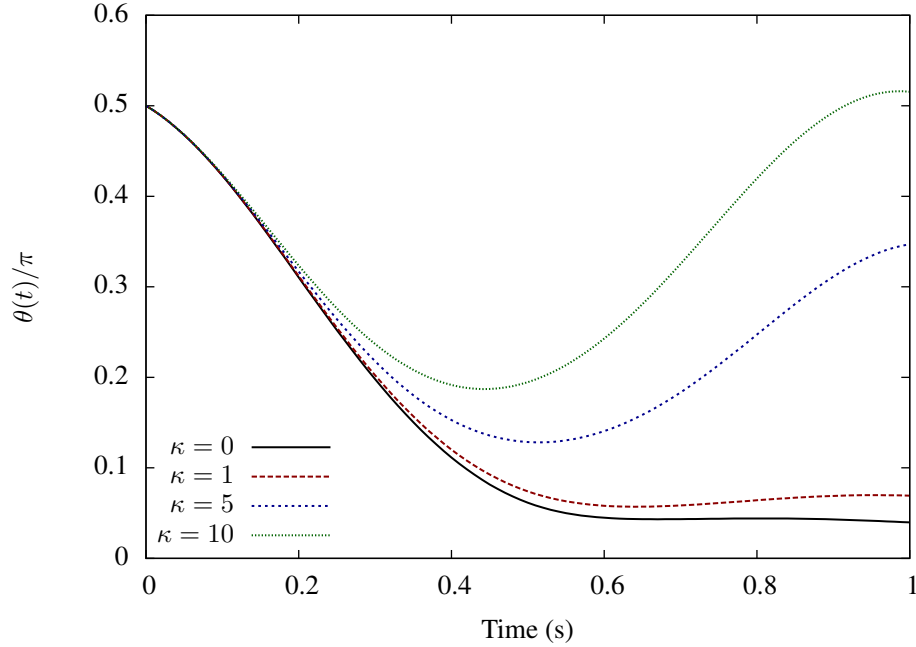
Figure 18. Test 3: test for five values of I . No rotational spring is attached to the leaflet.

Figure 19. Test 4: Geometric parameters

The fluid domain has the following dimensions: $[-2 \text{ cm}, 6 \text{ cm}] \times [0 \text{ cm}, 1.61 \text{ cm}]$, over which we use a 179×33 discretization. The length of the leaflet is $L = 0.8 \text{ cm}$. The geometry of the problem is depicted in Fig. 19.

Figure 20. Test 4: for different values of κ .

The moment of inertia of the leaflet is $I = 0.51 \text{ g} \cdot \text{cm}^2$. For the parameters of the fluid we choose $\mu = 0.03 \text{ g} \cdot \text{s}^{-1}$ and $\rho_f = 1.0 \text{ g} \cdot \text{cm}^{-2}$. The time step is set at $\delta t = 1/100 \text{ s}$ and the time range is from 0 to 1 s. The inflow boundary condition on $x = -2 \text{ cm}$ is given by the following equation:

$$\mathbf{u}(x, y, t) = \{5(\sin(2\pi t) + 1.1), 0\}^T. \quad (34)$$

We impose the no-slip boundary condition on $y = 0 \text{ cm}$ and $y = 1.61 \text{ cm}$, and a do-nothing outflow on $y = 6 \text{ cm}$. The initial condition is $\mathbf{u}_i = \mathbf{0} \text{ cm} \cdot \text{s}^{-1}$. The parameters for this problem are inspired from [44] and [33].

As a consistency check with expectations from physics, we can observe from Fig. 20 that, for a low stiffness, the spring is pushed much further than for the case with a higher stiffness. We may also notice that when the flow slows down (after $t = 0.25 \text{ s}$), for the case with $\kappa = 5 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$ and $\kappa = 10 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$ the spring is pushed back by the energy accumulated in the spring during the spring compression phase, while for $\kappa = 1 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$ only a slight return is observed, and none for the case $\kappa = 0 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$.

We may observe from Figs. 21 and 22 that at time $t = 0.25$ (i.e., at the inflow velocity peak) a large pressure jump is present across the leaflet, stressing the importance of allowing the pressure to be discontinuous across the leaflet. At times $t = 0.25$ and $t = 0.75$ a clear pressure jump across the leaflet is visible, without any spurious oscillations that may have resulted from continuous pressure and/or continuous velocity derivatives across the structure. At that time the inflow velocity decelerates and the leaflet starts to push back. This effect is particularly visible for $\kappa = 5$ and $\kappa = 10$. At time $t = 0.75$ (i.e., lowest velocity inflow) we can observe that a second vortex has been generated from the push back of the leaflet by the rotational spring.

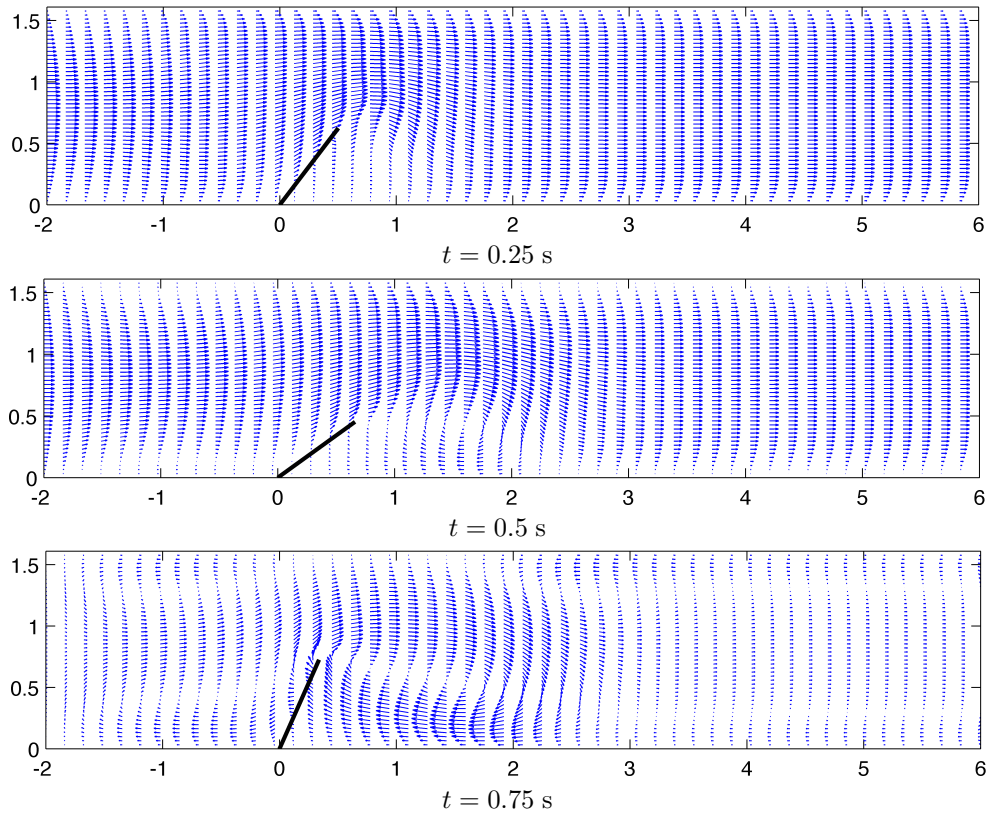


Figure 21. Test 4: Normalized velocity field using $\mathbf{P}_2^+/P_1$ with $\kappa = 10 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$ at various time snapshots.

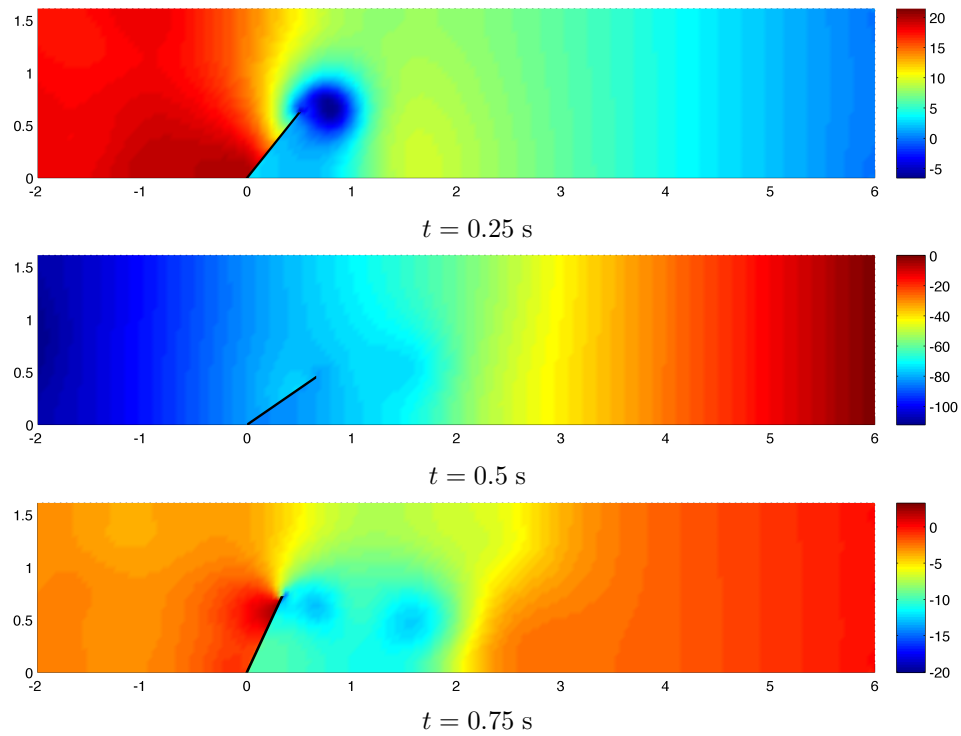


Figure 22. Test 4: Pressure field using the $\mathbf{P}_2^+/P_1$ element with $\kappa = 10 \text{ dyn} \cdot \text{cm} \cdot \text{rad}^{-1}$ at various time snapshots.

7. CONCLUSIONS

In this work we have presented an “immersed” type method based on a locally anisotropic remeshing strategy for thin structures, with an application to the case of a hinged rigid leaflet. The results are presented for two-dimensional problems. The method relies on remeshing only elements that are cut by the immersed structure such that the vertices of the triangles of the original mesh remain fixed during the simulation. Furthermore, it is possible to impose the kinematic constraints strongly and to build finite element spaces with a discontinuous pressure and discontinuous velocity derivatives across the leaflet in a fairly easy way.

An issue of the method is the presence of distorted elements near the structure. It is well known that anisotropic elements are allowed within the finite element method to deal with incompressibility. However, since we employ the mixed finite element method the fulfillment of an inf-sup condition is required, which may not be satisfied for anisotropic meshes even it is for isotropic ones.

In the present article we used different mixed finite element schemes, with both continuous (but discontinuous across the leaflet) and element-wise discontinuous pressures and continuous velocities. More precisely, we tested the $\mathbf{P}_2/P_0$, the $\mathbf{P}_2^+/P_1^d$, the $\mathbf{P}_2/P_1$, and the $\mathbf{P}_2^+/P_1$, mixed finite elements. We first validated the fluid-structure algorithm with results from the literature. We then compared the various mixed finite elements emphasizing the inf-sup stability and the conditioning of the linear system.

We showed inf-sup issues and strong conditioning issues for the $\mathbf{P}_2/P_0$ element and in particular for the $\mathbf{P}_2^+/P_1^d$ element, while both the $\mathbf{P}_2/P_1$ and $\mathbf{P}_2^+/P_1$ elements showed inf-sup stability in the considered cases. Because some elements are distorted, all finite elements had conditioning issues but stable elements show a much better conditioning than unstable ones, as expected from the theory. Moreover, we observed that the inf-sup condition issue does not have a major impact on the behavior of the leaflet, since spurious modes are localized on very small elements.

The fact that the leaflet is rigid may have an important role in that result. It was also observed the importance of having a discontinuous pressure field across the leaflet.

Further work will deal with conditioning issues (see, e.g., [45]), and to three-dimensional problems. For 3D problem, we have in mind the recently developed Virtual Element Method (VEM) (see, e.g., [46] and [47]). Since, the locally anisotropic remeshing strategy may be cumbersome in 3D due to sliver tetrahedra. The VEM provides instead a framework robust with distorted elements and allows avoiding the remeshing since the VEM works with arbitrary polyhedra.

8. ACKNOWLEDGMENTS

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