Fast Training of Implicit Networks with Applications in Inverse Problems

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What are Inverse Problems?

Inverse problems consist of recovering a signal x (e.g. an image, a parameter of a PDE, etc.) from indirect, noisy measurements d. This measurement process is usually modeled as an operator A, satisfying the following:

$$d = Ax + \text{"};$$

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Our task deals with image deblurring, i.e.,

$d \; 2 \; \mathrm{R}^{n \;\; n}$: blurred image with noise

$x \;\; 2 \; \mathrm{R}^{n \;\; n}$: original image

$\text{"} \; 2 \; \mathrm{R}^{n \;\; n}$: random noise (unknown) in $\mathrm{R}^{n \;\; n}$

From a Classical Approach

Direct Inverse:

$$d = Ax \quad + " =) \quad x \quad = A \quad ^1 d \quad A \quad ^1 "$$

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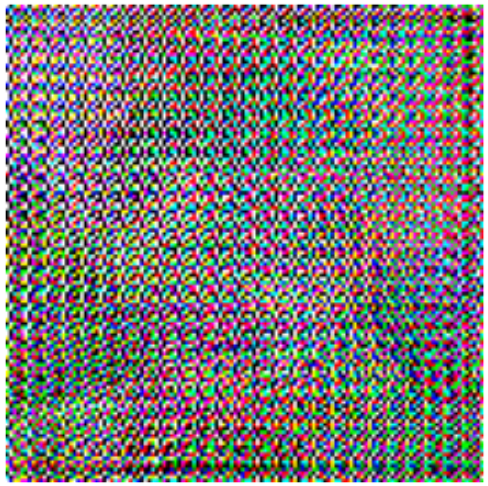
Original Image

Blurred Noisy Image

Apply Inverse



Classical Approach Cont.

Optimization: Formulate an optimization problem as follows:

$$x = \arg\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - d\|_{L^2}^2 + R(x)$$

where $R(x)$ is chosen based on prior knowledge of your data,
> 0 is a tunable parameter.

Classical Approach Cont.

Optimization: Formulate an optimization problem as follows:

$$x \quad = \arg\min_{x 2 \mathrm{R}^n \;\; n} \quad \frac{1}{2} jjAx \quad djj_{L^2}^2 + \quad R(x)$$

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E.g. $R(x) = 0 =) \;\; x \;\; = A \;\; {}^1 d$ when A invertible

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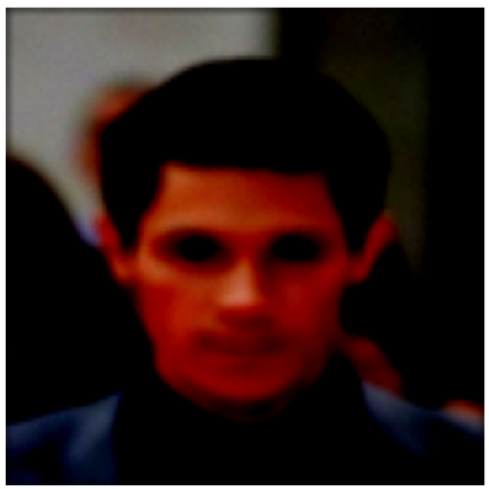
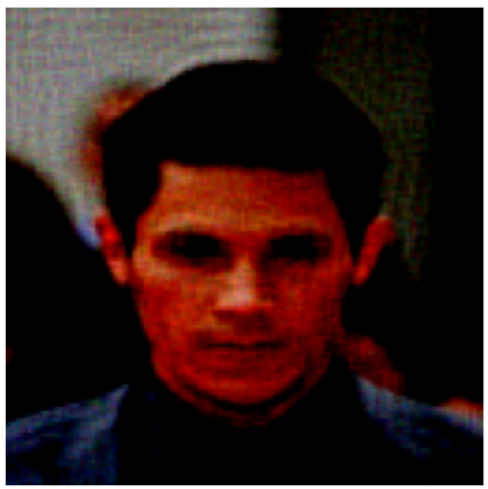
E.g. $R(x) = 0 \Rightarrow x = A^{-1}d$ when A invertible

E.g. Use gradient descent where $R(x) = \frac{}{2}\|x\|_{L^2}^2$:

Original Image Blurred Noisy Image Apply Gradient Descent

Implicit Deep Learning

Use dataset $f(d_i; x_i)g_{i=1}^m$ and physics (namely A)

[5] Davis Gilton, Gregory Ongie, and Rebecca Willett. "Deep equilibrium architectures for inverse problems in imaging." IEEE Transactions on Computational Imaging 7 (2021): 1123-1133.

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Mimic gradient descent [5], but replace $r_x R$ with a trainable network S_Θ: $8i$ and $0 \quad k \quad K \quad 1$,

$$x_i^{k+1} = x_i^k \underbrace{\quad r_x jjAx_i^k \quad d_i jj_{L^2}^2 + S_\Theta(x_i^k)}_{:=T_\Theta(x_i^k)}$$

where:

> 0 is step size

$T_\Theta(\ )$ is a layer of our neural network $N_\Theta(\ )$

K is the number of layers

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Problems: memory, choice of K

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- Convergent if $T_\Theta(\,)$ is a contraction mapping with Lipschitz constant $2\ [0;1)$, i.e,
$8y_1; y_2\ 2\ \mathrm{R}^{n^2}; jjT_\Theta(y_1) \quad T_\Theta(y_2)jj_{L^2} \qquad jjy_1 \quad y_2jj_{L^2}$

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Then, by *Banach xed-point theorem*, there exists $y \; 2 \; \mathrm{R}^{n^2}$ s.t. $T_\Theta(y\,) = y$

Implicit Backpropagation

Suppose we find a fixed point x^* for the previous update, i.e.,

$$x^* = T_\Theta(x^*)$$

With implicit differentiation,

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$$\frac{dx}{d\Theta} = \frac{dT_\Theta(x)}{dx}\frac{dx}{d\Theta} + \frac{@T_\Theta(x)}{@\Theta}$$

$$=) \quad I \quad \frac{dT_\Theta(x)}{dx} \quad \frac{dx}{d\Theta} = \frac{@T_\Theta(x)}{@\Theta} \tag{1}$$

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$$\Theta \quad \Theta \quad \frac{d`}{dx} \quad I \quad \frac{dT_\Theta(x)}{dx} \quad {}^{1} \frac{@T_\Theta(x)}{@\Theta};$$

where > 0 is the learning rate.

Implicit Backpropagation

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$$x = T_\Theta(x)$$

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$$\frac{dx}{d\Theta} = \frac{dT_\Theta(x)}{dx}\frac{dx}{d\Theta} + \frac{\partial T_\Theta(x)}{\partial \Theta}$$

$$\Rightarrow \quad I - \frac{dT_\Theta(x)}{dx}\frac{dx}{d\Theta} = \frac{\partial T_\Theta(x)}{\partial \Theta} \tag{1}$$

So the update rule of trainable parameters becomes:

$$\Theta \leftarrow \Theta - \eta \frac{d\ell}{dx}\left(I - \frac{dT_\Theta(x)}{dx}\right)^{-1}\frac{\partial T_\Theta(x)}{\partial \Theta},$$

where $\eta > 0$ is the learning rate.

Potential problem: solving (1) is highly nontrivial

Jacobian-Free Backpropagation (JFB)

Goal: alleviate memory requirement and avoid high computational cost.

[6] S. W. Fung, H. Heaton, Q. Li, D. McKenzie, S. Osher, and W. Yin (2021) Jfb: Jacobian-free back-propagation for implicit networks.

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Implicit Networks calculate the true gradient:

$$r_\Theta \ell = \frac{d\ell}{dx} \left(I - \frac{dT_\Theta(x)}{dx} \right)^{-1} \frac{\partial T_\Theta(x)}{\partial \Theta}$$

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JFB approximates the gradient: $p_\Theta = \frac{d\ell}{dx} \frac{\partial T_\Theta(x)}{\partial \Theta}$ which is a **descent direction** for ℓ if the following conditions hold (next slide) [6]:

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JFB Conditions

If:

i. T_Θ is contraction mapping with Lipschitz constant

ii. T_Θ is continuously differentiable w.r.t. Θ

iii. $M := \frac{@T_\Theta}{@\Theta}$ has full column rank

iv. M is well-conditioned, i.e., $(M^T M) < \frac{1}{}$

Then

$$p_\Theta = \frac{d`}{dx}\frac{@T_\Theta}{@\Theta}$$

is a **descent direction** for loss function `.

Numerical Experiments

Dataset: CelebA [7] (annotated celebrity faces)

[7] Liu, Ziwei, et al. "Deep learning face attributes in the wild." Proceedings of the IEEE international conference on computer vision. 2015.

Numerical Experiments

Generate blurred noisy images:

original image

blurred noisy image

PSNR = 21.57, SSIM=0.80

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Train with JFB

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Train with JFB

Preliminary results:

Loss v.s. number of SGD iterations Reconstructed image

PSNR = 25.69, SSIM=0.86

Future Work

- Train models using different learned optimization algorithms, e.g. Proximal Gradient Descent and Alternating Directions Method of Multipliers (ADMM)
- Experiment with fastMRI data[8]
- Compare training speeds and accuracy with Jacobian-based algorithms

[8] Zbontar, Jure, et al. "fastMRI: An open dataset and benchmarks for accelerated MRI." arXiv preprint arXiv:1811.08839 (2018).

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Thank you!

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Banach Fixed-Point Theorem

We demonstrate it in $\mathbb{R}^d$:
Suppose $T : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction map with Lipschitz constant $\alpha \in [0, 1)$.
$\forall x_0 \in \mathbb{R}^d$, iterate as follows:

$$
\begin{aligned}
x_1 &= T(x_0) \\
x_2 &= T(x_1) \\
&\vdots \\
x_{i+1} &= T(x_i) \\
&\vdots
\end{aligned}
$$

Then we obtain a sequence $\{x_m\}_{m \in \mathbb{N}}$

Banach Fixed-Point Theorem

Observe that

$$\begin{aligned}
\|x_2 - x_1\| &= \|T(x_1) - T(x_0)\| \le \|x_0 - x_1\| \\
\|x_3 - x_2\| &= \|T(x_2) - T(x_1)\| \le \|x_2 - x_1\| \le \|x_0 - x_1\| \\
&\vdots \\
\|x_{i+1} - x_i\| &\le \|x_0 - x_1\| \\
&\vdots
\end{aligned}$$

So $\lim_{m \to \infty} \|x_{m+1} - x_m\| \le \lim_{m \to \infty} \|x_0 - x_1\| = 0$

We also know that $0 \le \lim_{m \to \infty} \|x_{m+1} - x_m\|$.

$\Rightarrow \lim_{m \to \infty} \|x_{m+1} - x_m\| = 0$ by Squeeze Theorem

Banach Fixed-Point Theorem

By *Triangular Inequality*,

$$
\begin{aligned}
jjx_{m+k} \quad x_m jj \quad & jjx_m \quad x_{m+1} jj + jjx_{m+1} \quad xm + kjj \\
& jjx_m \quad x_{m+1} jj + (jjx_{m+1} \quad x_{m+2} jj + jjx_{m+2} \quad x_{m+k} jj) \\
& \vdots \\
& jjx_m \quad x_{m+1} jj + jjx_{m+1} \quad x_{m+2} jj + \\
& + jjx_{m+k\ \ 2} \quad x_{m+k\ \ 1} jj + jjx_{m+k\ \ 1} \quad x_{m+k} jj
\end{aligned}
$$

By *Squeeze Theorem* again, $fx_m g_{m2\mathrm{N}}$ is a Cauchy sequence, which is equivalent to $\lim_{m\ !\ 1} x_m = x$ exists.

$=)\ x\ = T(x\ )$ is a fixed point. □

Proximal Gradient Descent

With a function $h(\ )$, we can define a proximal operator

$$\text{prox}_h(x) = \arg\min_u \frac{1}{2} jju \quad xjj_{L^2}^2 + h(u)$$

Then the updating rule becomes:

$$x^{k+1} = \text{prox}_{h;} \quad x^k \qquad r_x jjAx^k \quad djj_{L^2}^2$$

We can replace this prox_h with a trainable network $R_\Theta(\ )$:

$$x^{k+1} = R_\Theta \ \ x^k \qquad r_x jjAx^k \quad djj_{L^2}^2$$