RIGIDITY OF LORENTZIAN METRICS WITH THE SAME SCATTERING RELATIONS

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Abstract. Consider small metric deformations of the Minkowski space. We prove that the scattering relation of null geodesics between two Cauchy surfaces uniquely determines the metric perturbation up to gauge obstructions.

1. Introduction

We study a scattering rigidity problem for Lorentzian manifolds. More precisely, let $z = (t, x^1, x^2, x^3)$ be coordinates for $\mathbb{R}^{3+1}$. Let g be a globally hyperbolic Lorentzian metric such that each hypersurface $\mathcal{M}_t \doteq \{t\} \times \mathbb{R}^3, t \in \mathbb{R}$ is a Cauchy surface and every complete null geodesic intersects $\mathcal{M}_t$ at one point. For $T > 0$, we consider null geodesics γ from $\mathcal{M}_0$ to $\mathcal{M}_T$ and obtain a well-defined scattering relation of null geodesics

$$S(\gamma(0), \dot\gamma(0)) = (\gamma(\tau_0), \dot\gamma(\tau_0)) \tag{1}$$

where $\gamma(0) \in \mathcal{M}_0, \gamma(\tau_0) \in \mathcal{M}_T$. See Figure 1. The question is whether S determines g on $\mathcal{M} \doteq (0, T) \times \mathbb{R}^3$ between $\mathcal{M}_0$ and $\mathcal{M}_T$. Recently, a closely related problem was studied by Eskin in [3, 4] when the metric g is independent of t.

In this work, we prove a positive result for compactly supported, small metric deformations of the Minkowski space. Let $g_0 = -dt^2 + \sum_{i=1}^3 (dx^i)^2$ be the Minkowski metric on $\mathbb{R}^{3+1}$. Let $\mathcal{K}$ be a simply connected compact set of $\mathcal{M}$ with connected smooth boundary. Given $\epsilon \in (0, 1)$, we consider Lorentzian metrics $g = g_0 + h$ on $\mathbb{R}^{3+1}$ in which h is a smooth covariant two tensor field satisfying

(a) h is supported in $\mathcal{K}$;

(b) $\|h\|_{C^3} = \sum_{i,j=0}^{3} \|h_{ij}\|_{C^3} < \epsilon.$

Under these assumptions, we have a simple parametrization of the scattering relation. For $v \in \mathbb{S}^2$, note that $\theta = (1, v)$ is a future pointing light-like vector at $(0, x), x \in \mathbb{R}^3$ for any g as described above. We denote by $\gamma_{x,v}(s)$ the unique geodesic with initial condition $\gamma_{x,v}(0) = (0, x), \dot\gamma_{x,v}(0) = (1, v)$. For ϵ sufficiently small, $\gamma_{x,v}$ intersects $\mathcal{M}_T$ at a unique point $\gamma_{x,v}(s_0)$ (see Section 2). The scattering relation

Date: November 5, 2023.

S for null geodesics is

$$S((0,x),(1,v)) = (\gamma_{x,v}(s_0), \dot\gamma_{x,v}(s_0)). \tag{2}$$

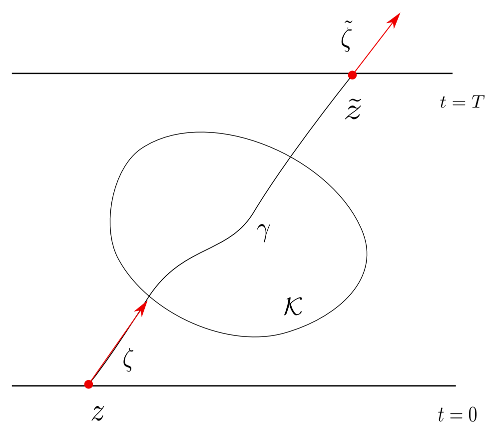

FIGURE 1. Scattering relation for light-like geodesics. γ is a future pointing light-like geodesic which goes from z in direction ζ and intersects $t = T$ at $\tilde z$ in direction $\tilde\zeta$. The metric perturbation is supported in $\mathcal{K}$.

To determine g from S, there are apparent gauge obstructions. It is known that null geodesics are invariant under conformal diffeomorphisms. We will show in Lemma 2.1 that the same holds for the scattering relation of null geodesics. In this work, we determine the metric perturbation h in the divergence and trace free gauge

$$\text{(c)}\ \ \operatorname{div} h = \operatorname{tr} h = 0$$

where $\operatorname{tr}, \operatorname{div}$ denote the Euclidean trace and divergence. Our gauge choice is analogous to the solenoidal gauge for the geodesic ray transform, see e.g. [10]. In fact, it is known that any Riemannian metrics sufficiently close to the Euclidean metric can be brought to the divergence free gauge (with respect to the Euclidean metric) via a diffeomorphism, see for example [6, Lemma 4.1]. By the same argument, we believe that any Lorentzian metric sufficiently close to the Minkowski metric can be transformed to the divergence and trace free gauge (with respect to the Euclidean metric) via a conformal diffeomorphism. However, we prefer to work in a fixed coordinate system and do not pursue the coordinate invariant statement. We denote the set of metric $g = g_0 + h$ with h satisfying (a), (b), (c) by $\mathcal{A}(\mathcal{K}, \epsilon)$.

For $\epsilon \in [0, 1)$, we consider a smooth one parameter family of symmetric two tensors h_ϵ in C^3 and $h_0 \neq 0$. Then we let $g_\epsilon = g_0 + \epsilon h_\epsilon$ and assume that $g_\epsilon \in \mathcal{A}(\mathcal{K}, \epsilon)$. Let S_ϵ be the corresponding scattering relation for null geodesics defined as in (2). Our main result is

Theorem 1.1. *There exists $\epsilon > 0$ such that for $\epsilon_1, \epsilon_2 \in (0, \epsilon)$, if $S_{\epsilon_1} = S_{\epsilon_2}$, then $g_{\epsilon_1} = g_{\epsilon_2}$.*

In the Riemannian setting, the scattering rigidity problem has been studied extensively especially in the case of simple manifolds for which the problem is equivalent to the boundary rigidity problem, see for instance [10, 13]. Our interest for the Lorentzian problem comes from some inverse problems in Lorentzian geometry and general relativity. One inspiring example is the nonlinear stability result of Christodoulou and Klainerman [2] which among other things demonsrates the completeness of null geodesics for Einstein vacuum spacetimes close to the Minkowski space. It is natural to ask whether the null geodesics or the scattering relation actually determines the metric perturbation, and this is partly why we consider the small perturbation problem in this work. There is another related problem studied by Guillemin [7] on the Zollfrei deformation of the Minkowski space, which in some sense concerns the scattering relation of null geodesics from the past null infinity to the future null infinity. It is conjectured that the $n+1, n \geq 3$ dimensional Minkowski space is rigid among Zollfrei deformations. One can also find applications in recovery of bulk geometry in the AdS/CFT correspondence, see for example [1].

We briefly discuss the ingredients in the proof and outline the structure of the paper. We consider the more general question of determining metrics in $\mathcal{A}(\mathcal{K}, \epsilon)$ from the scattering relation. Our starting point is an integral identity involving the difference of the Hamiltonian flows for two metrics in $\mathcal{A}(\mathcal{K}, \epsilon)$. This is done in Section 2 by following the approach of Stefanov and Uhlmann [12] for the boundary rigidity problem. The identity gives us some integral transforms of the metric perturbations along null geodesics. The main challenge is to prove the injectivity of the transforms. Let's explain the difficulty. After formal linearization at the Minkowski metric, the transforms become the light ray transform. It is known (see e.g. [7, 9, 16]) that this transform contains a microlocal kernel even after taking into account the gauge obstructions mentioned before. So it is not injective for general metric perturbations. For compactly supported perturbations, injectivity can be established, see [5, 11]. However, no stability estimate is available so one cannot extend the injectivity result via perturbation arguments to smooth perturbations of the light ray transform which we need. We resolve this issue by combining some partial stability estimates of the transform with the analytic continuation. We first analyze the transform on the Minkowski space, proving injectivity (up to gauge obstructions) in Section 3 and stability in Section 4. Then in Section 5, we analyze a light ray transform with both small metric and weight perturbations. We prove a stability estimate under some regularity assumptions which allows us to recover the Fourier transform of the metric difference on a non-empty open set which further yields the injectivity by analytic continuation because the metric difference is compactly supported.

Acknowledgement. The author wishes to thank Lauri Oksanen, András Vasy and Plamen Stefanov for their interest and advices in the early stage of the work about the light ray transform. This work is supported by NSF under grant DMS-2205266.

2. The scattering relation

Let $g = \sum_{i,j=0}^{3} g_{ij} dz^i dz^j$ be a Lorentzian metric. Consider the Hamiltonian

$$p(z,\zeta) = \frac{1}{2}\sum_{i,j=0}^{3} g^{ij}\zeta_i\zeta_j \tag{3}$$

Hereafter, we use $\zeta = (\tau, \xi), \tau \in \mathbb{R}, \xi \in \mathbb{R}^3$ to denote covectors at $z = (t, x), t \in \mathbb{R}, x \in \mathbb{R}^3$. Also, (g^{ij}) denotes the inverse of (g_{ij}). The Hamiltonian system is given by (with $m = 0, 1, 2, 3$)

$$\frac{dz^m}{ds} = \sum_{i=0}^{3} g^{im}\zeta_i, \quad \frac{d\zeta_m}{ds} = -\frac{1}{2}\sum_{i,j=0}^{3} \partial_{z^m} g^{ij}\zeta_i\zeta_j \tag{4}$$

Consider initial conditions

$$z|_{s=0} = z^{(0)} = (0, x^{(0)}), \quad \zeta|_{s=0} = \zeta^{(0)} = (\tau^{(0)}, \xi^{(0)}). \tag{5}$$

We denote $\mathsf{z} = (z, \zeta)$ and $\mathsf{z}^{(0)} = (z^{(0)}, \zeta^{(0)})$. We write the solution of (4) with (5) as $\mathsf{z} = \mathsf{z}(s, \mathsf{z}^{(0)})$, which is called a bicharacteristic. At this point, we do not assume that $\zeta^{(0)}$ is a null vector. For the Minkowski metric g_0, the system can be solved explicitly and the flow is

$$\mathsf{z}_0(s, \mathsf{z}^{(0)}) = (s\tau^{(0)}, x^{(0)} + s\xi^{(0)}, \tau^{(0)}, \xi^{(0)}), \quad s \in \mathbb{R}.$$

By a perturbation argument and the standard ODE well-posedness result (see e.g. [8, Section 1.2]), for ϵ sufficiently small and any $g \in \mathcal{A}(\mathcal{K}, \epsilon)$, there exists a unique C^∞ solution of (4) with initial condition $\mathsf{z}^{(0)}$ in a compact set $\mathcal{O}$ of $\mathbb{R}^3 \times \mathbb{R}^4$. The ϵ depends on T and $\mathcal{O}$. So the Hamiltonian flow (hence the scattering relation) is well-defined when ϵ is sufficiently small.

Now we prove the conformal invariance of the null scattering relation.

Lemma 2.1. *Suppose $g \in \mathcal{A}(\mathcal{K}, \epsilon)$ and $\tilde{g} = e^{\varphi} g$ for some $\phi \in C^\infty(\mathcal{M})$. Then the scattering relations $\tilde{S}, S$ for null geodesics defined as in (2) are the same.*

Proof. Let $p, \tilde{p}$ be the Hamiltonians of $g, \tilde{g}$ respectively, so $\tilde{p}(z, \zeta) = e^{-\varphi(z)} p(z, \zeta)$. Consider null geodesics for $\tilde{g}$ in the cotangent bundle which are null bicharacterisitics described by the Hamiltonian system. Using the fact that along null bicharacteristics $p = \tilde{p} = 0$, we can write the Hamiltonian system for $\tilde{g}$ as

$$\frac{dz^m}{ds} = e^{-\varphi}\sum_{i=0}^{3} g^{im}\zeta_i, \quad \frac{d\zeta_m}{ds} = -e^{-\varphi}\frac{1}{2}\sum_{i,j=0}^{3} \partial_{z^m} g^{ij}\zeta_i\zeta_j \tag{6}$$

with initial condition

$$z|_{\tilde{s}=0} = z^{(0)} = (0, x^{(0)}), \quad \zeta|_{\tilde{s}=0} = \zeta^{(0)} = (-1, \xi^{(0)}), \xi^{(0)} \in \mathbb{S}^2 \tag{7}$$

Note that $\zeta^{(0)}$ is light-like for both g and $\tilde{g}$. Let $\tilde{\gamma}(\tilde{s})$ be the projection of the bicharacteristics to $\mathcal{M}$. Define

$$s = \int_0^{\tilde{s}} e^{-\varphi(\tilde{\gamma}(\sigma))} d\sigma$$

We have $ds/d\tilde{s} = e^{-\varphi(\tilde{\gamma}(\tilde{s}))}$ along the null geodesics. After changing $\tilde{s}$ to s, (6) becomes (4). By the uniqueness of solutions of the Hamiltonian system, we have $\tilde{\gamma}(\tilde{s}) = \gamma(s)$. Finally, let $s_0, \tilde{s}_0$ be such that $\gamma(s_0), \tilde{\gamma}(\tilde{s}_0) \in \mathcal{M}_T$. We obtain that

$$\dot{\gamma}(s_0) = \frac{d}{ds}\tilde{\gamma}(\tilde{s})|_{s=s_0} = \dot{\tilde{\gamma}}(\tilde{s})|_{\tilde{s}=\tilde{s}_0}\frac{d\tilde{s}}{ds}|_{s=s_0} = \dot{\tilde{\gamma}}(\tilde{s})|_{\tilde{s}=\tilde{s}_0}$$

in which we used that $\varphi = 0$ outside $\mathcal{K}$. This proves that $S = \tilde{S}$. □

Next, we derive an integral identity following the idea in Stefanov and Uhlmann [12] for the Riemannian problem. Suppose $S = \tilde{S}$ for two metrics $g, \tilde{g} \in \mathcal{A}(\mathcal{K}, \epsilon)$. Let $\mathsf{z}, \tilde{\mathsf{z}}$ be the corresponding Hamiltonian flows for $g, \tilde{g}$. Starting from $\mathsf{z}^{(0)}$, we follow the flow z to $\mathsf{z}(s, \mathsf{z}^{(0)}) \in \mathcal{M}_t$ for some $t \in (0, T)$. Then we continue with the flow $\tilde{\mathsf{z}}$ until it reaches $\mathcal{M}_T$. There is an issue that the affine parameter s cannot be compared directly as the arc-length parameter in the Riemannian setting. We will use parameter $t \in (0, T)$ to connect the flows. Note that changing the parametrization from s to t along the flow will not change the Hamiltonian flow as a set, however the projections to $\mathcal{M}$ may not be geodesics anymore.

Lemma 2.2. *Let $\mathsf{z}(s, \mathsf{z}^{(0)}), \mathsf{z}_0(s, \mathsf{z}^{(0)})$ be the corresponding Hamiltonian flows for $g \in \mathcal{A}(\mathcal{K}, \epsilon)$ and g_0 respectively with $\mathsf{z}^{(0)}$ in some fixed compact set $\mathcal{O}$ of $\mathbb{R}^3 \times \mathbb{R}^4$. Then for ϵ sufficiently small (depending on $\mathcal{O}, T$), $\mathsf{z}(s, \mathsf{z}^{(0)})$ is well-defined for $s \in \mathbb{R}$. Moreover, for any $b > 0$, there exists ϵ small depending on $b, \mathcal{O}$ such that*

$$\|\mathsf{z} - \mathsf{z}_0\|_{C^2([0,b]\times\mathcal{O})} < C\epsilon \tag{8}$$

and $C > 0$ is uniform for $g \in \mathcal{A}(\mathcal{K}, \epsilon)$.

Proof. This again follows from standard ODE well-posedness results. We compare the Hamiltonian systems for $g \in \mathcal{A}(\mathcal{K}, \epsilon)$ and g_0. Consider $(z'(s), \zeta'(s)) \doteq \mathsf{z}(s, \mathsf{z}^{(0)}) - \mathsf{z}_0(s, \mathsf{z}^{(0)})$ which satisfies the following ODE system

$$\frac{d(z')^m}{ds} = a_m(z', \zeta', z_0, \zeta_0), \quad \frac{d\zeta'_m}{ds} = b_m(z', \zeta', z_0, \zeta_0), \quad m = 0, 1, 2, 3 \tag{9}$$

with zero initial condition. Here, a_m, b_m are smooth functions and for $|z'|, |\zeta'| \leq M$, we have $\|a_m\|_{C^3}$, $\|b_m\|_{C^2} < CM\epsilon$ with C depending on $\mathcal{O}, T$. Thus for ϵ sufficiently small depending on T and $\mathcal{O}$, we obtain a unique C^∞ solution of (9) on $\mathcal{M}$. Outside of $\mathcal{M}$, $\tilde{g} = g_0$ by our assumption. Thus the flows can be extended for all s. The C^2 estimates can be seen by considering the system (9) for $s \in [0, b], b > 0$. □

Let $s \in [0, s_0]$ be the affine parameter for z where s_0 is such that $\mathsf{z}(s_0, \mathsf{z}^{(0)}) \in T^*\mathcal{M}_T$. Let $t = \phi(s, \mathsf{z}^{(0)}) \in [0, T]$. For $\tilde{s} \in [0, \tilde{s}_0]$ the affine parameter of $\tilde{\mathsf{z}}$, we let $t = \tilde{\phi}(\tilde{s}, \mathsf{z}^{(0)}) \in [0, T]$. In view of the first equation of (4), both $\phi, \tilde{\phi}$ are smooth invertible functions for fixed $\mathsf{z}^{(0)}$. See Lemma 2.3 below. We first follow the flow z to $\mathsf{z}(s, \mathsf{z}^{(0)}) \in \mathcal{M}_t$. When connecting the two flows, note that the second flow by $\tilde{\mathsf{z}}$ does not necessarily correspond to a null geodesic. For any $(z, \zeta) \in T^*\mathcal{M}$ with ζ sufficiently close to some null vector $\hat{\zeta}$ for g, we know from the continuous dependence of the Hamiltonian flow on initial conditions that there exists $\mathsf{z}^{(0)}$ and $s \in \mathbb{R}$ such that $\mathsf{z}(s, \mathsf{z}^{(0)}) = (z, \zeta)$. We define $\mathsf{z}^{-1}(z, \zeta) = \mathsf{z}^{(0)}$. For $g, \tilde{g} \in \mathcal{A}(\mathcal{K}, \epsilon)$ and ϵ sufficiently small, we can use Lemma 2.2 to conclude that $\tilde{\mathsf{z}}^{-1}(\mathsf{z}(s, \mathsf{z}^{(0)}))$ is well-defined for all values of s. We set

$$\kappa(s, \mathsf{z}^{(0)}) = \tilde{\phi}^{-1}(\phi(s, \mathsf{z}^{(0)}), \tilde{\mathsf{z}}^{-1}(\mathsf{z}(s, \mathsf{z}^{(0)}))). \tag{10}$$

The function switches the affine parameters from s to $\tilde{s}$. Note that if $g = \tilde{g}$, then $\kappa(s, \mathsf{z}^{(0)}) = s$. We next show that $\kappa(s, \mathsf{z}^{(0)})$ is close to s if g is close to $\tilde{g}$.

Lemma 2.3. *For $g, \tilde{g} \in \mathcal{A}(\mathcal{K}, \epsilon)$, let $\mathsf{z}(s, \mathsf{z}^{(0)}), \tilde{\mathsf{z}}(s, \mathsf{z}^{(0)})$ be the corresponding Hamiltonian flows with $\mathsf{z}^{(0)}$ in a fixed compact set $\mathcal{O}$ of $\mathbb{R}^3 \times \mathbb{R}^4$. Then for ϵ sufficiently small (depending on $\mathcal{O}, T$) as in Lemma 2.2, there exists $C > 0$ independent of $g, \tilde{g}$ such that*

$$\|\kappa(s, \mathsf{z}^{(0)}) - s\|_{C^2([0, s_0] \times \mathcal{O})} < C\epsilon \tag{11}$$

Proof. The first equation of (4) can be written as

$$\frac{dt}{ds} = -\tau + O(\epsilon) = 1 + O(\epsilon)$$

in C^2. For the second equality, we used Lemma 2.2 and the initial condition $\tau = -1$. Write $t = \phi(s, \mathsf{z}^{(0)}) \in [0, T]$. By the inverse function theorem, ϕ is invertible for ϵ small and any fixed $\mathsf{z}^{(0)}$. We can use t as the parameter in (4) and follow the argument of Lemma 2.2 to conclude that $\|\phi^{-1}(t, \mathsf{z}^{(0)}) - t\|_{C^2} < C\epsilon$ with C uniform for $\mathsf{z}^{(0)} \in \mathcal{O}$ and $g \in \mathcal{A}(\mathcal{K}, \epsilon)$. For $\tilde{s} \in [0, \tilde{s}_0]$ the affine parameter of $\tilde{\mathsf{z}}$, we let $t = \tilde{\phi}(\tilde{s}, \mathsf{z}^{(0)}) \in [0, T]$. The same conclusion holds. Thus from the definition of $\kappa(s, \mathsf{z}^{(0)})$, we have

$$\begin{aligned} &\|\kappa(s, \mathsf{z}^{(0)}) - s\|_{C^2} \\ \leq &\|\tilde{\phi}^{-1}(\phi(s, \mathsf{z}^{(0)}), \tilde{\mathsf{z}}^{-1}(\mathsf{z}(s, \mathsf{z}^{(0)}))) - \phi(s, \mathsf{z}^{(0)})\|_{C^2} + \|\phi(s, \mathsf{z}^{(0)}) - s\|_{C^2} \leq C\epsilon. \end{aligned}$$

Here, we used that $\tilde{\mathsf{z}}^{-1}(\mathsf{z}(s, \mathsf{z}^{(0)}))$ stays in some compact set $\tilde{\mathcal{O}}$ of $\mathbb{R}^3 \times \mathbb{R}^4$ depending on $\mathcal{O}, \epsilon$. So the argument above applies. $\square$

Now we define a function

$$F(s) = \tilde{\mathsf{z}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)})) \tag{12}$$

on $[0, s_0]$. Note that

$$F(0) = \tilde{\mathsf{z}}(\tilde{s}_0 - \kappa(0, \mathsf{z}^{(0)}), \mathsf{z}^{(0)}) = \tilde{\mathsf{z}}(\tilde{s}_0, \mathsf{z}^{(0)})$$

because $\kappa(0, \mathsf{z}^{(0)}) = 0$ and

$$F(s_0) = \tilde{\mathsf{z}}(0, \mathsf{z}(s_0, \mathsf{z}^{(0)})) = \mathsf{z}(s_0, \mathsf{z}^{(0)})$$

because $\kappa(s_0, \mathsf{z}^{(0)}) = \tilde{s}_0$. For any $\mathsf{z}^{(0)} = (0, x, 1, \xi), x, \xi \in \mathbb{R}^3, |\xi| = 1$, we assume the scattering relations $S = \tilde{S}$ so $F(0) = F(s_0)$. This implies

$$\int_0^{s_0} F'(s)ds = 0 \tag{13}$$

Let $H_p(\mathsf{z}) = (\partial p/\partial\zeta, -\partial p/\partial z)$ be the Hamilton vector field at z. We use (12) to find

$$\begin{aligned} F'(s) = &- H_{\tilde{p}}(\tilde{\mathsf{z}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)})))\partial_s\kappa(s, \mathsf{z}^{(0)}) \\ &+ \frac{\partial\tilde{\mathsf{z}}}{\partial\mathsf{z}^{(0)}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)}))H_p(\mathsf{z}(s, \mathsf{z}^{(0)})) \end{aligned} \tag{14}$$

Next, we transform the first term on the right hand side of (14). For any $\tilde{s} \in [0, \tilde{s}_0]$, we have

$$0 = \frac{d}{ds}|_{s=0}\tilde{\mathsf{z}}(\tilde{s} - s, \tilde{\mathsf{z}}(s, \mathsf{z}^{(0)})) = -H_{\tilde{p}}(\tilde{\mathsf{z}}(\tilde{s}, \mathsf{z}^{(0)})) + \frac{\partial\tilde{\mathsf{z}}}{\partial\mathsf{z}^{(0)}}(\tilde{s}, \mathsf{z}^{(0)})H_{\tilde{p}}(\mathsf{z}^{(0)}) \tag{15}$$

Replace $\tilde{s}$ by $\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)})$ and $\mathsf{z}^{(0)}$ by $\mathsf{z}(s, \mathsf{z}^{(0)})$ we get

$$H_{\tilde{p}}(\tilde{\mathsf{z}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)}))) = \frac{\partial\tilde{\mathsf{z}}}{\partial\mathsf{z}^{(0)}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)}))H_{\tilde{p}}(\mathsf{z}(s, \mathsf{z}^{(0)})) \tag{16}$$

Putting identities (13), (14) and (16) together, we obtain

$$\int_0^{s_0} \frac{\partial\tilde{\mathsf{z}}}{\partial\mathsf{z}^{(0)}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)}))\partial_s\kappa(s, \mathsf{z}^{(0)})(H_p(\mathsf{z}(s, \mathsf{z}^{(0)})) - H_{\tilde{p}}(\mathsf{z}(s, \mathsf{z}^{(0)})))ds = 0. \tag{17}$$

We will use this identity to show $g = \tilde{g}$.

3. The linearization and injectivity

We consider the formal linearization of (17) at the Minkowski metric $g = g_0$:

$$\int_0^{s_0} \frac{\partial\mathsf{z}}{\partial\mathsf{z}^{(0)}}(s_0 - s, \mathsf{z}(s, \mathsf{z}^{(0)}))(H_p(\mathsf{z}(s, \mathsf{z}^{(0)})) - H_{\tilde{p}}(\mathsf{z}(s, \mathsf{z}^{(0)})))ds = 0 \tag{18}$$

We wish to demonstrate that $\tilde{g} = g_0$ from this identity. For the Minkowski metric, the Hamiltonian flow can be found explicitly. Take $\mathsf{z}^{(0)} = (z^{(0)}, \zeta^{(0)}) = (0, x, 1, \xi)$ with $|\xi| = 1$. We have

$$\mathsf{z}(s, \mathsf{z}^{(0)}) = (s, x + s\xi, 1, \xi) = \begin{pmatrix} \mathrm{Id} & s\,\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} z^{(0)} \\ \zeta^{(0)} \end{pmatrix}$$

thus

$$\frac{\partial\mathsf{z}}{\partial\mathsf{z}^{(0)}} = \begin{pmatrix} \mathrm{Id} & s\,\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{19}$$

Consider $\tilde g \in \mathcal{A}(\mathcal{K}, \epsilon)$. Set $u^{ij} = \tilde g^{ij} - \delta^{ij}, i, j = 0, 1, 2, 3$. Let $\zeta = (1, \xi)$. We have

$$
\begin{aligned}
H_{\tilde p} - H_p = (&\sum_{j=0}^3 u^{0j}\zeta_j, \sum_{0=1}^3 u^{1j}\zeta_j, \sum_{j=0}^3 u^{2j}\zeta_j, \sum_{j=0}^3 u^{3j}\zeta_j, \\
-\frac{1}{2}\sum_{i,j=0}^3 \partial_0 u^{ij}\zeta_i\zeta_j, -\frac{1}{2}&\sum_{i,j=0}^3 \partial_1 u^{ij}\zeta_i\zeta_j, -\frac{1}{2}\sum_{i,j=0}^3 \partial_2 u^{ij}\zeta_i\zeta_j, -\frac{1}{2}\sum_{i,j=0}^3 \partial_3 u^{ij}\zeta_i\zeta_j)
\end{aligned} \tag{20}
$$

Using (19) and (20), we find from the last four components of (18) that

$$
\int_0^{s_0} \sum_{i,j=0}^3 \partial_k u^{ij}(s, x + s\xi)\zeta_i\zeta_j ds = 0, \quad k = 0, 1, 2, 3. \tag{21}
$$

Here, $x \in \mathbb{R}^3, \zeta = (1, \xi), \xi \in \mathbb{S}^2$ and $s_0 = T$.

The integral in (21) can be regarded as the Minkowski light ray transform. Let $x \in \mathbb{R}^3, v \in \mathbb{S}^2$ and $\theta = (1, v)$ so that θ is a (future pointing) light-like vector. We can parametrize null geodesic γ on $(\mathcal{M}, g_0)$ as $\gamma(\tau) = (\tau, x + \tau v), \tau \in (0, T)$ so that $\gamma(0) = (0, x), \dot\gamma(0) = \theta$. For a covariant symmetric two tensor u, we define the Minkowski light ray transform as

$$
Lu(x, v) = \int_0^{s_0} \sum_{i,j=0}^3 u_{ij}(\tau, x + \tau v)\theta^i\theta^j d\tau. \tag{22}
$$

Thus, by raising and lowering indices using g_0, we can essentially view (21) as $L(\partial_k u) = 0$, $k = 0, 1, 2, 3$ in which u is a covariant symmetric two tensor. Below, we prove the injectivity of L which is sufficient to conclude $u = 0$ from (21) because the metric perturbations are all compactly supported. This shows that the formally linearized problem is solvable.

Let Sym^2 denote the vector bundle of covariant symmetric two tensors on $\mathcal{M}$ whose fiber can be identified with the space of symmetric matrices within the coordinate system we use. It is known that the light ray transform on symmetric two tensors $L : C_0^\infty(\mathbb{R}^{3+1}; \mathrm{Sym}^2) \to C^\infty(\mathcal{C})$ has a non-trivial null space given by

$$
\mathcal{N} = \{cg_0 + d^s w : c \in C_0^\infty(\mathbb{R}^{3+1}), w \in C_0^\infty(\mathbb{R}^{3+1}; \Lambda^1)\}, \tag{23}
$$

where d^s is the symmetric differential given by

$$
(d^s\omega)_{ij} = \frac{1}{2}((\nabla_i\omega)_j + (\nabla_j\omega)_i), \quad i, j = 0, 1, 2, 3,
$$

with ∇_i the covariant derivative, and Λ^1 denotes the bundle of one forms. See [9, Lemma 4.3]. It is essentially contained in Theorem 2 of [5] that this is the full null space. For $3 + 1$ dimensional Minkowski space, we give a somewhat shorter and different proof taken from [15].

Proposition 3.1. *Suppose $u \in C_0^\infty(\mathbb{R}^{3+1}, Sym^2)$. If $Lu = 0$, then there is unique $c \in C_0^\infty(\mathbb{R}^{3+1}), v \in C_0^\infty(\mathbb{R}^{3+1}, \Lambda_1)$ such that $u = cg_0 + d^s v$.*

Proof. We start with the Fourier slice theorem. For $u \in C_0^\infty(\mathbb{R}^{3+1}, \mathrm{Sym}^2)$, $Lu = 0$ if and only if

$$\sum_{i,j=0}^{3} \hat{u}_{ij}(\zeta)\theta^i\theta^j = 0 \tag{24}$$

where $\theta = (1, v), v \in \mathbb{S}^2$ and $\zeta = (\tau, \xi), \xi \neq 0$ such that $\tau + \xi\cdot v = 0$, that is $\theta \perp \zeta$ with respect to the Euclidean inner product. Hereafter, $\hat{}$ denotes the Fourier transform in (t, x) variables. Now we use Lemma 9.1 of [9]: for any space-like vector ζ, there exist $e = e(\zeta) \in \mathbb{R}, w = w(\zeta) \in \mathbb{R}^{3+1}$ such that

$$\hat{u}(\zeta) = e(\zeta)g_0 + \zeta \otimes w(\zeta) + w(\zeta) \otimes \zeta. \tag{25}$$

We claim that e, w can be uniquely solved from $\widehat{u}$.

Consider $\widehat{u}_{ij} = \xi_i w_j(\zeta) + \xi_j w_i(\zeta), i \neq j, i, j = 1, 2, 3$. We get three equations for $w_i, i = 1, 2, 3$

$$\begin{pmatrix} \xi_2 & \xi_1 & 0 \\ \xi_3 & 0 & \xi_1 \\ 0 & \xi_3 & \xi_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \widehat{u}_{12} \\ \widehat{u}_{13} \\ \widehat{u}_{23} \end{pmatrix}$$

The determinant of the coefficient matrix is $-2\xi_1\xi_2\xi_3$. When it is non-vanishing, we obtain that

$$\begin{gathered} w_1 = \frac{\xi_3\widehat{u}_{12} + \xi_2\widehat{u}_{13} - \xi_1\widehat{u}_{23}}{2\xi_2\xi_3}, \quad w_2 = \frac{\xi_3\widehat{u}_{12} - \xi_2\widehat{u}_{13} + \xi_1\widehat{u}_{23}}{2\xi_1\xi_3} \\ w_3 = \frac{-\xi_3\widehat{u}_{12} + \xi_2\widehat{u}_{13} + \xi_1\widehat{u}_{23}}{2\xi_1\xi_2} \end{gathered} \tag{26}$$

The Paley-Wiener theorem tells that $\widehat{u}_{ij}$ are Schwartz and analytic in ζ. So $\tilde{w} = (w_1, w_2, w_3)$ is analytic in ζ and (26) defines $\tilde{w}$ uniquely on $\mathbb{R} \times \mathbb{R}^3\backslash\{0\}$, and invariantly under orthogonal coordinate changes. In particular, the coordinate singularity at $\xi_i = 0, i = 1, 2, 3$ is removed but the singularity at $\xi = 0$ remains at this point. Note that the singularity at 0 is integrable. Next, we use $\hat{u}_{01} = \tau w_1 + \xi_1 w_0$ to get

$$w_0 = \frac{\hat{u}_{01}}{\xi_1} - \tau\frac{\xi_3\hat{u}_{12} + \xi_2\hat{u}_{13} - \xi_1\hat{u}_{23}}{2\xi_1\xi_2\xi_3} \tag{27}$$

Again, we can use the orthogonal invariance to get w_0 for $\zeta \in \mathbb{R} \times \mathbb{R}^3\backslash 0$. Finally, we can solve e from (25) and we proved the claim.

The above calculation also yields some rough regularity estimate of e, w. Recall that $\langle\xi\rangle^\kappa \mathscr{F}(\chi u_{ij})(\tilde{\xi}) \in L^2(\mathbb{R}^4)$ for $i, j = 1, 2, 3, i \neq j$. From the formula (26), we see that $\langle\xi\rangle^{\kappa+1} w_j(\tilde{\xi}) \in L^2(\mathbb{R}^4), j = 1, 2, 3$. Then from (27), we get $\langle\xi\rangle^{\kappa+1} w_0(\tilde{\xi}) \in L^2(\mathbb{R}^4)$. Again by using (25), we get $\langle\xi\rangle^\kappa e(\tilde{\xi}) \in L^2(\mathbb{R}^4)$. In particular, the inverse Fourier transforms of w, e, denoted by v, c belong to $L^2(\mathbb{R}; H^{\kappa+1}(\mathbb{R}^3))$ and $L^2(\mathbb{R}; H^\kappa(\mathbb{R}^3))$ respectively. By taking inverse Fourier transform of (25), we see that the decomposition

$$\chi u = cg_0 + d^s v \tag{28}$$

holds on $\mathbb{R}^4$ at least in the sense of distributions. Also, it is crucial to note that both v and c regarded as distributions are compactly supported. Suppose that u is supported in $[T_1, T_2] \times \mathbb{R}^3$ where $0 < T_1 < T_2 < T$. We translate $[T_1, T_2]$ to $[-a, a]$ for $a = (T_2 - T_1)/2 > 0$. For fixed $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we know from applying the Paley-Wiener theorem to $\mathscr{F}_x(\chi u_{ij})(s, \xi), i, j = 0, 1, 2, 3, i \neq j$ that

$$|\mathscr{F}(\chi u_{ij})(\xi_0, \xi)| \leq C(1 + |\xi_0|)^N e^{a|\operatorname{Im} \xi_0|}, \quad \xi_0 \in \mathbb{C}$$

for some $C, N \geq 0$. From (26), we see the same type of estimates holds for w_1. The Paley-Wiener theorem tells that v_1 is supported in $[T_1, T_2]$ after the translation. Because u is compactly supported in x^1, x^2, x^3 variables as well, we can repeat the argument for $\xi_j, j = 1, 2, 3$ with other ξ's fixed to conclude that v_1 is compactly supported. The same conclusion holds for $v_i, i = 0, 1, 2, 3$. Then it is easy to see from (28) that c is compactly supported.

By taking the inverse Fourier transform of (25), we find distributions c, v such that $u = cg_0 + d^s v$ in the sense of distribution. Let tr and div denotes the Euclidean trace and divergence. We have

$$\operatorname{tr}(u) = \operatorname{tr}(cg_0 + d^s v) = 2c + \operatorname{div} v \tag{29}$$

$$\operatorname{div} u = \operatorname{div}(cg_0 + d^s v) \tag{30}$$

Using (29) in (30), we obtain

$$c = \frac{1}{2}(\operatorname{tr}(u) - \operatorname{div} v) \tag{31}$$

and

$$\operatorname{div} u = \frac{1}{2} \operatorname{div}((\operatorname{tr}(u) - \operatorname{div} v) g_0) + \operatorname{div}(d^s v)$$

which gives four equations for v. In local coordinate, they are

$$\begin{pmatrix} \Delta v_0 \\ \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \end{pmatrix} = - \begin{pmatrix} -\partial_0(\operatorname{tr} u) + 2\sum_{j=0}^3 \partial_0 \partial_j v_j \\ \partial_1(\operatorname{tr} u) \\ \partial_2(\operatorname{tr} u) \\ \partial_3(\operatorname{tr} u) \end{pmatrix} + 2 \begin{pmatrix} (\operatorname{div} u)_0 \\ (\operatorname{div} u)_1 \\ (\operatorname{div} u)_2 \\ (\operatorname{div} u)_3 \end{pmatrix} \tag{32}$$

Here, $\Delta = \sum_{j=0}^3 \partial_j^2$ denotes the Laplacian on $\mathbb{R}^4$. Because we known a priori that v_j are compactly supported, we can solve the last three equations

$$\Delta v_j = -\partial_j \operatorname{tr} u + 2(\operatorname{div} u)_j, \quad j = 1, 2, 3 \tag{33}$$

by imposing Dirichlet condition. Given that u is smooth and compactly supported, v_j are smooth by standard regularity theory for elliptic equations. After that, we use v_1, v_2, v_3 to solve the first equation for v_0 in (32). This completes the proof of the proposition. □

Proposition 3.2. *Suppose $u \in C_0^\infty(\mathbb{R}^{3+1}, Sym^2)$. Then we can write $u = cg_0 + d^s v + w$ where $c \in C_0^\infty(\mathbb{R}^{3+1})$, $v \in C_0^\infty(\mathbb{R}^{3+1}, \Lambda_1), w \in C_0^\infty(\mathbb{R}^{3+1}, Sym^2)$ and* $\operatorname{tr} w = \operatorname{div} w = 0$. *In particular, if* $\operatorname{tr} u = \operatorname{div} u = 0$ *and* $Lu = 0$, *then* $u = 0$.

Proof. Let tr and div denotes the Euclidean trace and divergence. We look for $c \in C_0^\infty(\mathbb{R}^{3+1})$, $v \in C_0^\infty(\mathbb{R}^{3+1}, \Lambda_1)$ which satisfy (29) and (30). Then $w = u - cg_0 - d^s v$ is trace and divergence free. As in the proof of Proposition 3.1, we arrive at the equations (32). We can solve the last three equations

$$\Delta v_j = -\partial_j \operatorname{tr} u + 2(\operatorname{div} u)_j, \quad j = 1, 2, 3$$

by imposing Dirichlet conditions on a bounded set containing the support of u. Given that u is smooth and compactly supported, v_j are smooth by standard regularity theory for elliptic equations. After that, we use v_1, v_2, v_3 to solve the first equation for v_0 in (32).

Finally, if $Lu = 0$, by Proposition 3.1 we know that $u = cg_0 + d^s v$ for some $c, v \in C_0^\infty$. Also, c, v satisfy (29) and (30). Because $\operatorname{tr} u = \operatorname{div} u = 0$, we see from (32) with Dirichlet boundary condition that $c = v = 0$. Thus $u = 0$. $\square$

According to the proposition, it is natural to introduce

$$\mathcal{G} = \{h \in C_0^\infty(\mathbb{R}^{3+1}, \mathrm{Sym}^2) : h \text{ is supported in } \mathcal{K} \text{ and } \operatorname{tr} h = \operatorname{div} h = 0\} \tag{34}$$

which complements the kernel of the Minkowski light ray transform when acting on $\mathcal{A}(\mathcal{K}, \epsilon)$. The conclusion is that L is injective on $\mathcal{G}$ and we solved the formally linearized problem.

Remark 3.3. *The results we proved in this section also hold for compactly supported $H^\mu, \mu \in \mathbb{R}$ tensors. In Proposition 3.1, if $u \in H^\mu$ is compactly supported, then for $w_i, i = 1, 2, 3$ defined in (26), we have that $\langle \xi \rangle^\mu w_i$ are well-defined distributions. By repeating the rest of the argument, we conclude that there is $c \in H^\mu, v \in H^{\mu+1}$ so that $u = cg_0 + d^s v$. Similarly, Proposition 3.2 can be adapted.*

4. The stability analysis

We need some stability estimates for the light ray transform acting on two tensors. This is not simple because of the presence of the microlocal kernel. It is known that the normal operator of the light ray transform is not elliptic, see [9, 16, 17]. For the Minkowski case, we can use the kernel as a convolution operator and introduce a projection operator to obtain a stability estimate on the elliptic region.

As shown in [9, Lemma 4.1], L is an Fourier integral operator of order $-3/4$ associated with the canonical relation $N^*\mathcal{Z}'$ where $\mathcal{Z}$ is the point-line relation. Hence $L : \mathscr{E}'(\mathcal{M}, \mathrm{Sym}^2) \to \mathscr{D}'(\mathbb{R}^3 \times \mathbb{S}^2)$ is continuous where $\mathscr{E}'$ denotes the space of distributions with compact support and $\mathscr{D}'$ the space of distributions. We use standard product measure on $\mathbb{R}^3 \times \mathbb{S}^2$ to define the adjoint L^*. Let $N = L^*L$ be the normal operator. It is shown in [9, Section 8] that $N = (N^{jklm})$ is a convolution operator or a Fourier multiplier indeed. From [9, Lemma 8.1], we know that the Fourier transform of N^{jklm} are locally integral functions given by

$$\hat{N}^{jklm} = \begin{cases} 2\pi(|\xi|^2 - |\tau^2|)^{-\frac{1}{2}} \displaystyle\int_{\mathbb{S}^1_\zeta} \theta^j\theta^k\theta^l\theta^m dv, & \zeta = (\tau, \xi) \text{ space-like} \\ 0, \quad \text{otherwise.} \end{cases} \tag{35}$$

Here, $\mathbb{S}^1_\zeta = \{v \in \mathbb{S}^2 : \tau + \xi \cdot v = 0\}$ is a circle of radius $|\xi|^{-1}(|\xi|^2 - \tau^2)^{\frac{1}{2}}$, $\theta = (1, v)$. We introduce

$$\hat{K}^{jklm} = \begin{cases} \int_{\mathbb{S}^1_\zeta} \theta^j\theta^k\theta^l\theta^m dv, & \zeta \text{ space-like} \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

so we can write $\hat{N}^{jklm} = 2\pi(|\xi|^2 - |\tau^2|)^{-\frac{1}{2}}\hat{K}^{jklm}$. Note that $\hat{K}^{jklm}$ are homogeneous of degree zero in ζ. Some terms are computed explicitly in [9]. Let K be a Fourier multiplier defined by $(\hat{K}^{jklm})$. Observe that neither N nor K is elliptic for non-space-like ζ. On $(\mathbb{R}^{3+1}, g_0)$, we denote the cone of space-like vectors by $\Gamma^{sp} = \{(z, \zeta) \in T\mathbb{R}^{3+1} : \zeta = (\tau, \xi) \neq 0, \tau^2 \leq |\xi|^2\}$. Let $\chi(\zeta)$ be the characteristic function of $\Gamma^{sp} \subset \mathbb{R}^4$. Let $\chi(D)$ be the corresponding Fourier multiplier. The stability estimate we look for is

Proposition 4.1. *For $f \in \mathcal{G}$, there exists $C > 0$ such that*

$$\|\chi(D)f\|_{H^{-\frac{1}{2}+\mu}} \leq C\|Lf\|_{H^\mu}, \quad \mu \in \mathbb{R}. \tag{37}$$

Proof. We prove for $\mu = 0$. The general case is similar. For each ζ space-like, consider $\hat{K} : a_{lm} \to \hat{K}^{jklm}a_{lm}$ as a linear map on M_4, the vector space of 4×4 symmetric matrices. It is shown in [9, Lemma 9.2] that for space-like direction ζ, there exists a projection $\hat{P}$ on M_4 such that $\ker\hat{K} = \ker\hat{P}$ and $\hat{P}\hat{K} = \hat{K} = \hat{K}\hat{P}$. In particular, $\hat{P}$ is homogeneous of degree zero in ζ. The explicit expression of $\hat{P}$ can be found in [9] but we do not need it. For ζ space-like, we define $\hat{W} = \hat{K} + (\mathrm{Id} - \hat{P})$ and otherwise $\hat{W} = 0$. If $\hat{W}a = 0$, we see that

$$0 = \hat{P}\hat{W}a = \hat{P}\hat{K}a + 0 = \hat{P}a$$

This implies $\hat{K}a = \hat{P}a = 0$. Next,

$$\begin{aligned} 0 &= (\mathrm{Id} - \hat{P})\hat{W}a = \hat{K}a - \hat{P}\hat{K}a + (\mathrm{Id} - \hat{P})a \\ &= 0 - \hat{P}a + (\mathrm{Id} - \hat{P})a = (\mathrm{Id} - \hat{P})a \end{aligned}$$

Thus, we must have $a = 0$. So $\hat{W}$ is invertible in space-like directions. Let $\hat{Q}$ be the inverse of $\hat{W}$. Then $\hat{Q}$ is homogeneous of degree zero in ζ for ζ space-like. We introduce

$$\Omega(\zeta) = (2\pi)(|\xi|^2 - |\tau^2|)_+^{-\frac{1}{2}}, \quad \zeta \in \mathbb{R}^{3+1}$$

$$\Omega^{-1}(\zeta) = (2\pi)^{-1}(|\xi|^2 - |\tau^2|)_+^{\frac{1}{2}}, \quad \zeta \in \mathbb{R}^{3+1}.$$

Let $\Omega(D), \Omega^{-1}(D)$ be Fourier multipliers defined by Ω, Ω^{-1}. Observe that $N = \Omega(D)K = K\Omega(D)$. From (35) and (36), we get $Nf = K\Omega(D)f$. Then

$$\Omega(D)f = QW\Omega(D)f = Q(K + (I - P))\Omega(D)f = QNf + Q(I - P)\Omega(D)f$$

Thus we have

$$\chi(D)f = \Omega^{-1}(D)\Omega(D)f = \Omega^{-1}(D)QNf + \Omega^{-1}(D)Q(I - P)\Omega(D)f$$

which implies that

$$\|\chi(D)f\|_{H^{-\frac{1}{2}}} \le C\|Nf\|_{H^{\frac{1}{2}}} + C\|(I-P)f\|_{H^{\frac{1}{2}}} \tag{38}$$

Now take $f \in \mathcal{G}$ and consider $r = (I-P)f$. By the definition of P, we have $Kr = 0$ which is equivalent to $Lr = 0$. Thus we have $Pf \in \mathcal{G}$ so $r \in \mathcal{G}$ which implies that $r = 0$. So we get from (38)

$$\|\chi(D)f\|_{H^{-\frac{1}{2}}} \le C\|Nf\|_{H^{\frac{1}{2}}} \tag{39}$$

Finally, we use

$$\|N\chi(D)f\|_{H^1} \le C\|\chi(D)f\|_{L^2} \le C\|f\|_{L^2}$$

and the fact that $N(1-\chi(D))f = 0$ to derive that $L^* : L^2(\mathbb{R}^3 \times \mathbb{S}^2) \to H^{\frac{1}{2}}(\mathbb{R}^{3+1}, \mathrm{Sym}^2)$ is continuous, see also [17]. Thus we obtain that

$$\|\chi(D)f\|_{H^{-\frac{1}{2}}} \le C\|Lf\|_{L^2}$$

For general μ, we can add the $\langle\zeta\rangle^\mu$ factor to the proof to get (37). $\square$

5. The perturbed transform

We return to the identity (17) for $g, \tilde{g} \in \mathcal{A}(\mathcal{K}, \epsilon)$. Using Lemma 2.2 and 2.3, we have

$$\begin{aligned} &\frac{\partial \tilde{\mathsf{z}}}{\partial \mathsf{z}^{(0)}}(\tilde{s}_0 - \kappa(s, \mathsf{z}^{(0)}), \mathsf{z}(s, \mathsf{z}^{(0)}))\partial_s \kappa(s, \mathsf{z}^{(0)}) \\ =&\frac{\partial \mathsf{z}_0}{\partial \mathsf{z}^{(0)}}(\tilde{s}_0 - s, \mathsf{z}_0(s, \mathsf{z}^{(0)})) + O(\epsilon) = \begin{pmatrix} \mathrm{Id} & s\,\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix} + O(\epsilon) \end{aligned}$$

in C^1. Here, z_0 is the Hamiltonian flow for g_0. We repeat the calculation in Section 3 for $\tilde{g}, g \in \mathcal{A}(\mathcal{K}, \epsilon)$ with $\operatorname{tr} g = \operatorname{tr} \tilde{g}, \operatorname{div} g = \operatorname{div} \tilde{g}$. We still get (20) with $u^{ij} = \tilde{g}^{ij} - g^{ij}$. From the last four components of (17), we obtain that

$$\int_0^{s_0} \sum_{i,j=0}^{3} (1 + w_{ij}(z(s), \zeta(s)))\partial_k u^{ij}(z(s))\zeta_i(s)\zeta_j(s) ds = 0, \quad k = 0, 1, 2, 3 \tag{40}$$

where $w_{ij} \in O(\epsilon)$ in C^1 and we used $(z(s), \zeta(s)) = \mathsf{z}(s, \mathsf{z}^{(0)})$. In this section, we analyze a weighted light ray transform

$$L_w u(\gamma) = \int_0^{s_0} \sum_{i,j=0}^{3} (1 + w_{ij}(\gamma(s), \dot{\gamma}(s)))u_{ij}(\gamma(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) ds \tag{41}$$

where γ denotes light-like geodesics on $(\mathcal{M}, g)$ for $g \in \mathcal{A}(\mathcal{K}, \epsilon)$, and w_{ij} are smooth functions supported in a fixed compact set satisfying

$$\|w\|_{C^1} \doteq \sum_{i,j=0}^{3} \|w_{ij}\|_{C^1} < \epsilon \tag{42}$$

We assume that $\epsilon \in (0, 1)$ but later ϵ will be taken to be sufficiently small. By raising and lowering indices using g, the integral in (40) becomes the transform L_w. Our goal is to prove the injectivity of L_w which is the key ingredient for proving Theorem 1.1. Note that L_w is a perturbation of L but it involves both small weight and metric perturbations. The idea is to obtain a stability estimate similar to Proposition 4.1 which allows us to recover $\hat{u}$ on a non-empty open set of the phase space.

Consider the metric $g \in \mathcal{A}(\mathcal{K}, \epsilon)$ as a perturbation of g_0. Let $\gamma_{x,v}$ be the unique geodesic satisfying $\gamma_{x,v}(0) = (0, x)$ and $\dot{\gamma}_{x,v}(0) = (1, v)$. As shown in Lemma 9.2 of [14] (with slight modifications to adjust the parametrization of geodesics), we can write $\gamma_{x,v}(s) = (s, x+sv)+(\alpha(s, x, v), \beta(s, x, v))$ with $\beta = (\beta_1, \beta_2, \beta_3)$ such that α, β are smooth scalar functions satisfying

$$\|\alpha\|_{C^2(\mathcal{M})} < C_0\epsilon, \quad \|\beta_i\|_{C^2(\mathcal{M})} < C_0\epsilon \tag{43}$$

for some $C_0 > 0$ uniform for $g \in \mathcal{A}(\mathcal{K}, \epsilon)$. From now on, we use this parametrization in (41).

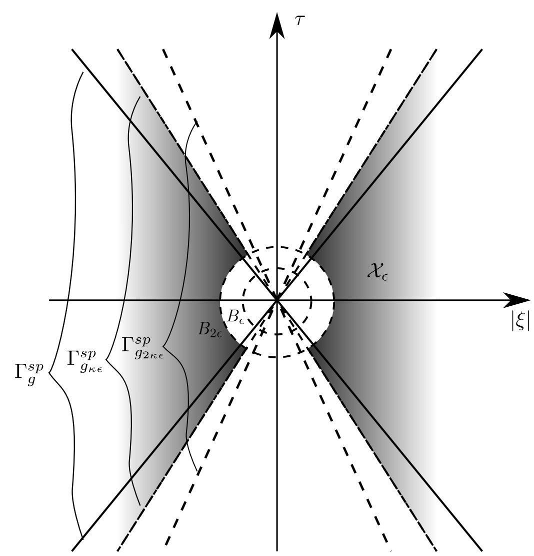

FIGURE 2. Construction of the cut-off function ψ in phase space. The picture shows the cones $\Gamma^{sp}_g, \Gamma^{sp}_{\kappa\epsilon}$ and $\Gamma^{sp}_{2\kappa\epsilon}$ projected to the τ-$|\xi|$ plane when ξ is represented in polar coordinates. The shaded region is $\mathcal{X}_\epsilon$ defined in (45) on which the function $\psi = 1$.

First, we construct a cut-off function in the phase space, denoted by ψ below. Note that for any $\zeta \in T_z\mathbb{R}^{3+1}$, we have $-3\epsilon|\zeta|^2 \leq g(\zeta, \zeta) - g_0(\zeta, \zeta)$. For $\epsilon > 0$, let

$$g_\epsilon = -(1 + 3\epsilon)dz_0^2 + (1 - 3\epsilon)\sum_{i=1}^{3} dz_i^2 \tag{44}$$

We have $g_\epsilon(\zeta, \zeta) \leq g(\zeta, \zeta)$ for all $\zeta \in \mathbb{R}^4$. For our construction, we will need $g_{\kappa\epsilon}, g_{2\kappa\epsilon}$ for some $\kappa \geq 1$ to be chosen later. Let $\Gamma^{sp}_g, \Gamma^{sp}_{\kappa\epsilon}, \Gamma^{sp}_{2\kappa\epsilon}$ be the space-like cones for $g, g_{\kappa\epsilon}, g_{2\kappa\epsilon}$. Note that Γ_g^{sp} varies as the base point varies but we always have $\Gamma_g^{sp} \subset \Gamma_{\kappa\epsilon}^{sp} \subset \Gamma_{2\kappa\epsilon}^{sp}$. See Figure 2. It is important to keep in mind that κ will be chosen independent of ϵ so when κ is fixed, the cones will be sufficiently close when ϵ is sufficiently small. Next, we avoid possible singularities at the vertex of the cone. Let $\mathbb{B}_\epsilon = \{\zeta \in \mathbb{R}^4 : |\zeta| < \epsilon\}$ and define

$$\mathcal{X}_\epsilon = \Gamma_{\kappa\epsilon}^{sp} \backslash \overline{\mathbb{B}_{2\epsilon}} \text{ and } \mathcal{X}_{2\epsilon} = \Gamma_{2\kappa\epsilon}^{sp} \backslash \overline{\mathbb{B}_\epsilon}. \tag{45}$$

Note that $\mathcal{X}_\epsilon \subset \mathcal{X}_{2\epsilon}$. See Figure 2. Now we let $\psi(\zeta)$ be C^∞ function such that $\psi = 1$ on $\mathcal{X}_\epsilon$ and $\psi = 0$ on $\mathbb{R}^4 \backslash \mathcal{X}_{2\epsilon}$. We define a Fourier multiplier via $\psi(D) = \mathscr{F}^{-1}\psi\mathscr{F}$.

Next, for $f \in C_0^\infty(\mathbb{R}^{3+1}, \mathrm{Sym}^2)$ supported in $\mathcal{K}$, we write (41) as

$$L_w f(x, v) = \int_0^{s_0} \sum_{i,j=0}^{3} \phi(s)(1 + w_{ij}(s, x, v)) f_{ij}(\gamma_{x,v}(s)) \dot{\gamma}_{x,v}^i(s) \dot{\gamma}_{x,v}^j(s) ds \tag{46}$$

in which $\phi(s)$ is compactly supported on $\mathbb{R}$ and $\phi(s) = 1$ on $[0, s_0]$. The choice of ϕ is not unique but it will not affect the transform. We consider

$$L_w f = L_w \psi(D) f + L_w(1 - \psi(D)) f \tag{47}$$

in which $\psi(D), 1 - \psi(D)$ are acting on each component of f. For our injectivity result, we make the following assumptions.

$$\|f\|_{H^{3/2}} \le C_0' \|\chi(D) f\|_{H^{-1/2}} \text{ for some } C_0' > 0. \tag{48}$$

In fact, this condition is more than what we need but it is easier to state. We deduce that

$$\|\psi(D) f\|_{H^{3/2}} \le C_0' \|\psi(D) f\|_{H^{-1/2}} + C_\rho \|f\|_{H^\rho} \tag{49}$$

for $\rho \in \mathbb{R}$ and some generic constants C_ρ. We will estimate the two terms on the right hand side.

Lemma 5.1. *Let $\kappa > 1$. For $\epsilon > 0$ sufficiently small and all w satisfying* (42) *and $g \in \mathcal{A}(\mathcal{K}, \epsilon)$, we have*

$$\|\psi(D) f\|_{H^{-\frac{1}{2}}} \le C_1 \|L_w \psi(D) f\|_{L^2} + C_\rho \|f\|_{H^\rho}, \quad f \in \mathcal{G}. \tag{50}$$

for some $C_1 > 0$ uniform in w, g.

Proof. We prove by using a perturbation argument. Hereafter, C denotes a generic constant (independent of ϵ). We do not keep track of it.

The idea is to compare L_w with the transform for a constant metric perturbation of g_0. In particular, let $L_{2\kappa\epsilon}$ be the light ray transform of $g_{2\kappa\epsilon}$

$$L_{2\kappa\epsilon} f(x, v) = \int_{\mathbb{R}} \phi(s) \sum_{i,j=0}^{3} f_{ij}(c^{-1}s, x + sv) \tilde{\theta}^i \tilde{\theta}^j ds, \quad \tilde{\theta} = (c^{-1}, v), v \in \mathbb{S}^2 \tag{51}$$

where $c = (\frac{1+6\kappa\epsilon}{1-6\kappa\epsilon})^{\frac{1}{2}}$ is close to 1 for ϵ small. Note that it suffices to assume $g_{2\kappa\epsilon} = g_0$ outside some compact set $\mathcal{W}$ in $\mathbb{R}^{3+1}$. This can be arranged by taking $\mathcal{W} = [0, T] \times \tilde{\mathcal{W}}$ with $\tilde{\mathcal{W}}$ compact in $\mathbb{R}^3$ and sufficiently large such that any light-like geodesic on $(\mathcal{M}, g)$ staring from $\{0\} \times \tilde{\mathcal{W}}^c$ does not intersect $\mathcal{K}$. Then we can make the arrangement $g_{2\kappa\epsilon} = g_0$ outside $\mathcal{W}$ by using a smooth cut-off function. The new metric is still a small perturbation of g_0 and the transform is unchanged.

Next, let $\gamma_{x,v}(s), s \in (0, s_0)$ be a light-like geodesic for metric g on $\mathcal{M}$ and write $\gamma_{x,v}(s) = (s, x+sv)+(\alpha(x, s, v), \beta(x, s, v))$ with estimate (43). For $f \in C_0^\infty(\mathcal{K}, \mathrm{Sym}^2)$ and $(x, v) \in \mathbb{R}^3 \times \mathbb{S}^2$, we have (with $\imath^2 = -1$)

(52)
$$
\begin{aligned}
&L_w\psi(D)f(x, v) - L_{2\kappa\epsilon}\psi(D)f(x, v)\\
=&\int_{\mathbb{R}} \phi(s) \sum_{i,j=0}^{3} ((1 + w_{ij})\psi(D)f_{ij}(\gamma_{x,v}(s))\dot{\gamma}^i_{x,v}(s)\dot{\gamma}^j_{x,v}(s) - \psi(D)f_{ij}(c^{-1}s, x + sv)\tilde{\theta}^i\tilde{\theta}^j)ds\\
=&(2\pi)^{-2} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(s)e^{\imath s\tau+\imath(x+sv)\cdot\xi} \sum_{i,j=0}^{3} \big((1 + w_{ij})(1 - e^{\imath(\alpha\tau+\beta\cdot\xi)}) - \tilde{w}_{ij}e^{\imath(c^{-1}-1)s\tau}\big)\\
&\cdot \psi(\tau, \xi)\hat{f}_{ij}(\tau, \xi)\theta^i\theta^j dsd\tau d\xi\\
&+ (2\pi)^{-2} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(s)e^{\imath s\tau+\imath(x+sv)\cdot\xi} \sum_{i,j=0}^{3} \big((1 + w_{ij})(1 - e^{\imath(\alpha\tau+\beta\cdot\xi)})\big)\\
&\cdot \psi(\tau, \xi)\hat{f}_{ij}(\tau, \xi)\hat{w}_{ij}dsd\tau d\xi
\end{aligned}
$$

where $\tilde{w}_{ij}$ are constants close to 1, and $\hat{w}_{ij}$ are smooth terms of order $O(\epsilon)$ in C^1 coming from the differences of θ and $\dot{\gamma}_{x,v}$, thanks to the estimate (43). Thus

$$
\begin{aligned}
&\|L_w\psi(D)f - L_{2\kappa\epsilon}\psi(D)f\|^2_{L^2}\\
\le& C\int_{\mathbb{R}^3}\int_{\mathbb{R}} \epsilon\langle(\tau, \xi)\rangle| \sum_{i,j=0}^{3} \psi(\tau, \xi)\hat{f}_{ij}(\tau, \xi)|^2 d\tau d\xi \qquad (53)\\
\le& C\epsilon\|\psi(D)f\|_{H^{-\frac12}}\|\psi(D)f\|_{H^{\frac32}} \le C\epsilon\|\psi(D)f\|^2_{H^{-\frac12}} + C_\rho\|f\|^2_{H^\rho}
\end{aligned}
$$

Here, we used that $g_{2\kappa\epsilon}$ is arranged to be equal to g_0 outside some compact set of $\mathbb{R}^{3+1}$. Thus $L_w\psi(D)f(x, v) - L_{2\kappa\epsilon}\psi(D)f(x, v)$ is compactly supported in x. Also, we used (49).

Next, we need a stability estimate like Proposition 4.1 for $L_{2\kappa\epsilon}$. This can be obtained by transforming $L_{2\kappa\epsilon}$ to L. We consider a diffeomorphism $\Phi : \mathbb{R}^{3+1} \to \mathbb{R}^{3+1}$ defined by $\Phi(t, x) = (c^{-1}t, x) = (s, x)$. We see that $\Phi^* f_{ij}(t, x) = f_{ij}(c^{-1}t, x), i, j = 1, 2, 3$, $\Phi^* f_{0j}(t, x) = c^{-1}f_{0j}(c^{-1}t, x), j = 1, 2, 3$ and $\Phi^* f_{00}(t, x) = c^{-2}f_{00}(c^{-1}t, x)$. Thus

$$
\begin{aligned}
L_{2\kappa\epsilon}f(x, v) &= \int_{\mathbb{R}} \phi(t) \sum_{i,j=0}^{3} f_{ij}(c^{-1}t, x + tv)\tilde{\theta}^i\tilde{\theta}^j dt\\
&= \int_{\mathbb{R}} \phi(t) \sum_{i,j=0}^{3} \Phi^* f_{ij}(t, x + tv)\theta^i\theta^j dt = L\Phi^* f(x, v)
\end{aligned}
$$

Note that if $f \in \mathcal{G}$ so $\operatorname{tr} f = \operatorname{div} f = 0$, then $\operatorname{tr} \Phi^* f = 0$ and $\operatorname{div} \Phi^* f = 0$. Applying the estimates for L, we get

$$\|\chi(D)\Phi^* f\|_{H^{-\frac{1}{2}}} \leq C\|L\Phi^* f\|_{L^2} + \|(\mathrm{Id} - P)\Phi^* f\|_{L^2} = C\|L_{2\kappa\epsilon} f\|_{L^2} \tag{54}$$

For the left hand side of (54), we use Plancherel's theorem to get

$$\begin{aligned}\|\chi(D)\Phi^* f\|^2_{H^{-\frac{1}{2}}} &= \int_{\mathbb{R}^4} \langle(\tau, \xi)\rangle^{-1} |\chi(\tau, \xi)\hat{f}(\tau/c, \xi)|^2 c^{-2} d\tau d\xi \\ &= \int_{\mathbb{R}^4} \langle(c\tau, \xi)\rangle^{-1} |\chi(c\tau, \xi)\hat{f}(\tau, \xi)|^2 c^{-1} d\tau d\xi \geq C_0 \|\chi_{2\kappa\epsilon}(D) f\|^2_{H^{-\frac{1}{2}}}\end{aligned}$$

Here, $\chi_{2\kappa\epsilon}$ is the characteristic function for the space-like cone $\Gamma^{sp}_{2\kappa\epsilon}$ and define $\chi_{2\kappa\epsilon}(D)$ as a Fourier multiplier. Also, the constant C_0 can be taken to be independent of ϵ. We thus get from (54) that

$$\|\chi_{2\kappa\epsilon}(D) f\|_{H^{-\frac{1}{2}}} \leq C\|L_{2\kappa\epsilon}\chi_{2\kappa\epsilon}(D) f\|_{L^2} + \epsilon\|\chi_{2\kappa\epsilon}(D) f\|_{H^{-\frac{1}{2}}}, \quad f \in \mathcal{G}. \tag{55}$$

Finally, note that $\chi_{2\kappa\epsilon}\psi = \psi$. We have $\chi_{2\kappa\epsilon}(D)\psi(D) f = \psi(D) f$. Thus combining (53) and (55), we derive that

$$\begin{aligned}\|\psi(D) f\|_{H^{-\frac{1}{2}}} &\leq C\|L_{2\kappa\epsilon}\chi_{2\kappa\epsilon}(D)\psi(D) f\|_{L^2} \\ &\leq C\|L_w\psi(D) f\|_{L^2} + \|L_w\psi(D) f - L_{2\kappa\epsilon}\psi(D) f\|_{L^2} \\ &\leq C\|L_w\psi(D) f\|_{L^2} + C\epsilon^{\frac{1}{2}}\|\psi(D) f\|_{H^{-\frac{1}{2}}} + C_\rho\|f\|_{H^\rho}\end{aligned}$$

For ϵ sufficiently small, we obtain

$$\|\psi(D) f\|_{H^{-\frac{1}{2}}} \leq C_1\|L_w\psi(D) f\|_{L^2} + C_\rho\|f\|_{H^\rho}, \quad f \in \mathcal{G}$$

which completes the proof. □

Lemma 5.2. *There exists $\kappa > 1$ such that for all $\epsilon > 0$ small, we have*

$$\|L_w(1 - \psi(D)) f\|_{L^2} \leq C_{-1}\|f\|_{H^{-1}}, \quad f \in \mathcal{G} \tag{56}$$

where the constant C_{-1} is uniform in w satisfying (42) and $g \in \mathcal{A}(\mathcal{K}, \epsilon)$.

Proof. We start from

$$\begin{aligned}&L_w(1 - \psi(D)) f(x, v) \\ &= \int_{\mathbb{R}} \phi(s) \sum_{i,j=0}^{3} (1 + w_{ij}(s, x, v))(1 - \psi(D)) f_{ij}(\gamma_{x,v}(s))\dot{\gamma}^i_{x,v}(s)\dot{\gamma}^j_{x,v}(s) ds\end{aligned} \tag{57}$$

We write $\gamma_{x,v}(s) = (s + \alpha(x, s, v), x + sv + \beta(x, s, v))$ and get

$$\begin{aligned}&L_w(1 - \psi(D)) f(x, v) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} e^{\imath(x+sv+\beta(x,s,v)-y)\cdot\xi + \imath(s+\alpha(x,s,v)-\sigma)\tau} \\ &\phi(s) \sum_{i,j=0}^{3} (1 + w_{ij}(s, x, v))(1 - \psi(\tau, \xi)) f_{ij}(\sigma, y)\dot{\gamma}^i_{x,v}(s)\dot{\gamma}^j_{x,v}(s) d\sigma dy d\tau d\xi ds\end{aligned}$$

This is an Fourier integral operator (FIO) with phase function

$$\Phi = (x + sv + \beta(x, s, v) - y) \cdot \xi + (s + \alpha(x, s, v) - \sigma)\tau$$

Using ξ, τ, s as parameters, we see that the critical set is given by

$$x + sv + \beta(x, s, v) = y, \quad s + \alpha(x, s, v) = \sigma, \quad v \cdot \xi + \partial_s\beta\xi + (1 + \partial_s\alpha)\tau = 0. \tag{58}$$

Note that $1-\psi(\tau, \xi)$ is supported in $|\tau| > (1+\kappa\epsilon)|\xi|$ for $|\tau|, |\xi|$ large. On the critical set, we obtain from the last equation of (58) that

$$|\xi| \geq |v \cdot \xi| \geq (1 + a)|\tau| - b|\xi|$$

where $a, b \in (0, C_2\epsilon)$ for some $C_2 > 0$ independent of ϵ and is uniform for metrics $g \in \mathcal{A}(\mathcal{K}, \epsilon)$. This implies that $|\tau| < (1 + C_3\epsilon)|\xi|$ for a fixed C_3. Now we can choose $\kappa > \max(C_3, 1)$ so the critical set is empty for $|\tau|, |\xi|$ large. Because $1 - \psi$ is supported in $\mathbb{B}_{2\epsilon} \cup \mathbb{R}^4 \backslash \mathcal{X}_\epsilon$, we see that the operator (57) is smoothing and we obtain the desired estimates. □

From now on we fix $\kappa > 1$ such that Lemma 5.2 holds. The key result of this section is

Proposition 5.3. *Suppose $f \in H^{3/2} \cap \mathcal{G}$ satisfies (48) and $\|f\|_{H^{-1/2}} \leq C_0''$ for some $C_0'' > 0$. Then there exist $\epsilon > 0$ sufficiently small and $C_4 > 0$ such that for w satisfying (42), the light ray transform L_w satisfies*

$$\|\psi(D)f\|_{H^{-\frac{1}{2}}} \leq C_4\|L_w f\|_{L^2}, \quad f \in \mathcal{G}. \tag{59}$$

Proof. Using Lemma 5.1 and 5.2, we arrive at

$$\|\psi(D)f\|_{H^{-\frac{1}{2}}} \leq C_1\|L_w f\|_{L^2} + C_{-1}\|f\|_{H^{-1}}. \tag{60}$$

We argue by contradiction and assume that (59) is not true. Then for $n = 1, 2, \cdots$, there exists (i) metric $g^n \in \mathcal{A}(\mathcal{K}, 1/n)$; (ii) weight function w^n on $\mathcal{M}$ supported in $\mathcal{K}$ with $\|w^n\|_{C^1} < 1/n$; (iii) $f^n \in H^{-\frac{1}{2}} \cap \mathcal{G}$ supported in $\mathcal{K}$ with $\|f^n\|_{H^{-1/2}} \leq C_0''$ and $\|\psi(D)f^n\|_{H^{-\frac{1}{2}}} = 1$, such that

$$\|\psi(D)f^n\|_{H^{-\frac{1}{2}}} \geq n\|L_{(w^n,g^n)}f^n\|_{L^2} \tag{61}$$

where we added g^n to the notation L_{w^n} to emphasize its dependency. Because $H^{-\frac{1}{2}}(\mathcal{K})$ is compactly embedded in $H^{-1}(\mathcal{K})$, there exists a subsequence still denoted by f^n which converges to f in H^{-1}. Taking $n \to \infty$ in (61), we arrive at $\|Lf\|_{L^2} = 0$ so $Lf = 0$. Note that f also satisfies $\operatorname{tr} f = \operatorname{div} f = 0$. By the injectivity of L on compactly supported tensors (see Remark 3.3), we get $f = 0$.

Finally, using (60), we get

$$C_1\|L_{g^n,w^n}f^n\|_{L^2} + C_{-1}\|f^n\|_{H^{-1}} \geq 1$$

which implies that $\|f\|_{H^{-1}} \geq C > 0$ by taking $n \to \infty$. This contradicts to $f = 0$. □

Now we prove the injectivity of L_w.

Proposition 5.4. *Suppose $u \in \mathcal{G}$ satisfies* (48)*. Then there exists $\epsilon > 0$ such that and $L_w u = 0$, then $u = 0$.*

Proof. By rescaling u, we can assume that $\|u\|_{H^{-1/2}} \leq C_0''$ and u satisfies $L_w u = 0$ and (48). With $L_w u = 0$, $u \in \mathcal{G}$, we deduce $\psi(D)u = 0$ from Proposition 5.3. By the definition of $\psi(D)$, we conclude that $\hat{u} = 0$ on $\mathcal{X}_\epsilon$. Thus we obtain that $\hat{u} = 0$ on $\mathbb{R}^4$ by the analyticity of $\hat{u}$. So $u = 0$. □

6. Proof of the theorem

Consider $g_{\epsilon_1}, g_{\epsilon_2}$ in the statement of Theorem 1.1. Following the argument in Section 2, we arrive at the identity (40). We need to check that $\partial_k u^{ij} = \partial_k g^{ij}_{\epsilon_2} - \partial_k g^{ij}_{\epsilon_1}$ satisfies condition (48) in order to apply Proposition 5.4. We observe that $\lim_{\epsilon_2 \to \epsilon_1} (\partial_k g^{ij}_{\epsilon_2} - \partial_k g^{ij}_{\epsilon_1})/(\epsilon_2 - \epsilon_1) = \partial_\epsilon \partial_k g^{ij}_\epsilon|_{\epsilon=\epsilon_1}$. Note that $\partial_\epsilon \partial_k g^{ij}_\epsilon|_{\epsilon=0}$ is a fixed non-zero tensor hence satisfies the condition (48) for some constant C'. By continuity of the norms in ϵ, we conclude that $\partial_k u^{ij}$ satisfies (48) for ϵ_1, ϵ_2 sufficiently small with possibly a different constant C'. Then we can apply Proposition 5.4 to obtain $\epsilon > 0$ such that $\partial_k u = 0$ if $\epsilon_1, \epsilon_2 < \epsilon$. By the fact that u is compactly supported, we conclude that $u = 0$ so $g_{\epsilon_1} = g_{\epsilon_2}$. This completes the proof of Theorem 1.1.

Finally, we remark that one can state a more general version of Theorem 1.1. Let $g, \tilde{g}$ be Lorentzian metrics in $\mathcal{A}(\mathcal{K}, \epsilon)$ and $S, \tilde{S}$ be the corresponding scattering relation for null geodesics defined as in (2). Suppose that $\partial_k(g - \tilde{g}), k = 1, 2, 3$ satisfy (48), namely

$$\|\partial_k(g - \tilde{g})\|_{H^{3/2}} \leq C_0' \|\chi(D)(\partial_k(g - \tilde{g}))\|_{H^{-1/2}} \text{ for some } C_0' > 0$$

Then there exists $\epsilon > 0$ such that, if $S = \tilde{S}$, then $g = \tilde{g}$.

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