COSMIC BACKGROUND RADIATION IN SCALAR GRAVITY

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Abstract. Consider the Cosmic Microwave Background (CMB) in Nordström's scalar gravity theory. The photon distribution is governed by the massless Nordström-Vlasov system. We show that the gravitational perturbation is uniquely determined by the photon distribution observed later at a Cauchy surface.

1. Introduction

Cosmic Microwave Background (CMB) is the thermal radiation remnant from the Big Bang. It is considered as a primary source of information regarding the early universe. For example, the EGS (Ehlers-Geren-Sachs) theorem roughly states that the isotropy of the observed CMB implies the isotropy and spatial homogeneity of the universe, see [4] and [11] for more discussions. The CMB radiation can be modeled upon the Einstein-Boltzmann equation coupled with matters which form a complicated nonlinear system. Due to the complexity, it is often analyzed on the linearization level in cosmology literatures, such as the derivation of the Sachs-Wolfe effects [12]. Recently, it is proved in [13] (see also [14]) that on the linearization level, the integrated Sachs-Wolfe effect uniquely determines the metric perturbations of scalar type in a Friedman-Lemaître-Robertson-Walker (FLRW) universe. This result is further generalized in [15] to include photon scattering and absorptions by using the linear Boltzmann equation model. It is natural to ask whether such determination results still hold when the nonlinear effects are taken into account. In this short note, we give a positive answer in Nördstrom's scalar gravitation theory. Despite being non-physical, the theory is often considered as a test field for the Einstein-Vlasov system, see e.g. [10], which we aim to pursue in the future.

In 1913, Nordström introduced a gravitation theory, see Calogero [1] and Rendall [10] for more discussions. In this theory, one considers on $\mathbb{R}^{3+1}$ a Lorentz metric g conformal to the Minkowski metric g_0, that is $g = e^{2\phi} g_0$ for some scalar function ϕ. One assumes that $\Box\phi = 0$ where $\Box$ is the d'Alembertian on $(\mathbb{R}^{3+1}, g_0)$. We consider the distribution of photons on $(\mathbb{R}^{3+1}, g)$. Let p be a null vector (light-like vector) at $z \in \mathbb{R}^{3+1}$ which satisfies $g_0(p, p) = 0$. Let $f(z, p)$ be the photon distribution function. Hereafter, we use $z = (t, x), t \in \mathbb{R}, x \in \mathbb{R}^3$ as local coordinates. Let $T > 0$ and $\mathcal{M} = (0, T) \times \mathbb{R}^3$. We consider the massless

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Vlasov-Nordström system in $\mathcal{M}$

$$
\begin{aligned}
&\Box\phi = 0 \\
&p^\alpha \frac{\partial f}{\partial z^\alpha} - 2p^\alpha \frac{\partial \phi}{\partial z^\alpha} p^i \frac{\partial f}{\partial p^i} = 0
\end{aligned}
\tag{1}
$$

with initial conditions

$$
\phi(0,x) = \phi_0(x), \quad \partial_t \phi(0,x) = \phi_1(x), \quad f(0,x,p) = f_0(x,p), \tag{2}
$$

see [5, Section 4.1]. Hereafter, we use Einstein summation convention in which Greek letters take values $0,1,2,3$ and Lattin letters take values $1,2,3$. The first equation in (1) is the standard wave equation. The second equation in (1) is the (Liouville–)Vlasov equation, the left hand side of which is the Lie derivative of f under the geodesic vector field on $(\mathcal{M}, g)$, see [2] or Section 4 for more details. Note that the two equations in (1) are decoupled which facilitates the analysis. We remark that (1) can be obtained from the Einstein-Vlasov equation for metrics conformal to the Minkowski metric if one ignores certain nonlinear terms, see [5].

It is convenient to write $p = (p^0, p^0 v), v \in \mathbb{S}^2, p^0 > 0$. Here, p^0 is the photon energy. For the CMB problem, it is reasonable to assume that the initial data satisfies the Planck distribution

$$
f_0(z,p) = (e^{p^0/T_0} + 1)^{-1} \tag{3}
$$

where T_0 is the background temperature (a non-zero constant). For our purpose, it suffices to consider a fixed energy level say $p^0 = 1$. Let the photon distribution observed at $t = T$ with $p^0 = 1$ be

$$
f_T(x,v) = f(T,x,v), \quad x \in \mathbb{R}^3, v \in \mathbb{R}^3 \tag{4}
$$

where f is the solution of (1). The question is whether f_T determines ϕ_0, ϕ_1. We point out that this is certainly impossible if $f_0 = 0$. Below, we write $\vec{\phi} = (\phi_0, \phi_1)$ and denote the corresponding f_T by f_T^ϕ. Our main result is

Theorem 1.1. *Let $T > 0$ and $\mathcal{X}$ be a compact set of $\mathbb{R}^3$. Let $\vec{\phi} \in C^\infty(\mathbb{R}^3; \mathbb{R}^2)$ with supp $\vec{\phi} \subset \mathcal{X}$ and $\phi_0 \neq 0$. Then there exists a neighborhood $\mathcal{W}$ of $\vec{\phi}$ in $H^\mu(\mathbb{R}^3; \mathbb{R}^2) \cap C^\infty(\mathbb{R}^3; \mathbb{R}^2), \mu > 5$ such that for any $\vec{\psi} \in \mathcal{W}$, if $f_T^\phi = f_T^\psi$, then $\vec{\phi} = \vec{\psi}$.*

Roughly speaking, the theorem says that in scalar gravity, gravitational perturbations are locally determined by the CMB at $t = T$. We remark that the regularity requirement on $\vec{\phi}$ in Theorem 1.1 is by no means optimal but it is not our concern. However, it is not clear at least from our proof whether or not the condition $\phi_0 \neq 0$ can be dropped. The condition is used to derive certain regularity conditions (see Proposition 5.1 and Remark 5.2) which seem to be related to those needed for the almost EGS theorems, see discussions in [11, Section 4.1]. Finally, we remark that one can replace the Vlasov equation in (1) by the linear Boltzmann equation to include photon scattering and absorption, and we expect that Theorem 1.1 still holds for generic absorption coefficients and scattering kernels in view of the results in [15].

The note is organized as follows. In Section 2, we describe the structure of the proof which is based on the implicit function theorem of Nash-Moser. We establish some well-posedness result in Section 4 and derive the linearized problems. The main part of the proof is in Section 5 where we prove the injectivity and stability of the linearized operator. Finally, we finish the proof of Theorem 1.1 in Section 6.

2. The plan

We formulate the forward problem as a nonlinear map

$$\Phi(\vec{\phi}) = f_T \tag{5}$$

where f_T is defined in (4). To solve the inverse problem, we attempt to apply the inverse function theorem. For the linearized operator $\Phi'(\vec{\phi})$, we can prove injectivity with some stability estimates. In fact, the problem boils down to an integral transform of solutions of wave equations, which is simpler than those studied in [13, 15]. However, we do not know where $\Phi'(\vec{\phi})$ is surjective thus the inverse function theorem for Banach spaces cannot be applied directly. We will resort to a version of the implicit function theorem of Nash-Moser which only requires the injectivity of Φ', see [6, 7]. In particular, we use the theorem in Hörmander's paper [7, Theorem 2.3.1]. Because of the technical nature of the theorem, we recall it below for reader's convenience.

Let $\mathcal{X}$ be a compact C^∞ manifold and Φ a map from $C^\infty(\mathcal{X};\mathbb{R}^N)$ to $C^\infty(\mathcal{X};\mathbb{R}^{N'})$ with N, N' positive integers. Denote the Sobolev space on $\mathcal{X}$ of order μ by H^μ and the Sobolev norm by $\|\cdot\|_\mu$. Notice that this is different from Hörmander's paper where H^μ stands for Hölder spaces. However, we will explain that the theorem holds for Sobolev spaces as well, which is already known, see e.g. [6].

Let $\mathcal{V}$ be a H^μ neighborhood of u_0. We assume the following:

(A) The second differential of Φ satisfies for $u \in \mathcal{V} \cap C^\infty$

$$\begin{aligned}&\|\Phi''(u; v, w)\|_{\lambda_0+a} \leq C\{\|v\|_{m_1+a}\|w\|_{m_2} + \|v\|_{m_2}\|w\|_{m_1+a}\\ &+(\|v\|_{m_3}\|w\|_{m_4} + \|v\|_{m_4}\|w\|_{m_3})\|u\|_{m_5+a}, \quad 0 \leq a \leq a_\Phi,\end{aligned}$$

where $\lambda_0, m_1, \cdots, m_5$ are non-negative numbers. See [7, (2.1.5)].

(B) $\Phi'(u)$ when $u \in \mathcal{V} \cap C^\infty$ has a left inverse $\psi(u)$ and that the following is valid

$$\|\psi(u)g\|_{\mu_1+a} \leq C(\|g\|_{\lambda_1+a} + \|g\|_{\lambda_2}\|u\|_{\mu_2+a}), \quad 0 \leq a \leq a_\psi$$

where λ_1, μ_1, μ_2 are non-negative numbers. See [7, (2.1.6)].

(C) Denote $b^+ = \max(0, b)$ and set

$$M_j = m_j - \lambda_1, j = 1, 2, \cdots, 5, \quad M = \mu_2 - \mu_1, \quad \Lambda_j = \lambda_j - \lambda_{j-1}.$$

The following relations of the constants hold, see [7, (2.3.8)].

$$\alpha > \max(0, M_1, \cdots, M_5, M_1 + M_2 + \Lambda_1, \max(M_1, M_5) + \Lambda_1 + \Lambda_2^+)$$
$$2\alpha > M_3 + M_4 + \max(M, \Lambda_1 + \max(M_1, M_5)),$$
$$2\alpha > M + M_1 + M_2 + (\Lambda_1 + \Lambda_2)^+$$
$$\alpha + \alpha_\Phi > M_3 + M_4 + \Lambda_1$$
$$\alpha + \min(a_\psi, \beta) > \max(M_3 + M_4, M_1 + M_2 + (\Lambda_1 + \Lambda_2)^+)$$
$$\alpha \geq M, \quad \beta \geq \Lambda_2^+, \quad a_\Phi \geq \Lambda_1 + \Lambda_2^+.$$

(D) Let $\Phi_{\alpha\beta}^{-1}(f)$ be the set of all $u \in H^{\alpha+\mu_1}$ such that for some sequence $u_k \in \mathcal{V} \cap C^\infty$

$$u_k \to u \text{ in } H^{\alpha+\mu_1}, \quad \Phi(u_k) \to \Phi(u_0) + f \text{ in } H^{\beta+\lambda_1}.$$

Finally, given (A)–(D), the conclusion of Theorem 2.3.1 of [7] is that for every bounded set $\mathcal{B}$ in $H^{\alpha+\mu_1}$, one can find a constant N such that $\Phi_{\alpha\beta}^{-1}(f) \cap \mathcal{B}$ never has more than N elements and $\|u - v\|_0 > 1/N$ for any two different element.

We return to the issue that Theorem 2.3.1 of [7] was proved for Hölder spaces, denoted by C^μ below. The argument in Section 2.3 of [7] works if one replaces the Hölder norm by the Sobolev norm. In fact, the properties of Hölder spaces used in the proof are contained in Theorem A.5 of [7] which says that (i) C^μ is a Banach space which decreases when μ decreases, and (ii) there is the interpolation estimate for C^μ. Both hold for Sobolev functions in the same form.

3. The Sobolev estimates

To derive the linearized operator and the estimates, we need some well-posedness result for (1) and (2). Because the two equations in (1) are decoupled, it suffices to first solve the Cauchy problem for the wave equation and then solve the Vlasov equation. In general, the energy estimates for the Vlasov equation can be obtained by integrating the geodesic vector field as discussed in Choquet-Bruhat [2, Section X.4]. We carry out the calculation explicitly.

For our purpose, it suffices to take (ϕ_0, ϕ_1) smooth and supported in a compact set $\mathcal{X} \subset \mathbb{R}^3$. It is well-known that there is a unique solution $\phi \in C^\infty(\mathcal{M})$ for the wave equation in (1) and

$$\|\phi\|_{H^{s+1}(\mathcal{M})} \leq C(\|\phi_0\|_{H^{s+1}(\mathbb{R}^3)} + \|\phi_1\|_{H^s(\mathbb{R}^3)}), \quad s \in \mathbb{R}.$$

Next, the Christoffel symbols on $(\mathcal{M}, g)$ are given by

$$\Gamma^\gamma_{\alpha\beta} = \delta_{\gamma\alpha}\frac{\partial\phi}{\partial z^\beta} + \delta_{\gamma\beta}\frac{\partial\phi}{\partial z^\alpha} - g_{0,\alpha\beta}\frac{\partial\phi}{\partial z^\gamma} \tag{6}$$

in which $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and otherwise 0. We consider the transport equation associated with the geodesic flow

$$T_\phi = p^\alpha\left(\frac{\partial}{\partial z^\alpha} - p^\beta\Gamma^i_{\alpha\beta}\frac{\partial}{\partial p^i}\right) = p^\alpha\frac{\partial}{\partial z^\alpha} - 2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i\frac{\partial}{\partial p^i} \tag{7}$$

see [5, Section 4.1]. If $\phi \in C^\infty$, a classical C^∞ solution of $T_\phi f = 0$ can be found by integrating along the characteristics provided that $f_0 \in C^\infty$. In fact, the characteristics $\sigma(\tau) = (z(\tau), p(\tau)), \tau \geq 0$ satisfy

$$\frac{dz^\gamma}{d\tau} = p^\gamma, \quad \frac{dp^\gamma}{d\tau} = -\Gamma^\gamma_{\alpha\beta} p^\alpha p^\beta. \tag{8}$$

By using (6), the second equation above can be reduced to $d[e^{\phi(\sigma(\tau))} p^\gamma]/d\tau = 0$. Let $p(0) = \tilde{p}$. Then $p(\tau) = e^{-\phi(\sigma(\tau))}\tilde{p}$. The first equation in (8) can be solved by integrating in $\tau \geq 0$. Note that if ϕ is compactly supported in $\mathcal{M}$, then f is supported in $T\mathcal{K}$ for some compact set $\mathcal{K}$ of $\mathcal{M}$.

We consider the energy estimate of the solution. Below, we write $p = (p^0, \theta), \theta \in \mathbb{R}^3$. Then $p^0 = |\theta|$ for light-like vectors. We will use p and θ interchangeably in the follows. We consider the photon distribution f as a function of t, x, θ variables. Let

$$\mathcal{E}(f)(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^2(t, x, \theta)|\theta| d\theta dx.$$

For $u \in C^k(\mathcal{M})$, $k \geq 0$ integer, we use $|u|_{C^k} = \sup_{z \in \mathcal{M}} \sum_{|\beta| \leq k} |\partial^\beta u(z)|$ to denote the seminorms.

Lemma 3.1. *Let $\phi \in C^\infty(\mathcal{M})$ be compactly supported in $\mathcal{M}$ and $|\phi|_{C^1} \leq C_0$. Suppose that $f \in C^\infty$ satisfies $T_\phi f = |\theta| h$ on $[0, T] \times \mathbb{R}^3_x \times \mathbb{R}^3_\theta$. Then there exists a constant $C > 0$ independent of h such that for all $t \in [0, T]$,*

$$\mathcal{E}(f)(t) \leq (\mathcal{E}(f)(0) + \int_0^T \mathcal{E}(h)(s) ds) e^{Ct}$$

Proof. Suppose that $g \in C^\infty$ is supported in $T\mathcal{K}$ for some compact set $\mathcal{K}$ of $\mathcal{M}$. Consider $T_\phi g = |\theta| b$. We integrate the equation and apply integration by parts to get

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g(t, x)|\theta| dx d\theta &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g(0, x)|\theta| dx d\theta \\ + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b(s, x)|\theta| dx d\theta ds &+ 6 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (T_0 \phi) g(s, x) dx d\theta ds \end{aligned} \tag{9}$$

See also [2, Section X.4] and Lemma 4.3 of [5]. Now using $T_\phi f = |\theta| h$, we have

$$T_\phi(f^2) = 2f T_\phi f = 2f|\theta| h \tag{10}$$

We can apply (9) to (10) to get

$$\begin{aligned} \mathcal{E}(f)(t) \leq \mathcal{E}(f)(0) &+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |2f(s, x)h(s, x)||\theta| dx d\theta ds \\ &+ 6 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |T_0 \phi(s, x)| f^2(s, x) dx d\theta ds \\ &\leq \mathcal{E}(f)(0) + \int_0^t \mathcal{E}(f)(s) ds + \int_0^t \mathcal{E}(h)(s) ds \\ &+ 6 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |T_0 \phi(s, x)| f^2(s, x) dx d\theta ds \end{aligned}$$

By our assumption on ϕ, we know that $|T_0\phi|_{C^0} < C_1|\theta|$ for some $C_1 > 0$. We get

$$\mathcal{E}(f)(t) \leq (\mathcal{E}(f)(0) + \int_0^T \mathcal{E}(h)(t)dt) + C\int_0^t \mathcal{E}(f)(s)ds$$

for some constant $C > 0$. Finally, we can apply Gronwall's inequality to complete the proof. □

Next we consider higher order estimates. We introduce some notations. Let $Z = (z_0, z_1, z_2, z_3, \theta_1, \theta_2, \theta_3), z_0 \in (0,T), z_i, \theta_j \in \mathbb{R}, i, j = 1,2,3$. Denote $\partial^\beta f = \frac{\partial^\beta f}{\partial Z_0^{\beta_0}\cdots\partial Z_6^{\beta_7}}, \beta = \beta_0 + \cdots + \beta_7$. For $k = 1, 2, \cdots$, let

$$\mathcal{E}^{(k)}(f)(t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{|\beta|\leq k}(\partial^\beta f(t,x,\theta))^2|\theta|^{\beta_\theta+1}d\theta dx \tag{11}$$

where $\beta_\theta \in \mathbb{Z}$ is the number of derivatives in θ in $(\partial^\beta f)^2$ that are less than or equal to 4, see Definition 4.5 and 4.7 in Section X.4 of [2]. For example, if $\partial^\beta = \partial_{z^\alpha}$ for some $\alpha = 0,1,2,3$. Then $\beta_\theta = 0$. If $\partial^\beta = \partial_{p^i}$ for some $i = 1,2,3$, then $\beta_\theta = 2$. We remark that β_θ is needed because the terms have different asymptotic behaviors in fiber directions.

Lemma 3.2. *Let $\phi \in C^\infty(\mathcal{M})$ be compactly supported in $\mathcal{M}$ and $|\phi|_{C^{k+1}} \leq C_0$. Let $f \in C^\infty$ satisfy $T_\phi f = |\theta|h$ on $[0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\theta$ for $T > 0$. Then for all $t \in [0,T]$, there exists $C_k > 0$ such that*

$$\mathcal{E}^{(k)}(f)(t) \leq (\mathcal{E}^{(k)}(f)(0) + \int_0^T \mathcal{E}^{(k)}(h)(t)dt)e^{C_k t}, \quad k = 1,2,\cdots \tag{12}$$

Proof. We start with some calculations. For $k \geq 0$, we have

$$\begin{aligned}
&T_\phi(\partial^\beta f|\theta|^k)^2 = 2(\partial^\beta f|\theta|^k)T_\phi(\partial^\beta f|\theta|^k)\\
&= 2(\partial^\beta f|\theta|^k)(p^\alpha\frac{\partial}{\partial z^\alpha}(\partial^\beta f|\theta|^k) - 2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i\frac{\partial}{\partial p^i}(\partial^\beta f|\theta|^k))\\
&= 2(\partial^\beta f|\theta|^k)(|\theta|^k\partial^\beta(p^\alpha\frac{\partial f}{\partial z^\alpha}) - |\theta|^k(\partial^\beta p^\alpha)\frac{\partial f}{\partial z^\alpha} - |\theta|^k\partial^\beta(2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i\frac{\partial f}{\partial p^i}) \qquad (13)\\
&+ |\theta|^k\partial^\beta(2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i)\frac{\partial f}{\partial p^i} - 2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i\frac{\partial|\theta|^k}{\partial p^i}\partial^\beta f)\\
&= 2(\partial^\beta f|\theta|^k)|\theta|^k\partial^\beta(T_\phi f) + Q(f)
\end{aligned}$$

where

$$Q(f) = 2(\partial^\beta f|\theta|^k)(-|\theta|^k(\partial^\beta p^\alpha)\frac{\partial f}{\partial z^\alpha} + |\theta|^k\partial^\beta(2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i)\frac{\partial f}{\partial p^i} - 2p^\alpha\frac{\partial\phi}{\partial z^\alpha}p^i\frac{\partial|\theta|^k}{\partial p^i}\partial^\beta f)$$

Note that in the above calculation, there is no summation over k. Now consider the proof for $\mathcal{E}^{(1)}$ for which we apply (9) to $(\partial^\beta f)|\theta|^k$ for $|\beta| = 1$ with proper k. There are two cases to consider. If $\partial^\beta = \partial_{z^\alpha}$ for some $\alpha = 0,1,2,3$, we take $k = 0$. Then

$$|Q(f)(z,p)| \leq C|\partial^\beta f(z,p)|\sum_{i=1}^3|\theta|^2|\partial_{p^i}f(z,p)|$$

Here, the constant depends on $|\phi|_{C^2}$. Second, if $\partial^\beta = \partial_{p^i}, i = 1, 2, 3$, we take $k = 1$ to get

$$|Q(f)(z,p)| \leq C|\theta|^{2+1}|\partial^\beta f(z,p)| \sum_{i=1}^{3} |\partial_{p^i} f(z,p)| + |\theta|^2 |\partial^\beta f(z,p)| \sum_{\alpha=0}^{3} |\partial_{z^\alpha} f(z,p)|$$

Now we use (13) and (9) to get

$$\mathcal{E}^{(1)}(f)(t) \leq \mathcal{E}^{(1)}(f)(0) + C\int_0^t \mathcal{E}^{(1)}(f)(s)ds + \int_0^t \mathcal{E}^{(1)}(h)(s)ds$$

Then we obtain (12) by applying Gronwall's inequality.

Next we prove for $\mathcal{E}^{(2)}$. We again apply (9) to $(\partial^\beta f)|\theta|^k$ for $|\beta| = 2$ with suitable k. There are three cases to consider. First, if $\partial^\beta = \partial_{z^\alpha}\partial_{z^\gamma}$ for $\alpha, \gamma = 0, 1, 2, 3$, we take $k = 0$ to get

$$|Q(f)(z,p)| \leq C|\theta|^2 |\partial^\beta f(z,p)| \sum_{i=1}^{3} |\partial_{p^i} f(z,p)|$$

where C depends on $|\phi|_{C^3}$. Next, if $\partial^\beta = \partial_{p^i}\partial_{p^j}$ for $i, j = 1, 2, 3$, we take $k = 2$ to get

$$|Q(f)(z,p)| \leq C(|\theta|^4 |\partial^\beta f(z,p)| \sum_{i=1}^{3} |\partial_{p^i} f(z,p)| + |\theta|^{4+1} |\partial^\beta f(z,p)||\partial^\beta f(z,p)|)$$

Finally, if $\partial^\beta = \partial_{z^\alpha}\partial_{p^i}$ for some $\alpha = 0, 1, 2, 3, i = 1, 2, 3$, we take $k = 1$ to get

$$|Q(f)(z,p)| \leq C(|\theta|^{2+1} |\partial^\beta f(z,p)| \sum_{i=1}^{3} |\partial_{p^i} f(z,p)| + |\theta|^{2+1} |\partial^\beta f(z,p)||\partial^\beta f(z,p)|)$$

Again, we use (13) and (9) to get

$$\mathcal{E}^{(2)}(f)(t) \leq \mathcal{E}^{(2)}(f)(0) + C\int_0^t \mathcal{E}^{(2)}(f)(s)ds + \int_0^t \mathcal{E}^{(2)}(h)(s)ds.$$

We obtain (12) by applying Gronwall's inequality.

For $k \geq 3$, the proof is similar. Actually, if $\partial^\beta = \partial_z^\gamma \partial_p^\delta$ with $|\delta| \leq 2$, the estimates of $|Q(f)(z,p)|$ are of the form we analyzed for $k = 1, 2$. The only difference is that the constant C now depends on $|\phi|_{C^{k+1}}$. Next, we note that the form of the estimates does not change for $|\delta| \geq 3$ because the terms in $Q(f)$ are at most quadratic in p^i without the $|\theta|^k$ factors. $\square$

For $k = 0, 1, 2, \cdots$, we denote by $\mathcal{H}^k$ the completion of $C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3_\theta)$ under the norm $(\mathcal{E}^{(k)}(u)(T))^{\frac{1}{2}}$. It is convenient to use standard Sobolev spaces instead of the weighted spaces $\mathcal{H}^k$. Consider the solution $f(t, x, \theta)$ of (1) with initial condition (3) at a fixed energy level $|\theta| = p^0 = 1$. We let $v \in \mathbb{S}^2$ and denote the solution at $t = T$ by $f(T, x, v)$. If ϕ_0, ϕ_1 are smooth, we see that $f(T, x, v)$ belongs to $H^k(\mathbb{R}^3 \times \mathbb{S}^2)$ for integer $k \geq 0$ using Lemma 3.1 and 3.2.

4. The linearization

Now we are ready to derive the linearized operator. Let $f(t, x, \theta)$ be the solution of (1) with initial condition (2). Consider the nonlinear map (5). First, we find Φ'. Let $\lambda \in \mathbb{R}$ and compute

$$\Phi'(\vec{\phi})\vec{\psi} = \lim_{\lambda \to 0} \frac{\Phi(\vec{\phi} + \lambda\vec{\psi}) - \Phi(\vec{\phi})}{\lambda}$$

We find that $\Phi'(\vec{\phi})\vec{\psi} = f|_{t=T,|\theta|=1}$ where f is the solution of

$$\Box\psi = 0$$

$$p^\alpha \frac{\partial f}{\partial z^\alpha} - 2p^\alpha \frac{\partial \phi}{\partial z^\alpha} p^i \frac{\partial f}{\partial p^i} = 2p^\alpha \frac{\partial \psi}{\partial z^\alpha} p^i \frac{\partial f_0}{\partial p^i} \tag{14}$$

with initial conditions

$$\psi(0, x) = \psi_0(x), \quad \partial_t\psi(0, x) = \psi_1(x), \quad f(0, x, \theta) = 0$$

and ϕ satisfies

$$\Box\phi(t, x) = 0, \quad \phi(0, x) = \phi_0(x), \quad \partial_t\phi(0, x) = \phi_1(x).$$

Using the energy estimate Lemma 3.2, we see that

$$\|\Phi'(\vec{\phi})\vec{\psi}\|_{H^k} \leq C\|\psi\|_{H^{k+1}(\mathcal{M})} \leq C(\|\psi_0\|_{H^{k+1}(\mathbb{R}^3)} + \|\psi_1\|_{H^k(\mathbb{R}^3)}), \quad k \geq 0.$$

Notice that in (14), if we use the form of f_0 in (3), we get

$$2p^i \frac{\partial f_0}{\partial p^i} = -2p^i \frac{p^i}{p^0}(e^{1/T_0} + 1)^{-2} T_0^{-1} e^{1/T_0} = -2(e^{1/T_0} + 1)^{-2} T_0^{-1} e^{1/T_0},$$

which is a constant. Without loss of generality, we set $C_0 = 1$ for the rest of the paper.

Next, we compute the second derivative

$$\Phi''(\vec{\phi})(\vec{\psi}, \vec{\omega}) = \frac{\partial^2}{\partial\lambda\partial\mu} \Phi(\vec{\phi} + \lambda\vec{\psi} + \mu\vec{\omega})|_{\lambda=\mu=0}.$$

This is given by $f|_{t=T,|\theta|=1}$ where f is the solution of

$$\Box\phi = 0$$

$$p^\alpha \frac{\partial f}{\partial x^\alpha} - 2p^\alpha \frac{\partial \phi}{\partial x^\alpha} p^i \frac{\partial f}{\partial p^i} = 2p^\alpha \frac{\partial \psi}{\partial x^\alpha} p^i \frac{\partial f^\omega}{\partial p^i} + 2p^\alpha \frac{\partial \omega}{\partial x^\alpha} p^i \frac{\partial f^\psi}{\partial p^i} \tag{15}$$

with initial conditions

$$\phi(0, x) = \phi_0(x), \quad \partial_t\phi(0, x) = \phi_1(x), \quad f(0, x, \theta) = 0.$$

Here, ψ satisfy

$$\Box\psi(t, x) = 0, \quad \psi(0, x) = \psi_0(x), \quad \partial_t\psi(0, x) = \psi_1(x).$$

and ω satisfy

$$\Box\omega(t, x) = 0, \quad \omega(0, x) = \omega_0(x), \quad \partial_t\omega(0, x) = \omega_1(x).$$

Also, f^ψ, f^ω denote solutions of the linearized problem (14) with initial condition $\vec{\psi}$ and $\vec{\omega}$ respectively. Using Lemma 3.2 again, we know that

$$\mathcal{E}^{(k)}(f^\psi)(t) \leq C\|\psi\|_{H^{k+1}(\mathcal{M})} \leq C(\|\psi_0\|_{H^{k+1}(\mathbb{R}^3)} + \|\psi_1\|_{H^k(\mathbb{R}^3)})$$
$$\mathcal{E}^{(k)}(f^\omega)(t) \leq C\|\omega\|_{H^{k+1}(\mathcal{M})} \leq C(\|\omega_0\|_{H^{k+1}(\mathbb{R}^3)} + \|\omega_1\|_{H^k(\mathbb{R}^3)})$$

for $t \in [0, T]$ and $k = 0, 1, 2\cdots$. Applying these estimates and Lemma 3.2 to (15), we get

$$\|\Phi''(\vec{\phi})(\vec{\psi}, \vec{\omega})\|_{H^k} \leq C(\|\vec{\psi}\|_{H^{k+1}}\|\vec{\omega}\|_{H^{k+2}} + \|\vec{\omega}\|_{H^{k+1}}\|\vec{\psi}\|_{H^{k+2}}) \tag{16}$$

provided that $k+1 > 2$ so $H^{k+1}(\mathcal{M})$ is an algebra. Here, we used vector valued Sobolev spaces for $\vec{\psi}$ and $\vec{\omega}$.

5. Construction of the left inverse

We first consider the linearized map $\Phi'(\vec{\phi})$ at $\vec{\phi} = \vec{0}$ and try to construct a left inverse. We recall that $\Phi'(\vec{0})\vec{\psi} = f|_{t=T,|\theta|=1}$ where f is the solution of (14). Note that the equation for f in (14) for $p = (1, v), v \in \mathbb{S}^2$ becomes

$$\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} = p^\alpha \frac{\partial \psi}{\partial x^\alpha}$$

which can be integrated along characteristics to yield

$$f(T, x, v) = \int_0^T (\frac{\partial \psi}{\partial t} + v^i \frac{\partial \psi}{\partial x^i})(t, x + tv)dt, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{S}^2. \tag{17}$$

Observe that the integral can be evaluated explicitly

$$f(T, x, v) = \int_0^T \frac{d}{dt}\psi(t, x + tv)dt = \psi(T, x + Tv) - \psi(0, x). \tag{18}$$

Now consider the solution of the Cauchy problem for ψ

$$\begin{aligned} \square\psi = 0, &\quad \text{on } \mathcal{M} = (0, T) \times \mathbb{R}^3 \\ \psi = \psi_0, \quad \partial_t\psi = \psi_1, &\ \text{ at } \{0\} \times \mathbb{R}^3 \end{aligned} \tag{19}$$

Let $(\tau, \xi), \xi \in \mathbb{R}^n$ be the dual variables in $T^*\mathcal{M}$ to $(t, x), x \in \mathbb{R}^3$. Let $\hat{\psi}$ denote the Fourier transform of ψ in x variables. Using Fourier transform in the x variable, we get

$$\begin{aligned} \psi(t, x) &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi + t|\xi|)}\hat{h}_0(\xi)d\xi + (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi - t|\xi|)}\hat{h}_1(\xi)d\xi \\ &= E_+h_0 + E_-h_1, \end{aligned} \tag{20}$$

where

$$\hat{h}_0 = \frac{1}{2}(\hat{\psi}_0 + \frac{1}{i|\xi|}\hat{\psi}_1), \quad \hat{h}_1 = \frac{1}{2}(\hat{\psi}_0 - \frac{1}{i|\xi|}\hat{\psi}_1).$$

Here, h_0, h_1 are the re-parametrized Cauchy data for the Cauchy problem. We write

$$h_0 = \frac{1}{2}(\psi_0 + T\psi_1), \quad h_1 = \frac{1}{2}(\psi_0 - T\psi_1)$$

where T is a pseudo-differential operator of order -1 on $\mathbb{R}^3$ modulo a smoothing operator. We note that $E_\pm$ are represented by oscillatory integrals

$$E_\pm f(t,x) = (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((x-y)\cdot\xi \pm t|\xi|)} f(y)dyd\xi. \tag{21}$$

Using these expressions in (18), we get

$$\Phi'(\vec{0})\vec{\psi}(x,v) = E_+^T h_0(x,v) + E_-^T h_1(x,v) - (h_0(x) + h_1(x)) \tag{22}$$

where $E_\pm^T$ are defined as

$$E_\pm^T f(x,v) = (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((x+Tv-y)\cdot\xi \pm T|\xi|)} f(y)dyd\xi$$

To construct the left inverse, we will adopt the idea in [13] to integrate the v variable in (22). Let I be the integration operator on $C^\infty(\mathbb{R}^3\times\mathbb{S}^2)$ defined by

$$If(y) = \int_{\mathbb{S}^2} f(y,v)dv.$$

Then we consider $I\circ\Phi'(\vec{0})\vec{\psi}$. Note that

$$I\circ E_+^T f(x) = (2\pi)^{-3}\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((x+Tv-y)\cdot\xi + T|\xi|)} f(y)dyd\xi dv \tag{23}$$

For v integration, the phase function $Tv\cdot\xi$ has critical points at $v = \pm\xi/|\xi|$. For $|\xi|$ large, we can use stationary phase argument, see [9, Section 1.3], to obtain that

$$\begin{aligned} I\circ E_+^T f(x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((x-y)\cdot\xi + 2T|\xi|)} a_+(\xi) f(y)dyd\xi \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(x-y)\cdot\xi} a_-(\xi) f(y)dyd\xi = K_1^+ f + K_2^+ f \end{aligned} \tag{24}$$

where $a_\pm(\xi)$ has the asymptotic expansion

$$a_\pm(\xi) \sim |\xi|^{-1}\sum_{j=0}^{\infty} |\xi|^{-j} a_{\pm,j}(\xi/|\xi|) \tag{25}$$

and $a_{\pm,j}$ are smooth on $\mathbb{S}^2$. In (24), the operators K_1^+, K_2^+ are defined by the two integrals. For $I\circ E_-^T$, we have a similar result that

$$\begin{aligned} I\circ E_-^T f(x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(x-y)\cdot\xi} b_+(\xi) f(y)dyd\xi \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((x-y)\cdot\xi - 2T|\xi|)} b_-(\xi) f(y)dyd\xi = K_1^- f + K_2^- f \end{aligned} \tag{26}$$

where $b_\pm(\xi)$ has similar expansion as (25). Using (24) and (26), we find that

$$I\circ\Phi'(\vec{0})\vec{\psi} + I(h_0 + h_1) = (K_1^+ + K_2^+)h_0 + (K_1^- + K_2^-)h_1 \tag{27}$$

We note that K_1^-, K_2^+ are pseudo-differential operators of order -1 on $\mathbb{R}^3$, while K_1^+, K_2^- are Fourier integral operators from $\mathcal{E}'(\mathbb{R}^3)$ to $\mathcal{D}'(\mathbb{R}^3)$ of order -1 and they are of the canonical graph type. We are ready to prove the main result of this section.

Proposition 5.1. *Let* $\mathcal{W} \subset H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3), s \geq 0$ *consisting of* $\vec{\phi}$ *such that*

(1) $\vec{\phi}$ *is supported in a fixed compact set* $\mathcal{X}$ *of* $\mathbb{R}^3$*;*
(2) $\|\phi_1\|_{H^{s-1}} \leq C_0 \|\phi_0\|_{H^{s-1}}$ *for some fixed* C_0*.*

There exist an operator A *such that*

$$A\Phi'(\vec{0})\vec{\psi} = \vec{\psi}$$

for $\vec{\psi} \in \mathcal{W}$*. Also, for* u *in the range of* $\Phi'(\vec{0})$ *on* $\mathcal{W}$*, there exists* $C > 0$ *such that*

$$\|Au\|_{H^s} \leq C\|u\|_{H^s}$$

Remark 5.2. *The condition (2) in the definition of* $\mathcal{W}$ *can be replaced by* $\|\phi_1\|_{\dot{H}^{s-1}} \leq C_0\|\phi_1\|_{H^{s-2}}$ *for some* $C_0 > 0$ *where* $\dot{H}^{s-1}$ *denotes the homogeneous Sobolev space.*

Proof. We write (27) in terms of ψ_0, ψ_1 and get

$$I \circ \Phi'(\vec{0})\vec{\psi} + I\psi_0 = (K_1^+ + K_2^+)\frac{1}{2}(\psi_0 + T\psi_1) + (K_1^- + K_2^-)\frac{1}{2}(\psi_0 - T\psi_1)$$

Then we have

$$\begin{aligned}\|\psi_0\|_{H^s} = C\|I\psi_0\|_{H^s} &\leq \|I \circ \Phi'(\vec{0})\vec{\psi}\|_{H^s} \\ &+ \|(K_1^+ + K_2^+)\frac{1}{2}(\psi_0 + T\psi_1) + (K_1^- + K_2^-)\frac{1}{2}(\psi_0 - T\psi_1)\|_{H^s} \\ &\leq \|\Phi'(\vec{0})\vec{\psi}\|_{H^s} + C\|\psi_0\|_{H^{s-1}} + C\|\psi_1\|_{H^{s-2}}\end{aligned}$$

from which we obtain

$$\|\psi_0\|_{H^s} + \|\psi_1\|_{H^{s-1}} \leq \|\Phi'(\vec{0})\vec{\psi}\|_{H^s} + C\|\psi_0\|_{H^{s-1}} + C\|\psi_1\|_{H^{s-2}} \tag{28}$$

where we used condition (2) and $\vec{\psi} \in \mathcal{W}$.

To remove the last term on the right hand side of (27), we use the injectivity of $\Phi'(\vec{0})$. Suppose that ψ satisfies the Cauchy problem (19) with $(\psi_0, \psi_1) \in H^{s+1} \times H^s, s \geq 0$ supported in a compact set $\mathcal{X}$ of $\mathbb{R}^3$. If $\Phi'(\vec{0})\vec{\psi} = 0$, we claim that $\psi = 0$ in $\mathcal{M}$. We use (18) to get $\psi(0, x) = \psi(T, x+Tv)$ for any $x \in \mathbb{R}^3, v \in \mathbb{S}^2$. We first observe that $\psi(0, x) = \psi(T, x) = 0$ using the fact that ψ_1 is compactly supported. In fact, suppose $\psi(T, y) \neq 0$. Then we have

$$\begin{aligned}\psi(T, y) = \psi(0, y - Tv) = \psi(T, y - 2Tv) = \psi(0, y - 3Tv) \\ = \cdots \psi(0, y - (2n+1)Tv)\end{aligned}$$

for any $n \geq 0$. For n sufficiently large, we get a contradiction.

Next, we arrive at the following problem

$$\begin{aligned}\Box\psi = -\partial_t^2\psi + \Delta_x\psi = 0 \text{ in } \mathcal{M} \\ \psi(0, x) = \psi(T, x) = 0\end{aligned} \tag{29}$$

where $\Delta_x = \sum_{i=1}^3 \partial^2_{x^i}$. Taking Fourier transform of the wave equation in (29) in the x variable, we get

$$-\partial_t^2\hat{\psi}(t,\xi) + |\xi|^2\hat{\psi}(t,\xi) = 0, \quad t \in (0,T), \xi \in \mathbb{R}^3$$
$$\hat{\psi}(0,\xi) = \hat{\psi}(T,\xi) = 0$$

If $|\xi|^2$ is not an eigenvalue of the Sturm-Liouville problem for the second order operator ∂_t^2 with Dirichlet boundary condition at $t = 0, T$, we see that $\hat{\psi}(t,\xi) = 0$. Because the eigenvalues form a discrete set with no accumulation points, we conclude that $\hat{\psi}(t,\xi) = 0$ for $t \in [0,T]$ and $\xi \in \mathbb{R}^3$. This implies that $\psi = 0$ in $H^{s+1}, s \geq 0$. So we proved the claim.

Let $\mathcal{N}^s = H^s(\mathcal{X}) \times H^{s-1}(\mathcal{X})$. The inclusion of $\mathcal{N}^s$ into $\mathcal{N}^{s-1}$ is compact. We can write (28) as

$$\|\vec{\psi}\|_{\mathcal{N}^s} \leq \|\Phi'(\vec{0})\vec{\psi}\|_{H^s} + C\|\vec{\psi}\|_{\mathcal{N}^{s-1}} \tag{30}$$

We claim that

$$\|\vec{\psi}\|_{\mathcal{N}^s} \leq C_1\|\Phi'(\vec{0})\vec{\psi}\|_{H^s} \tag{31}$$

for some $C_1 > 0$. We argue by contradiction and assume the above is not true. We can get a sequence $\vec{\psi}^j = 1, 2, \cdots$ with unit norm in $\mathcal{N}^s$ such that $\Phi'(\vec{0})\vec{\psi}$ goes to 0 in $H^s(\mathbb{R}^3 \times \mathbb{S}^2)$ as $j \to \infty$. By (30) for $\vec{\psi}$ supported in $\mathcal{X}$, we conclude that $1 = \|\vec{\psi}^j\|_{\mathcal{N}^s} \leq C_1\|\vec{\psi}^j\|_{\mathcal{N}^{s-1}}$. This gives a weak limit $\vec{\psi}$ in $\mathcal{N}^s$ along a subsequence, which thus converges strongly in $\mathcal{N}^{s-1}$. Therefore, $\|\vec{\psi}\|_{\mathcal{N}^{s-1}}$ is bounded below by $1/C_1$, thus non-zero. However, $\Phi'(\vec{0})\vec{\psi} = 0$ so $\vec{\psi} = 0$ by the injectivity of $\Phi'(\vec{0})$. So $\vec{\psi} = 0$ a contradiction. This finishes the proof of (31). ☐

6. Proof of Theorem 1.1

First, we show that $\Phi'(\vec{\phi})$ has a left inverse.

Proposition 6.1. *Let $\vec{\phi} \in C^\infty$ and $\mathcal{W}$ be defined as in Proposition 5.1. Then there exists an operator A such that*

$$A\Phi'(\vec{\phi})\vec{\psi} = \vec{\psi}$$

for $\vec{\psi} \in \mathcal{W}$. Also, for u in the range of $\Phi'(\vec{\phi})$ on $\mathcal{W}$, there exists $C > 0$ such that for $s \geq 0$

$$\|Au\|_{H^s} \leq C\|u\|_{H^s} \tag{32}$$

Proof. Recall that $\Phi'(\vec{\phi})\vec{\psi} = f|_{t=T,|\theta|=1}$ where f satisfies (14). Using the characteristic equation (8), we can write the equation for f as

$$\frac{d}{d\tau}f(\sigma(\tau)) = e^{-\phi(\sigma(\tau))}\tilde{p}^\alpha\frac{\partial\psi}{\partial z^\alpha} = \frac{d}{d\tau}\psi(z(\tau))$$

where $\sigma(\tau) = (z(\tau), p(\tau))$. We thus obtain that

$$\Phi'(\vec{\phi})\vec{\psi} = \psi(z(\tau_0)) - \psi(z(0))$$

where τ_0 is such that $z^0(\tau_0) = T$. In fact, if we use τ for t, we get

$$\Phi'(\vec{\phi})\vec{\psi}(x, v) = \psi(\tau_0, x + \tau_0 v) - \psi(0, x)$$

for $x \in \mathbb{R}^3, v \in \mathbb{S}^2$. The proof is finished by following the argument in Proposition 5.1. □

Proof of Theorem 1.1. To complete the proof, we check the tame estimate and constants in Nash-Moser's theorem. We recall the tame estimates (16), (32)

$$\|\Phi''(\vec{\phi})(\vec{\psi}, \vec{\omega})\|_{H^s} \leq C(\|\vec{\psi}\|_{H^{s+1}}\|\vec{\omega}\|_{H^{s+2}} + \|\vec{\omega}\|_{H^{s+1}}\|\vec{\psi}\|_{H^{s+2}}) \leq C\|\vec{\psi}\|_{H^{s+2}}\|\vec{\omega}\|_{H^{s+2}}$$

for $s > 1$ and

$$\|Au\|_{H^s} \leq C\|u\|_{H^s}$$

for $s \geq 0$. Comparing with the indices in Section 2, we see that it suffices to take

$$a_\Phi = 0, \quad a_\psi = 0, \quad \lambda_0 = s, \quad \lambda_1 = \lambda_2 = s,$$
$$m_1 = m_2 = \cdots = m_5 = s + 2, \quad \mu_1 = \mu_2 = s$$

and $s > 1$. This implies that $\Lambda_1 = 0, \Lambda_2 = 0, M = 0, M_1 = \cdots = M_5 = 2$. Finally, it suffices to take

$$\alpha > 4, \quad \beta \geq 0.$$

Thus the Nash-Moser theorem in Section 2 tells that if $\vec{\phi}, \vec{\psi} \in H^\mu, \mu = \alpha + s > 5$ and $\Phi(\vec{\phi}) = \Phi(\vec{\psi}) \in H^{\frac{1}{2}+s}, s > 1$, then $\vec{\phi} = \vec{\psi}$ in a sufficiently small neighborhood of $\vec{\psi}$. □

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