MATH 427: COMPLEX ANALYSIS (SUMMER 2018) NOTE TO 08-01 LECTURE

Today we talked about functions defined by power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{0.1}$$

assuming the radius of convergence R. Then we showed that $f'(z)$ exists and

$$f'(z) = \sum_{k=1}^{\infty} k c_k (z - z_0)^{k-1} \tag{0.2}$$

In this note, I want to explain that the radius of convergence for this series is also R, which is an important result. The argument applies to the series of the integral of $f(z)$ and $f^{(n)}(z)$ as well.

First of all, by the root test of (0.1), we know that $R = (\limsup |c_k|^{1/k})^{-1}$. For simplicity, I assume below $0 < R < \infty$. Now we consider (0.2) and compute

$$\limsup |kc_k|^{1/k} = \limsup k^{1/k}|c_k|^{1/k} = \lim_{n\to\infty} u_n$$

where

$$u_n = \sup\{k^{1/k}|c_k|^{1/k}, k \geq n\}$$

Because we know that

$$\lim_{k\to\infty} k^{1/k} = 1,$$

(for the proof of this fact, use $\lim k^{1/k} = \lim e^{\frac{1}{k}\log k}$ and show $\lim \frac{\log k}{k} = 0$ using L'Hospital rule. This is exercise 3.1.11), we conclude that for any $\epsilon > 0$, there is $N > 0$ such that

$$1 - \epsilon < k^{1/k} < 1 + \epsilon, \text{ for } k \geq N.$$

Thus we get for $n \geq N$,

$$(1-\epsilon)\sup\{|c_k|^{1/k}, k \geq n\} \leq u_n \leq (1+\epsilon)\sup\{|c_k|^{1/k}, k \geq n\}$$

Then use sandwich theorem to conclude that

$$(1-\epsilon)R^{-1} \leq \lim_{n\to\infty} u_n \leq (1+\epsilon)R^{-1}$$

Sine this is true for any ϵ, we get

$$\lim_{n\to\infty} u_n = R^{-1}.$$

This finishes the proof of the claim.

By the same argument, you can try to prove that

Lemma 0.1. *If* $\lim_{k\to\infty} a_k = a$ *is finite and* $\limsup b_k = b$*, then*
$$\limsup(a_k b_k) = ab.$$

This is a case of the exercise 3.1.12.