MATH 427: COMPLEX ANALYSIS (SUMMER 2018)

Homework 2: **New due date Monday, July 9th.**

- Section 1.4: 4, 7, 12, 16, 17
- Section 2.1: 4, 6, 8, 16

Some solutions:

Sec 2.1 #4

Proof. By the definition of open sets, we need to show that for any $w \in A$, there exists $\delta > 0$ such that the open disk $D_\delta(w) \subset A$.
Because $w \in A$, we know that $\mathrm{Re}(w) > 0$. Choose $\delta < \mathrm{Re}(w)$. Then for any $z \in D_\delta(w)$, we have

$$|z - w| < \delta.$$

But we have

$$|\mathrm{Re}(z) - \mathrm{Re}(w)| \leq |z - w|$$

so we derive

$$-\delta + \mathrm{Re}(w) < \mathrm{Re}(z) < \mathrm{Re}(w) + \delta$$

Thus $\mathrm{Re}(z) > 0$. This proves that $z \in A$ hence $D_\delta(w) \subset A$. □

Sec 2.1 #16

Proof. "$\Longrightarrow$": If f is continuous at a, then for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(z) - f(a)| < \epsilon, \text{ if } |z - a| < \delta$$

(Here, because the domain of f is open, we can take δ small so that $D_\delta(a) \subset U$.) Now for any sequence $z_n \to a$, we can choose N_0 large so that

$$|z_n - a| < \delta \text{ if } n > N_0.$$

Thus for $n > N_0$, we have

$$|f(z_n) - f(a)| < \epsilon$$

hence the sequence $f(z_n)$ converge to $f(a)$ by the definition of convergence of complex sequences.

"$\Longleftarrow$": we prove by contradiction. Suppose f is not continuous at a. Then there exists a $\epsilon > 0$ such that for any $\delta > 0$, there is some $z \in U$ with $|z - a| < \delta$ such that

$$|f(z) - f(a)| \geq \epsilon.$$

Now we apply this to $\delta = 2^{-n}, n = 1, 2, \cdots$, and find the corresponding z_n so that

$$|z_n - a| < 2^{-n}$$

and $|f(z_n) - f(a)| \geq \epsilon$. It is clear that z_n converges to a but $f(z_n)$ does not converge to $f(a)$. The contradiction shows that f is continuous at a. $\square$