MATH 427 MIDTERM, SUMMER 2018
SOLUTION KEY

1. (10 pts) Find $(1+i)^8$ in the standard form $z = x + iy$.

Proof. Answer is $16 + 0i$ □

2. (10 pts) Recall that $z^a, a \in \mathbb{C}$ is defined using the principal branch of the complex logarithmic function. Consider the function

$$f(z) = (z+1)^i.$$

For $z = x + iy$, find the real and imaginary parts of $f(z)$, that is $u(x,y), v(x,y)$ in $f(z) = u(x,y) + iv(x,y)$. (You can use the function $\arg_{(-\pi,\pi]}(z) = \arg_{(-\pi,\pi]}(x,y)$ in your answer.) Also, find where $f(z)$ is not continuous.

Proof. By definition of z^a, we get

$$\begin{aligned}(z+1)^i = e^{i\mathrm{Log}(z+1)} &= e^{i(\log|z+1| + i\arg_{(-\pi,\pi]}(z+1))} \\ &= e^{i\log|z+1| - \arg_{(-\pi,\pi]}(z+1)} \\ &= e^{-\arg_{(-\pi,\pi]}(z+1)}(\cos(\log|z+1|) + i\sin(\log|z+1|))\end{aligned}$$

In terms of function of x, y, this is

$$e^{-\arg_{(-\pi,\pi]}(x+1,y)}\cos(\log\sqrt{(x+1)^2+y^2}) + ie^{-\arg_{(-\pi,\pi]}(x+1,y)}\sin(\log\sqrt{(x+1)^2+y^2})$$

Because the principal branch of $\arg(z)$ is discontinuous at $(-\infty, 0]$, we see that $(z+1)^i$ is discontinuous at $(-\infty, -1]$. (More precisely, you should find the jump of the function across $(-\infty,-1]$) to show the function is discontinuous there.) □

3. Consider the set $E = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

(a) (7pts) Show that E is an open set.

Proof. By definition of open sets, we need to show that for any $w \in E$, there is $\delta > 0$ such that the open disk $D_\delta(w) \subset E$. We can choose $0 < \delta < \min(\mathrm{Re}(w), \mathrm{Im}(w))$ because $\mathrm{Re}(w), \mathrm{Im}(w) > 0$. Then for any $z \in D_\delta(w)$, we get

$$|z - w| < \delta$$

So we get

$$|\mathrm{Re}(z) - \mathrm{Re}(w)| < |z - w| < \delta,$$

and

$$|\operatorname{Im}(z) - \operatorname{Im}(w)| < |z - w| < \delta$$

Then we get

$$-\delta + \operatorname{Re}(w) < \operatorname{Re}(z) < \operatorname{Re}(w) + \delta, \quad -\delta + \operatorname{Im}(w) < \operatorname{Im}(z) < \operatorname{Im}(w) + \delta$$

so that $\operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0$. This shows $z \in E$ hence $D_\delta(w) \subset E$. This finishes the proof.

Note: for this part, you need to give a rigorous proof using the definition. There is a hw problem similar to this. □

(b) (3 pts) Find the boundary of E. (Briefly state why the points are boundary points. No need for rigorous proofs.)

Proof. The boundary is

$$\partial E = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0, \operatorname{Im}(z) = 0\} \cup \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0, \operatorname{Re}(z) = 0\}$$

For any point $z \in \partial E$, we see that any open disk $D_\delta(z)$ has points in E and the complement of E. So this is the boundary by the characterization of boundary sets. □

(c) (5 pts) Find the pre-image $f^{-1}(E)$ of the set E for $f(z) = e^z$. Sketch the pre-image on the complex plane.

Proof. Let $z = x + iy$, we see that

$$f(z) = e^{x+iy} = e^x e^{iy}.$$

To find $f^{-1}(E)$, we need to find $z = x + iy$ such that $f(z) \in E$. By the polar form of e^{iy}, we see that

$$e^x > 0, \quad y \in [2k\pi, \pi/2 + 2k\pi], k = 0, \pm 1, \pm 2, \cdots$$

This implies that $x \in \mathbb{R}$ and $y \in [2k\pi, \pi/2 + 2k\pi]$. So the pre-image is a union of horizontal strips. The graph is omitted. □

4.(10 pts) Consider $f(z) = z\operatorname{Re}(z)$ defined on $\mathbb{C}$. Use Cauchy-Riemann equation to determine where the function is complex differentiable and find the derivative $f'(z)$ at those points.

Proof. $f(z) = (x + iy)x = x^2 + xyi$. So $u(x, y) = x^2, v(x, y) = xy$. These two functions are differentiable on $\mathbb{R}^2$. We find

$$u_x = 2x, \quad u_y = 0,$$

$$v_x = y, \quad v_y = x.$$

The CR equation holds at $(x, y) = (0, 0)$ because

$$2x = x, \quad y = 0$$

So the function $f(z)$ has complex derivative at $z = 0$. The derivative is

$$f'(0) = (2x + yi)|_{x=y=0} = 0.$$

□

5. Let γ be the closed path $\gamma(t) = e^{it}, t \in [-\pi, 3\pi]$. Evaluate the following contour integrals. You can use any results you've learned so far.

(a) (10 pts) $\int_\gamma \frac{\text{Log}(z)}{z} dz$, where $\text{Log}(z)$ is the principal branch of the logarithmic function. (Hint: use $\gamma(t)$ to compute and note that $\text{Log}(z)$ is not continuous at the cut-line.)

Proof.

$$\int_\gamma \frac{\text{Log}(z)}{z} dz = \int_{-\pi}^{3\pi} \frac{\text{Log}(e^{it})}{e^{it}} (e^{it})' dt = \int_{-\pi}^{3\pi} \text{Log}(e^{it}) i dt$$

$$= \int_{-\pi}^{3\pi} [\log |e^{it}| + i \arg_{(-\pi,\pi]}(e^{it})] i dt$$

$$= -\int_{-\pi}^{3\pi} \arg_{(-\pi,\pi]}(e^{it}) dt$$

$$= -\int_{-\pi}^{\pi} \arg_{(-\pi,\pi]}(e^{it}) dt - \int_{\pi}^{3\pi} \arg_{(-\pi,\pi]}(e^{it}) dt$$

$$= -\int_{-\pi}^{\pi} t dt - \int_{\pi}^{3\pi} (t - 2\pi) dt$$

Then do the integral. □

(b) (5 pts) $\int_\gamma \frac{1}{100 - z^2} dz$.

Proof. We see that the integrand

$$f(z) = \frac{1}{100 - z^2}$$

is analytic in $\mathbb{C} \setminus \{\pm 10\}$. The path γ is closed and contained in the disk $D_2(0)$ which is convex. By Cauchy theorem for convex sets, the integral is 0.

□