MATH 175: ELEMENTARY FUNCTIONAL ANALYSIS (WINTER 2019)

Homework 4: due Wednesday, Feb. 6

(1) Let $f \in C([a,b];\mathbb{R})$ be a continuous function on $[a,b]$. Show that $f \in \mathcal{L}^1$ and show that its Riemann integral agrees with the Lebesgue integral, namely

$$\int_a^b f(x)dx = \int f.$$

(2) Let $x_0 \in [a,b]$ and $Y = \{f \in C([a,b];\mathbb{R}) : f(a) = f(b) = 0\}$. Show that Y is dense in $L^1([a,b])$.

(3) Suppose $f \in L^1([-\pi,\pi])$. Show that $\int f(x)e^{-inx} \to 0$ as $n \to \infty$. Here the integral is the Lebesgue integral on $[-\pi,\pi]$.

(4) Find a sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{L}^1$ such that $\int |f_k|$ converges to 0 (namely f_k converges to 0 in $\mathcal{L}^1$) as $k \to \infty$ but f_k does not converge to 0 almost everywhere.

(5) Consider a series $\sum_{k=1}^{\infty} f_k(x)$, $f_k(x) \in \mathcal{L}^1$, $f_k \geq 0$. Suppose that $\sum_{k=1}^{\infty} \int f_k$ is finite. Show that $\sum_{k=1}^{\infty} f_k(x)$ converges a.e. and

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k$$