NOTES ON LEBESGUE INTEGRATION

These notes record the lectures on Lebesgue integration, which is a topic not covered in the textbook. It is largely inspired by the approach of Leon Simon when he previously taught this course. (All mistakes are my own of course!) The notes will be posted after each lecture. **Any comments or corrections, even very minor ones, are very much appreciated!**

Our goal is to define the notion of Lebesgue integration on a compact interval $[a,b] \subset \mathbb{R}$. The main purpose is to show that Lebesgue integration has a "completeness property" and this will eventually allow us to view (appropriately defined) L^1 and L^2 spaces as Banach spaces.

1. Measure zero

We begin with the notion of "measure zero".

Definition 1.1. A set $S \subset [a,b]$ has *Lebesgue measure zero* (or, simply, measure zero) if for every $\epsilon > 0$, there exists a countable collection of open intervals $\{I_j\}_{j=1}^\infty$ such that $S \subset \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty |I_j| < \epsilon$. (Here, for an open interval $I_j = (a_j, b_j)$, $|I_j| = b_j - a_j$.)

Lemma 1.2. *Let $\{S_j\}_{j=1}^\infty$ be a countable collection of measure zero sets. Then $S = \cup_{j=1}^\infty S_j$ has measure zero.*

Proof. Let $\epsilon > 0$. For every S_j, there exist open intervals $\{I_{j,i}\}_{i=1}^\infty$ such that $S_j \subset \cup_{i=1}^\infty I_{j,i}$ and $\sum_{i=1}^\infty |I_{j,i}| < 2^{-j}\epsilon$. Therefore, $\{I_{j,i}\}_{i,j=1}^\infty$ is a countable collection of open intervals such that $S \subset \cup_{j=1}^\infty \cup_{i=1}^\infty I_{j,i}$ and $\sum_{j=1}^\infty \sum_{i=1}^\infty |I_{j,i}| < \sum_{j=1}^\infty (2^{-j}\epsilon) = \epsilon$. □

Remark 1.3. Notice that a point in $[a,b]$ obviously has measure zero. As a consequence of the previous lemma, any countable set of points also has measure zero.

Definition 1.4. We say that a property holds *Lebesgue almost everywhere* (or simply almost everywhere, or even more simply, a.e.) in $[a,b]$, if it holds on the complement of a measure zero set $S \subset [a,b]$.

2. Definition of integration

After introducing the notion of measure zero. We now begin our discussion of integration. The approach will be to build up the notion of integration from the most basic building blocks, for which it is "obvious" how integration should be defined, and use that to define integration for more general functions. The basic building blocks are *step functions*.

Definition 2.1. A function $\phi : [a,b] \to \mathbb{R}$ is a *step function* if there exists a partition $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ such that $\phi \restriction_{(x_i, x_{i+1})} = c_i$ for some constant $c_i \in \mathbb{R}$ for $i = 0, 1, \dots, N-1$.

Notice that step functions are closed under addition, subtraction and multiplication. For step functions, we define the integral in the following way:

Definition 2.2. Let ϕ be a step function. Using notations in Definition 2.1, we define

$$\int \phi := \sum_{i=0}^{N-1} c_i (x_{i+1} - x_i).$$

Of course we want to define integration for more general functions. We now consider a class of functions which is "well-approximated" by step functions from below.

Definition 2.3. Let $\mathcal{L}_0$ be the set of functions $f : [a,b] \to \mathbb{R}$ such that there exists an increasing sequence[1] of step functions $\{\phi_k\}_{k=1}^\infty$ such that $\phi_k \to f$ almost everywhere and $\{\int \phi_k\}_{k=1}^\infty$ is bounded.

[1] That is, $\phi_1(x) \leq \phi_2(x) \leq \phi_3(x) \leq \dots$ for every $x \in [a,b]$.

Definition 2.4. Let $f \in \mathcal{L}_0$. Define

$$\int f := \lim_{k\to\infty} \int \phi_k,$$

where ϕ_k is an increasing sequence of step functions as in Definition 2.3.

Remark 2.5. Let us note that the limit exists since $\{\int \phi_k\}_{k=1}^\infty$ is an increasing and bounded sequence.

However, it is not clear that $\int f$ is well-defined. In particular, does the definition depend on the particular choice of the sequence of step functions? The answer is no and $\int f$ is well-defined by the following lemma:

Lemma 2.6. *Let $f \in \mathcal{L}_0$ and $\{\phi_k\}_{k=1}^\infty$, $\{\psi_\ell\}_{\ell=1}^\infty$ be increasing sequences of step functions as in Definition 2.3. Then*

$$\lim_{k\to\infty} \int \phi_k = \lim_{\ell\to\infty} \int \psi_\ell.$$

Proof. It suffices to show that for every $\epsilon > 0$,

$$\lim_{k\to\infty} \int \phi_k \geq \lim_{\ell\to\infty} \int \psi_\ell - \epsilon.$$

In turn, it suffices to show that for every $\epsilon > 0$ and every $\ell \in \mathbb{N}$,

$$\lim_{k\to\infty} \int \phi_k \geq \int \psi_\ell - \epsilon. \tag{2.1}$$

Now, let $\epsilon > 0$ and $\ell \in \mathbb{N}$ be fixed. Our goal will be to prove (2.1).

Define the set $A_k := \{x \in [a,b] : \phi_k(x) \geq \psi_\ell(x) - \frac{\epsilon}{b-a}\}$. Notice the following two properties:

(1) A_k is a finite union of intervals and points. Similarly for $(A_k)^c$. Therefore $\mathbb{1}_{A_k}$ and $\mathbb{1}_{(A_k)^c}$ are step functions.
(2) $S \setminus \cup_{k=1}^\infty A_k$ has measure zero and $A_1 \subset A_2 \subset A_3 \subset \dots$. In particular[2], $|(A_k)^c| \to 0$ as $k \to \infty$.

We now compute[3]

$$\begin{aligned}\int \phi_k &= \int \phi_k \mathbb{1}_{A_k} + \int \phi_k \mathbb{1}_{(A_k)^c}\\ &\geq \int \psi_\ell \mathbb{1}_{A_k} + \int \phi_k \mathbb{1}_{(A_k)^c} - \epsilon\end{aligned}$$

As a consequence,

$$\lim_{k\to\infty} \int \phi_k \geq \underbrace{\liminf_{k\to\infty} \int \psi_\ell \mathbb{1}_{A_k}}_{=:I} + \underbrace{\liminf_{k\to\infty} \int \phi_k \mathbb{1}_{(A_k)^c}}_{=:II} - \epsilon$$

We now consider each term on the right hand side. First, we claim that $I = \int \psi_\ell$. To see this

$$\left| \int \psi_\ell \mathbb{1}_{A_k} - \int \psi_\ell \right| \leq \sup_{x\in[a,b]} |\psi_\ell(x)| |(A_k)^c| \to 0$$

as $k \to \infty$ since $|(A_k)^c| \to 0$. Second, we claim the $II \geq 0$. To see this, we note that ϕ_k is bounded below by ϕ_1. Therefore,

$$\int \phi_k \mathbb{1}_{(A_k)^c} \geq -\left(\sup_{x\in[a,b]} |\phi_1(x)| \right) |(A_k)^c| \to 0$$

as $k \to \infty$ since $|(A_k)^c| \to 0$. Combining the above estimates, we obtain (2.1). □

Using this definition, we can prove some basic properties regarding integration of functions in $\mathcal{L}_0$.

Proposition 2.7. *(1) If $f \in \mathcal{L}_0$ and $f = \tilde{f}$ a.e., then $\tilde{f} \in \mathcal{L}_0$ and $\int f = \int \tilde{f}$.*
(2) If $f, g \in \mathcal{L}_0$ and $\alpha, \beta \geq 0$, then $(\alpha f + \beta g) \in \mathcal{L}_0$ and $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$.
(3) If $f, g \in \mathcal{L}_0$, then $\max\{f, g\}, \min\{f, g\} \in \mathcal{L}_0$.
(4) If $f, g \in \mathcal{L}_0$ and $f \leq g$ a.e., then $\int f \leq \int g$.

[2]Here, since $(A_k)^c$ is a finite union of intervals and points, one can define $|(A_k)^c|$ as the sum of the lengths of the intervals.
[3]Note that every function involved here is a step function and the manipulation of the integrals of step functions is easy.

Proof. Part (1) is trivial and follows from the definition. For part (2), notice that if $\phi_k \to f$ and $\psi_k \to g$ are increasing sequences of step functions with bounded integrals, then $\alpha\phi_k + \beta\psi_k \to \alpha f + \beta g$ is an increasing sequence of step functions with bounded integrals. For part (3), notice that given step functions ϕ and ψ, $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\}$ are step functions. Hence, given increasing sequences of step functions $\phi_k \to f$ and $\psi_k \to g$ (with bounded integrals), we have $\min\{\phi_k, \psi_k\}$ an increasing sequence of step functions with bounded integrals such that $\min\{\phi_k, \psi_k\} \to \min\{f, g\}$. Hence $\min\{f, g\} \in \mathcal{L}_0$. Similarly for $\max\{f, g\}$. Finally, for part (4), take increasing sequences of step functions $\phi_k \to f$ and $\psi_k \to g$. Now, note that

$$\int f \leftarrow \int \min\{\phi_k, \psi_k\} \leq \int \max\{\phi_k, \psi_k\} \to \int g.$$

□

Definition 2.8. Let $\mathcal{L}^1$ be the set of functions f such that $f = g - h$ for some $g, h \in \mathcal{L}_0$.

Definition 2.9. Let $f \in \mathcal{L}^1$. For $f = g - h$ with $g, h \in \mathcal{L}_0$, define $\int f = \int g - \int h$.

Again, this is not obviously well-defined and requires a proof:

Lemma 2.10. *$\int f$ as defined in Definition 2.9 is well-defined.*

Proof. Let $f \in \mathcal{L}^1$. Suppose $f = g_1 - h_1 = g_2 - h_2$ for $g_1, g_2, h_1, h_2 \in \mathcal{L}_0$. Then $g_1 + h_2 = g_2 + h_1$. By part (2) in Proposition 2.7, we have

$$\int g_1 + \int h_2 = \int g_2 + \int h_1.$$

Re-arranging, we get

$$\int g_1 - \int h_1 = \int g_2 - \int h_2.$$

Hence $\int f$ is well-defined. □

We first prove some easy properties regarding integration of $\mathcal{L}^1$ functions.

Proposition 2.11. *(1) If $f \in \mathcal{L}^1$ and $f = \tilde{f}$ a.e., then $\tilde{f} \in \mathcal{L}^1$ and $\int f = \int \tilde{f}$.*
(2) $\mathcal{L}^1$ is a vector space over $\mathbb{R}$. Moreover, if $f, g \in \mathcal{L}_0$ and $\alpha, \beta \in \mathbb{R}$, then $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$.
(3) If $f_1, f_2 \in \mathcal{L}^1$ and $f_1 \leq f_2$ a.e., then $\int f_1 \leq \int f_2$.
(4) If $f \in \mathcal{L}^1$, then $|f| \in \mathcal{L}^1$ and $|\int f| \leq \int |f|$.

Proof. (1) and (2) are easy and follow from the definitions.

For (3), we want to use the analogous result for $\mathcal{L}_0$ in Proposition 2.7. Then $f_1 = g_1 - h_1$ and $f_2 = g_2 - h_2$, where $g_1, h_1, g_2, h_2 \in \mathcal{L}_0$. $f_1 \leq f_2$ a.e. implies $g_1 + h_2 \leq g_2 + h_1$ a.e. Hence $\int(g_1 + h_2) \leq \int(g_2 + h_1)$ by Part (4) in Proposition 2.7. The conclusion follows from linearity of integration.

For (4), note that since $f = g - h$ for $g, h \in \mathcal{L}_0$, $|f| = \max\{g, h\} - \min\{g, h\} \in \mathcal{L}^1$ by Part (3) in Proposition 2.7. Finally, since $f \leq |f|$ and $-f \leq |f|$, we have $|\int f| = \max\{\int f, -\int f\} \leq \int |f|$ by part (3). □

Our next goal is to show that $\int |f| = 0$ if and only if $f = 0$ a.e. For this, we need an important technical lemma, whose proof will be deferred.

Lemma 2.12. *Let $\{\phi_k\}_{j=1}^{\infty}$ be an increasing sequence of step functions such that $\{\int \phi_k\}_{k=1}^{\infty}$ is bounded. Then, for a.e. $x \in [a, b]$, $\{\phi_k(x)\}_{k=1}^{\infty}$ is bounded (and therefore also convergent).*

We now prove some properties regarding integration of $\mathcal{L}^1$ functions.

Proposition 2.13. *(1) Let $f \in \mathcal{L}^1$. Then there exists a decreasing sequence $\{g_k\}_{k=1}^{\infty} \subset \mathcal{L}_0$ such that $g_k \to f$ a.e. and $\int g_k \to \int f$.*
(2) If $\{f_k\}_{k=1}^{\infty} \subset \mathcal{L}^1$ are non-negative and such that $\int f_k \to 0$, then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ such that $f_{k_j} \to 0$ a.e.
(3) If $f \in \mathcal{L}^1$ and $\int |f| = 0$, then $f = 0$ a.e.

Proof. For part (1), since $f \in \mathcal{L}^1$, $f = g - h$ for $g, h \in \mathcal{L}_0$. In particular, there exists an increasing sequence of step functions $\{\phi_k\}_{k=1}^\infty$ such that $\phi_k \to h$ and $\int \phi_k \to \int h$. A desired sequence $\{g_k\}_{k=1}^\infty$ is given by $g_k = g - \phi_k$. First, it is easy to check that it is decreasing. Second, since $g \in \mathcal{L}_0$ and ϕ_k is a step function, $g_k \in \mathcal{L}_0$. Third, it is easy to see that $g_k \to f$ a.e. and $\int g_k \to \int f$.

For part (2), choosing a subsequence f_{k_j} such that $\int f_{k_j} \leq 2^{-j}$. By the previous part, there exists a sequence $\{g_j\}_{j=1}^\infty \subset \mathcal{L}_0$ such that $g_j \geq f_{k_j}$ and $\int g_j \leq \int f_{k_j} + 2^{-j}$. Since $g_j \in \mathcal{L}_0$, for each j, there exists $\{\psi_{j,i}\}_{i=1}^\infty$ an increasing sequence of step functions such that $\psi_{j,i} \to g_j$ a.e. and $\lim_{i\to\infty} \int \psi_{j,i} = \int g_j$. Without loss of generality, assume that $\psi_{j,i} \geq 0$ (Otherwise, take $\max\{0, \psi_{j,i}\}$). Let $\psi_i := \sum_{j=1}^i \psi_{j,i}$. Let's note the following:

(1) $\psi_i \leq \psi_{i+1}$ since $\sum_{j=1}^i \psi_{j,i} \leq \sum_{j=1}^i \psi_{j,i+1} \leq \sum_{j=1}^{i+1} \psi_{j,i+1}$.
(2) $\int \psi_i = \sum_{j=1}^i \int \psi_{j,i} \leq \sum_{j=1}^i \int g_j \leq \sum_{j=1}^i \left(\int f_{k_j} + 2^{-j}\right) \leq \sum_{j=1}^i 2^{-j+1} \leq 2$.

Therefore, by Lemma 2.12, ψ_i is bounded a.e. Let's unwind what we have achieved: For a.e. x, there exists an $M_x > 0$ such that

$$\sum_{j=1}^i \psi_{j,i} \leq M_x.$$

In particular, if $i > N$, for a.e. x,

$$\sum_{j=1}^N \psi_{j,i} \leq M_x.$$

Taking sup in i, this implies that for a.e. x,

$$\sum_{j=1}^N g_j \leq M_x.$$

Since this holds for all N, we have

$$\sum_{j=1}^\infty g_j \leq M_x.$$

Therefore, for a.e. x, $g_j \to 0$. Hence, since $g_j \geq f_{k_j}$, for a.e. x, $f_{k_j} \to 0$.

Part (3) is just a consequence of part (2) after choosing $f_j = |f|$ for all j. □

3. Completeness

We now prove completeness. First, let us define the following:

Definition 3.1. For $f \in \mathcal{L}^1$, define $\|f\|_{\mathcal{L}^1} := \int |f|$.

Notice that $\|f\|_{\mathcal{L}^1} = \int |f|$ is *not* a norm on $\mathcal{L}^1$, since $\|f\|_{\mathcal{L}^1} = 0$ does *not* imply $f = 0$ (It only implies $f = 0$ a.e.) Nevertheless, $\|\cdot\|_{\mathcal{L}^1}$ is still a *seminorm*, i.e., a function that satisfies all the other axioms of a norm (i.e., $\|f\|_{\mathcal{L}^1} \geq 0$ for all $f \in \mathcal{L}^1$, $\|f + g\|_{\mathcal{L}^1} \leq \|f\|_{\mathcal{L}^1} + \|g\|_{\mathcal{L}^1}$ for all $f, g \in \mathcal{L}^1$ and $\|\lambda f\|_{\mathcal{L}^1} = |\lambda|\|f\|_{\mathcal{L}^1}$ for all $\lambda \in \mathbb{R}$, $f \in \mathcal{L}^1$.) Nonetheless, we can still discuss the notion of a Cauchy sequence, although it limit points (even if they exist) may not be unique.

We begin with an easy lemma:

Lemma 3.2. *Let $f \in \mathcal{L}^1$. For every $\epsilon > 0$, there exists a step function ϕ such that $\|f - \phi\|_{\mathcal{L}^1} < \epsilon$.*

Proof. $f = g - h$ for $g, h \in \mathcal{L}_0$. By definition, there exist step functions ψ and η such that $\|g - \psi\|_{\mathcal{L}^1}$, $\|h - \eta\|_{\mathcal{L}^1} < \frac{\epsilon}{2}$. Let $\phi = \psi - \eta$, which is a step function. Then

$$\|f - \phi\|_{\mathcal{L}^1} = \int |g - h - (\psi - \eta)| \leq \|g - \psi\|_{\mathcal{L}^1} + \|h - \eta\|_{\mathcal{L}^1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

We are now ready to prove completeness. Notice that at this point, we are still assuming the technical lemma (Lemma 2.12) without proof. The following is the main theorem.

Theorem 3.3. *Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{L}^1$ in the sense that for every $\epsilon > 0$, there exists $N > 0$ such that $\|f_k - f_\ell\| := \int |f_k - f_\ell| < \epsilon$. Then, there exists $f \in \mathcal{L}^1$ such that $\|f_k - f\| \to 0$ as $k \to \infty$.*

Proof. Consider a subsequence $\{f_{k_j}\}_{j=1}^\infty$ such that $\|f_{k_j} - f_\ell\| < 2^{-j}$ for every $\ell \leq k_j$. Without loss of generality, we can assume $k_{j+1} > k_j$. For each $j \in \mathbb{N}$, we now approximate each f_{k_j} by a step function ϕ_j such that

$$\|\phi_j - f_{k_j}\|_{\mathcal{L}^1} < 2^{-j}.$$

These step functions exist by Lemma 3.2. Define moreover $\phi_0 = 0$. Hence,

$$\phi_j = \sum_{i=1}^{j}(\phi_i - \phi_{i-1}) = \sum_{i=1}^{j}(\phi_i - \phi_{i-1})_+ - \sum_{i=1}^{j}(\phi_i - \phi_{i-1})_-,$$

where for every $x \in \mathbb{R}$, $(x)_+ = \max\{x, 0\}$, $(x)_- = -\min\{x, 0\}$. Now $\psi_j = \sum_{i=1}^{j}(\phi_i - \phi_{i-1})_+$ and $\eta_j = \sum_{i=1}^{j}(\phi_i - \phi_{i-1})_-$ are both increasing sequence of step functions and

$$\int \psi_j \leq \sum_{i=1}^{j}(\phi_i - \phi_{i-1})_+ \leq \sum_{i=1}^{j}\left(\|\phi_i - f_{k_i}\|_{\mathcal{L}^1} + \|\phi_{i+1} - f_{k_{i+1}}\|_{\mathcal{L}^1} + \|f_{k_i} - f_{k_{i+1}}\|_{\mathcal{L}^1}\right) \leq \sum_{i=1}^{j}(2^{-i} + 2^{-i-1} + 2^{-i}) \leq \frac{5}{2}.$$

Similarly,

$$\int \eta_j \leq \frac{5}{2}.$$

We can therefore apply Lemma 2.12 to get $\psi, \eta \in \mathcal{L}_0$ such that $\psi_j \to \psi$ and $\eta_j \to \eta$ a.e. and $\int \psi_j \to \int \psi$ and $\int \eta_j \to \int \eta$. Define $f = \psi - \eta$. We check that

$$\|f_{k_j} - f\|_{\mathcal{L}^1} \leq \|f_{k_j} - \phi_j\|_{\mathcal{L}^1} + \|\psi_j - \psi\|_{\mathcal{L}^1} + \|\eta_j - \eta\|_{\mathcal{L}^1} \to 0$$

as $j \to \infty$. Moreover, by the Cauchy property

$$\|f_{k_j} - f_j\|_{\mathcal{L}^1} \to 0$$

as $j \to \infty$. Therefore,

$$\|f_j - f\|_{\mathcal{L}^1} \leq \|f_{k_j} - f\|_{\mathcal{L}^1} + \|f_{k_j} - f_j\|_{\mathcal{L}^1} \to 0$$

as $j \to \infty$, as desired. □

To complete our discussion, we need to prove Lemma 2.12:

Proof of Lemma 2.12. It is convenient to define $\tilde{\phi}_k := \phi_k - \phi_1$ so that $\tilde{\phi}_k \geq 0$ for all k. Let

$$S := \{x \in [a, b] : \{\tilde{\phi}_k\}_{k=1}^\infty \text{ is unbounded}\}.$$

Our goal is to show that S has measure zero. Let $\alpha > 0$ be a constant to be chosen later. Define

$$S_k := \{x \in [a, b] : \tilde{\phi}_k > \alpha\}.$$

It is clear that for every $\alpha > 0$, $S \subset \cup_{k=1}^\infty S_k$. Notice that we can write (since $S_1 = \emptyset$)

$$\begin{aligned} S_N =&(S_N \setminus S_{N-1}) \cup (S_{N-1} \setminus S_{N-2}) \cup \cdots \cup (S_2 \setminus S_1) = \cup_{i=1}^{N-1}(S_{i+1} \setminus S_i) \\ =& \cup_{i=1}^{N-1}\{x \in [a, b] : \tilde{\phi}_i(x) \leq \alpha < \tilde{\phi}_{i+1}(x)\}. \end{aligned}$$

Now each $(S_{i+1} \setminus S_i)$ is given by a finite union of open intervals and points. Let's call the collection of open intervals $\mathcal{Q}_i$ and the set of all the points E_i, i.e.,

$$(S_{i+1} \setminus S_i) = (\cup_{I \in \mathcal{Q}_i} I) \cup E_i.$$

We want to estimate the size of all the intervals in $\mathcal{Q}_i$ for all $i \leq N - 1$. Now note that $\tilde{\phi}_N(x) > \alpha$ on S_N and that if $I \in \mathcal{Q}_i$ and $J \in \mathcal{Q}_j$ for $i \neq j$, then $I \cap J = \emptyset$. Therefore,

$$\tilde{\phi}_N > \alpha \sum_{i=1}^{N-1} \sum_{I \in \mathcal{Q}_i} \mathbb{1}_I.$$

Integrating, we get

$$\int \tilde{\phi}_N > \alpha \sum_{i=1}^{N-1} \sum_{I \in \mathcal{Q}_i} |I|.$$

But by assumption of the lemma, the left hand side is uniformly bounded for all N, i.e., there exists $C > 0$ such that

$$\sum_{i=1}^{N-1} \sum_{I \in \mathcal{Q}_i} |I| < \frac{C}{\alpha}.$$

We can now estimate the size of S. First, we note that

$$S_N \subset \cup_{i=1}^{N-1} (\cup_{I \in \mathcal{Q}_i} I \cup E_i).$$

By definition, we have $S \subset S_N$ for some N. Therefore,

$$S \subset \cup_{N=1}^{\infty} S_N \subset (\cup_{i=1}^{\infty} \cup_{I \in \mathcal{Q}_i} I) \cup (\cup_{i=1}^{\infty} E_i).$$

Let $\epsilon > 0$. Since $\cup_{i=1}^{\infty} E_i$ is a countable set of points, it has measure zero (see Remark 1.3). Hence, there exists a countable collection of intervals $J_1, J_2, \dots$ such that $\cup_{i=1}^{\infty} E_i \subset \cup_{i=1}^{\infty} J_i$ and $\sum_{i=1}^{\infty} |J_i| < \frac{\epsilon}{2}$. Therefore,

$$S \subset (\cup_{i=1}^{\infty} \cup_{I \in \mathcal{Q}_i} I) \cup (\cup_{i=1}^{\infty} J_i)$$

and

$$\sum_{i=1}^{\infty} \sum_{I \in \mathcal{Q}_i} |I| + \sum_{i=1}^{\infty} |J_i| < \frac{C}{\alpha} + \frac{\epsilon}{2}.$$

Finally, let $\alpha = \frac{2C}{\epsilon}$, we obtain

$$\sum_{i=1}^{\infty} \sum_{I \in \mathcal{Q}_i} |I| + \sum_{i=1}^{\infty} |J_i| < \epsilon.$$

Therefore, S has measure zero. □

Now we have proven the technical lemma, and hence we have completed the proof that $\mathcal{L}^1$ is complete. However, as we noted before, $\mathcal{L}^1$ is not a normed vector space, since $\|f\|_{\mathcal{L}^1} = 0$ only implies that $f = 0$ a.e. We therefore consider the following equivalence classes.

Definition 3.4. Let $L^1 := \mathcal{L}^1 / \sim$, where $\sim$ is the equivalence relation $f \sim g$ if $f = g$ a.e. Define the norm on $\mathcal{L}^1$ by

$$\|f\|_{L^1} = \|f\|_{\mathcal{L}^1}.$$

Remark 3.5. Notice that the norm above is well-defined since $\|f\|_{\mathcal{L}^1} = \|\tilde{f}\|_{\mathcal{L}^1}$ if $f = \tilde{f}$ a.e.

With the definition of L^1, we thus have the following important corollary of Theorem 3.3:

Corollary 3.6. *L^1 is a Banach space.*

4. L^2 SPACE

As in our approach for defining the L^1, we first define $\mathcal{L}^2$, which is a space of functions. L^2 will then be defined later as a space of equivalence classes of functions.

Definition 4.1. Define $\mathcal{L}^2$ by

$$\mathcal{L}^2 := \{f : [a,b] \to \mathbb{R} : f \in \mathcal{L}^1, f^2 \in \mathcal{L}^1\}.$$

Remark 4.2. Unlike for $\mathcal{L}^1$, which can easily be seen as a vector space, the fact that $\mathcal{L}^2$ is a vector space is harder to see. In particular, we will need Lemma 4.3 below.

Lemma 4.3 (Fatou's lemma). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative real-valued functions $f_n : [a,b] \to \mathbb{R}$ such that $f_n \to f$ a.e. and $\sup_n \int f_n$ is bounded. Then $f \in \mathcal{L}^1$ and*

$$\int f \leq \liminf_{n \to \infty} \int f_n.$$

Proof. This is a homework problem, which uses the fact that $\mathcal{L}^1$ is complete. [More precisely, in the homework problem, you are only asked to show the inequality, *assuming that* $f \in \mathcal{L}^1$. Nevertheless, the same proof shows that the limit is in $\mathcal{L}^1$.] □

Proposition 4.4. *$\mathcal{L}^2$ is a vector space.*

Proof. The only non-trivial axiom to prove is that $f, g \in \mathcal{L}^2$ imples $f+g \in \mathcal{L}^2$. To see this, let $f, g \in \mathcal{L}^2$. Notice that since $f \in \mathcal{L}^2$, there exist sequences of step functions $\{\phi_k\}_{k=1}^\infty$ and $\{\tilde{\phi}_k\}_{k=1}^\infty$ such that $\phi_k \to f$, $\tilde{\phi}_k \to f^2$ a.e. and $\int |\phi_k - f|, \int |\tilde{\phi}_k - f^2| \to 0$. Similarly, there exist sequences of step functions $\{\psi_k\}_{k=1}^\infty$ and $\{\tilde{\psi}_k\}_{k=1}^\infty$ such that $\psi_k \to g$, $\tilde{\psi}_k \to g^2$ a.e. and $\int |\psi_k - g|, \int |\tilde{\psi}_k - g^2| \to 0$. Define

$$\Phi_k = sign(\phi_k)\sqrt{|\tilde{\phi}_k|}, \quad \Psi_k = sign(\psi_k)\sqrt{|\tilde{\psi}_k|},$$

where $sign(\phi)(x) = \begin{cases} 1 & \text{if } \phi(x) > 0 \\ 0 & \text{if } \phi(x) = 0 \\ -1 & \text{if } \phi(x) < 0 \end{cases}$. Then we have $\Phi_k \to f$ a.e., $\Phi_k^2 \to f^2$ a.e. and $\int |\Phi_k^2 - f^2| \to 0$ and similarly for g. Now, $(\Phi_k + \Psi_k)^2 \to (f+g)^2$ a.e. We check that

$$\int (\Phi_k + \Psi_k)^2 \leq \int \Phi_k^2 + 2\int |\Phi_k \Psi_k| + \int \Psi_k^2 \leq 2\left(\int \Phi_k^2 + \int \Psi_k^2\right),$$

which is uniformly bounded since $\int |\Phi_k^2 - f^2|, \int |\Psi_k^2 - g^2| \to 0$. Therefore, by Fatou's lemma, we have $(f+g)^2 \in \mathcal{L}^1$. Obviously, we also have $f, g \in \mathcal{L}^1 \implies (f+g) \in \mathcal{L}^1$. We thus have $(f+g) \in \mathcal{L}^2$. This concludes the proof. □

Definition 4.5. We define a semi-inner product[4] on $\mathcal{L}^2$ by

$$\langle f, g \rangle := \int fg.$$

The semi-inner product induces a semi-norm in the usual way. The main result is that this is complete.

Theorem 4.6. *$\mathcal{L}^2$ is complete.*

Proof. The idea is to use the completeness of $\mathcal{L}^1$. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{L}^2$.

Step 1: Convergence of f_k^2 in $\mathcal{L}^1$ We check that $\{f_k^2\}_{k=1}^\infty$ is Cauchy:

$$\|f_k^2 - f_\ell^2\|_{\mathcal{L}^1} \leq \|(f_k + f_\ell)(f_k - f_\ell)\|_{\mathcal{L}^1} \leq \|f_k + f_\ell\|_{\mathcal{L}^2}\|f_k - f_\ell\|_{\mathcal{L}^2} \leq 2\left(\sup_n \|f_n\|_{\mathcal{L}^2}\right)\|f_k - f_\ell\|_{\mathcal{L}^2}.$$

Since f_n is Cauchy in $\mathcal{L}^2$, this implies $(\sup_n \|f_n\|_{\mathcal{L}^2})$ is bounded. (This follows from the general fact that a Cauchy sequence is bounded.) This implies f_k^2 is Cauchy in $\mathcal{L}^1$. By completeness of $\mathcal{L}^1$, there exists g such that

$$\|f_k^2 - g\|_{\mathcal{L}^1} \to 0$$

as $k \to \infty$. Moreover, by part (2) of Proposition 2.13, there exists a subsequence of f_{k_j} such that $f_{k_j}^2 \to g$ almost everywhere.

Step 2: Convergence of f_{k_j} in $\mathcal{L}^1$

We now consider the subsequence f_{k_j} as in the previous step. By Cauchy–Schwarz inequality,

$$\|f_{k_i} - f_{k_j}\|_{\mathcal{L}^1} \leq \|f_{k_i} - f_{k_j}\|_{\mathcal{L}^2}\sqrt{b-a},$$

Since f_{k_j} is Cauchy in $\mathcal{L}^2$, the above inequality shows that f_{k_j} is Cauchy in $\mathcal{L}^1$. By completeness of $\mathcal{L}^1$, there exists $f \in \mathcal{L}^1$ such that

$$\|f_{k_j} - f\|_{\mathcal{L}^1} \to 0$$

as $j \to \infty$. Moreover, there exists a further subsequence $f_{k_{j_\ell}}$ such that $f_{k_{j_\ell}} \to f$ a.e. This in particular implies that $g = f^2$ a.e.

Step 3: $f \in \mathcal{L}^2$ Showing $f \in \mathcal{L}^2$ is straightforward since $f \in \mathcal{L}^1$ and $f^2 = g$ a.e., $g \in \mathcal{L}^1$.

Step 4: Showing convergence of f_k to f in $\mathcal{L}^2$ To show convergence, we need to split into the positive and negative parts. We write $f_k = f_{k,+} - f_{k,-}$, where $f_{k,+}, f_{k,-} \geq 0$. Similarly, we write $f = f_+ - f_-$. Since $\{f_{k,+}\}$ and $\{f_{k,-}\}$ are both Cauchy, by the above argument, there exist $\tilde{f}_+$ and $\tilde{f}_-$ and there exist subsequences of $\{f_{k,+}\}$ and $\{f_{k,-}\}$ such that

- $f_{k_j,\pm} \to \tilde{f}_\pm$ a.e. and in $\mathcal{L}^1$,
- $f_{k,\pm}^2 \to \tilde{f}_\pm^2$ in $\mathcal{L}^1$.

[4]Note that this is not standard terminology. For a vector space V over $\mathbb{R}$, we use this to mean a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that all of the axioms for inner products hold, **except** for $\langle v, v \rangle = 0$ if and only if $v = 0$.

By uniqueness of limits, we must have $f_\pm = \tilde{f}_\pm$ a.e. Therefore,

$$\begin{aligned}\|f_k - f\|_{\mathcal{L}^2}^2 &\leq (\|f_{k,+} - f_+\|_{\mathcal{L}^2} + \|f_{k,-} - f_-\|_{\mathcal{L}^2})^2 \\ &\leq 2\left(\int |f_{k,+} - f_+|^2 + \int |f_{k,-} - f_-|^2\right) \\ &\leq 2\left(\int |f_{k,+} - f_+||f_{k,+} + f_+| + \int |f_{k,-} - f_-||f_{k,-} + f_-|\right) \\ &= 2\left(\int |f_{k,+}^2 - f_+^2| + \int |f_{k,-}^2 - f_-^2|\right) \to 0\end{aligned}$$

as $k \to 0$. This concludes the proof. □

To conclude our discussion, we define L^2.

Definition 4.7. Let $L^2 := \mathcal{L}^2/\sim$, where $\sim$ is the equivalence relation $f \sim g$ if $f = g$ a.e. Define the norm on $\mathcal{L}^2$ by

$$\|f\|_{L^2} = \|f\|_{\mathcal{L}^2}.$$

Remark 4.8. Notice that the norm above is well-defined since $\|f\|_{\mathcal{L}^2} = \|\tilde{f}\|_{\mathcal{L}^2}$ if $f = \tilde{f}$ a.e.

With the definition of L^2, we thus have the following important corollary of Theorem 4.6:

Corollary 4.9. *L^2 is a Hilbert space.*