REMARK ABOUT FOURIER TRANSFORMS

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In this note, we clarify some notations of Fourier transforms in the book, especially the convolution. The Fourier and inverse Fourier transforms are defined as

$$\mathcal{F}[f](\xi) = F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx,$$

$$\mathcal{F}^{-1}[F](x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} F(\xi)d\xi.$$

(Other books may define them differently. You need to be careful about the $\sqrt{2\pi}$ factors.)

In this book, the convolution of two functions $f(x), g(x)$ is defined by

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

with the $\frac{1}{\sqrt{2\pi}}$ factor. Then we should have the property

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$$

This is equation (12.5). The formula #11 of Table A in the appendix is wrong and should be replaced by this one. We give a proof here.

$$\begin{aligned}
\mathcal{F}[f]\mathcal{F}[g] &= (\frac{1}{\sqrt{2\pi}})^2 \int_{-\infty}^{\infty} e^{ix\xi} f(x)dx \int_{-\infty}^{\infty} e^{iy\xi} g(y)dy \\
&= (\frac{1}{\sqrt{2\pi}})^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+y)\xi} f(x)g(y)dxdy \\
&= (\frac{1}{\sqrt{2\pi}})^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iz\xi} f(z-y)g(y)dzdy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iz\xi} (f * g)(z)dz = \mathcal{F}[f * g].
\end{aligned}$$

Another common definition of convolution is

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

without the $\frac{1}{\sqrt{2\pi}}$ factor. Then we should have the property

$$\mathcal{F}[f * g] = \sqrt{2\pi}\mathcal{F}[f]\mathcal{F}[g]$$

by re-examine the above calculation. You can use either definition but should pay attention to where to put the $\sqrt{2\pi}$ factor.