# MATH 131P: PRACTICE MIDTERM (WINTER 2019)

Note: please let me know if there is any typo etc. Updated: February 10, 2019.
Note: This is a closed book, no calculator exam. You have 50 min to complete the exam.

**Problem 1:** State the type of each of the following equations (linear or nonlinear, homogeneous or non-homogeneous, elliptic, hyperbolic or parabolic, order).

(1) $u_{xx} - u_{yy} = 0$, where $u = u(x, y)$.
(2) $u_t + uu_{xxx} = 0$, where $u = u(t, x)$.
(3) $u_t = u_{xx} + f(x, t)$, where $u = u(x, t)$.

**Solution:** (i) Linear, 2nd order, homogeneous, hyperbolic;
(ii) nonlinear, 3rd order;
(iii) linear, 2nd order, non-homogeneous, parabolic.

**Problem 2:** Solve the equation ($\alpha > 0$)

$$\begin{aligned} u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\ u(0, t) = 1, \quad t > 0 \\ u(L, t) = 1, \quad t > 0 \\ u(x, 0) = 1 - 3\sin(2\pi x/L), \quad 0 \le x \le L. \end{aligned}$$

**Solution:** The equation has non-homogeneous boundary conditions. We can transform it to homogeneous boundary conditions. In this case, we can find a steady state solution $v(x, t) = 1$. So $w(x, t) = u(x, t) - v(x, t)$ satisfies

$$\begin{aligned} w_t = \alpha^2 w_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\ w(0, t) = 0, \quad t > 0 \\ w(L, t) = 0, \quad t > 0 \\ w(x, 0) = -3\sin(2\pi x/L), \quad 0 \le x \le L. \end{aligned}$$

Separation of variables gives the solution with general IC

$$w(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) e^{-\alpha^2 (n\pi/L)^2 t}$$

with

$$A_n = \frac{2}{L} \int_0^L \sin(n\pi x/L)\phi(x)dx.$$

For $\phi(x) = -3\sin(2\pi x/L)$, we can compute $A_n$ or compare the coefficients to get

$$w(x, t) = -3\sin(2\pi x/L) e^{-\alpha^2 (2\pi/L)^2 t}$$

so that

$$u(x,t) = 1 - 3\sin(2\pi x/L)e^{-\alpha^2(2\pi/L)^2 t}.$$

**Problem 3:** Consider the equation ($\alpha > 0$)

$$\begin{gathered}
u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\
u_x(0,t) = 0, \quad t > 0 \\
u(L,t) = 0, \quad t > 0 \\
u(x,0) = \phi(x), \quad 0 \le x \le L
\end{gathered}$$

(1) Find the steady state solution.
(2) Solve the equation using separation of variables. (You need to find the coefficients of the expansion in terms of integrals but you do not need to evaluate the integrals.)
**Solution:** (1). Let $v(x,t)$ be steady state solution. It satisfies

$$\begin{gathered}
0 = \alpha^2 v_{xx}, \quad 0 < x < L, \quad 0 < t < \infty \\
v_x(0,t) = 0, \quad t > 0 \\
v(L,t) = 0, \quad t > 0.
\end{gathered}$$

The solution is $v = 0$.
(2). Let $u(x,t) = X(x)T(t)$. Separating variables, we get (for $k$ constant)

$$T' + \alpha^2 kT = 0,$$

and an eigenvalue problem

$$\begin{gathered}
X'' + kX = 0 \\
X'(0) = X(L) = 0.
\end{gathered}$$

We first solve the eigenvalue problem. For $k = 0$, the solution is $X(x) = Ax + B$. Using boundary condition, we find $A = B = 0$. So $k = 0$ is not eigenvalue. For $k < 0$, let $k = -\lambda^2, \lambda > 0$. The general solution is $X(x) = Ae^{-\lambda x} + Be^{\lambda x}$. Boundary conditions imply

$$Ae^{-\lambda L} + Be^{\lambda L} = 0, \quad -\lambda A + \lambda B = 0$$

so $B(e^{-\lambda L} + e^{\lambda L}) = 0$. This implies $B = 0$ hence $A = 0$. For $k > 0$, let $k = \lambda^2, \lambda > 0$. We get the solution

$$X_n(x) = \cos(\lambda_n x), \quad \lambda_n = (n + \frac{1}{2})^2 \frac{\pi^2}{L^2}, n = 1, 2, \cdots.$$

We solve the ODE for $T$ to get

$$T_n(t) = e^{-\alpha^2 \lambda_n^2 t}.$$

So the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \lambda_n^2 t} \cos(\lambda_n x).$$

Using the orthogonality of $\cos(\lambda_n x)$ and the initial condition $u(x,0) = \phi(x)$, we get

$$A_n \int_0^L \cos^2(\lambda_n x) dx = \int_0^L \phi(x) \cos(\lambda_n x) dx,$$

$$A_n = \frac{2}{L} \int_0^L \cos(\lambda_n x) \phi(x) dx.$$

**Problem 4:** Solve the following equation using the Fourier transform ($\alpha > 0$)

$$u_t = \alpha^2 u_{xx} - u, \quad -\infty < x < \infty, 0 < t < \infty$$

$$u(x,0) = \phi(x), \quad -\infty < x < \infty.$$

Give the answer as a convolution in terms of $\phi$. Find the solution explicitly when $\phi(x) = e^{-x^2}$.
**Solution:** Take Fourier transform in $x$ variable. Let $U = \mathcal{F}[u(x,t)]$. We get

$$U_t + U = -\alpha^2 \xi^2 U,$$

$$U(\xi, 0) = \Phi(\xi)$$

where $\Phi(\xi) = \mathcal{F}[\phi(x)]$. Solve the ODE using integrating factor method, we get

$$U(\xi, t) = e^{-t} \Phi(\xi) e^{-\alpha^2 \xi^2 t}.$$

Finally, take the inverse Fourier transform to get

$$u(x,t) = \mathcal{F}^{-1}[U(\xi,t)] = e^{-t} \phi(x) * \mathcal{F}^{-1}[e^{-\alpha^2 \xi^2 t}]$$

From the table, we can find the inverse transform and get

$$u(x,t) = e^{-t} \phi(x) * \left(\frac{1}{\alpha\sqrt{2t}} e^{-x^2/(4\alpha^2 t)}\right) = \frac{e^{-t}}{\alpha\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/(4\alpha^2 t)} \phi(y) dy.$$

When $\phi(x) = e^{-x^2}$, you can compute the convolution. Another more convenient way is to use the table and find $\Phi(\xi) = \frac{1}{\sqrt{2}} e^{-\xi^2/4}$ so

$$U(\xi, t) = \frac{1}{\sqrt{2}} e^{-t} e^{-(\alpha^2 t + 1/4)\xi^2}.$$

Then use table again to get

$$u(x,t) = \frac{e^{-t}}{\sqrt{1 + 4\alpha^2 t}} e^{-x^2/(1+4\alpha^2 t)}.$$