Math 112z, Fall 2019
Practice Midterm 3
Solution Key

Name: ____________________

Student ID Number: ____________________

- There are 6 pages of questions. Make sure your exam contains all these questions.
- This is a closed book, closed note, no calculator exam. You must show your work on all problems. The correct answer with no supporting work may result in no credit.
- **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- If you need more room, use the backs of the pages and indicate clearly that you have done so.
- Raise your hand if you have a question.
- Remember the **Honor Code**. Avoid suspicion of cheating by keeping your eyes on your paper and clearly showing your work on each problem!
- The problems are not ordered according to their difficulties, so please take a look at all problems and do not waste too much time on one problem. Budget your time wisely.
- You have 75 minutes to complete the exam.

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GOOD LUCK!

1. (20 pts) Determine whether the series are absolutely convergent, conditionally convergent or divergent.

   (a) $\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$.

   Try divergence test. $a_n = (-1)^n \cos(1/n^2)$. When n even, $a_n \to 1$ but when n odd, $a_n \to -1$ so limit does not exist. The series diverges.

   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$

   Here, $a_n = (-1)^n \frac{n}{n^2+4}$. First, consider absolute convergence, hence we look at $|a_n| = \frac{n}{n^2+4}$. We can use limit comparison theorem with $b_n = 1/n$. We get

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n \cdot n}{n^2+4} = 1$$

   Since $\sum_{n=1}^{\infty} 1/n$ is divergent, by comparison theorem, the series $\sum_{n=1}^{\infty} |a_n|$ is divergent. Hence the series is not absolutely convergent. But we see that $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$ is alternating. We can check that the test for alternating series works and the series converges. So the series is conditionally convergent.

   (c) $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

   We can use ratio test for this one.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{((n+1)^2+1)5^n}{5^{n+1}(n^2+1)} = \frac{1}{5} < 1.$$

   So the series is absolutely convergent.

2. (10 pts) Use the Maclaurin series $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ to evaluate the integral

$$\int \sin(x^2)dx.$$

First find the Maclaurin series for $\sin x^2$ by substitution, which is

$$\sin x^2 = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2(2n+1)}}{(2n+1)!}$$

Then integrate term by term.

$$\int \sin x^2 dx = C + \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!}.$$

3. (20 pts) Consider the power series:

$$\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$$

(a) Find the radius of convergence.

Use ratio test. Let $a_n = \frac{(x+2)^n}{2^n \ln n}$. We have

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{|x+2| 2^n \ln n}{2^{n+1} \ln(n+1)}.$$

We can use L'Hospital's rule to get

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = 1$$

so

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x+2|/2$$

The series converges if $|x+2|/2 < 1$ so that $|x+2| < 2$. The convergence radius is 2.

(b) Find the interval of convergence.

From part (a), we get that the series converges in $-4 < x < 0$. We need to check the convergence at $x = 0, x = -4$.

When $x = 0$, the series is

$$\sum_{n=2}^{\infty} \frac{2^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

We can use integral test to see the series diverges. Comparison test also works here.

When $x = -4$, the series is

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

We can use alternating series test to see this converges. So the interval of convergence is $(0, 4]$.

4. (20 pts) Use the power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ for this problem.

(a) Find a power series for $f(x) = \frac{2}{3-5x}$.

We rewrite

$$f(x) = \frac{2}{3} \cdot \frac{1}{1 - \frac{5}{3}x} = \sum_{n=0}^{\infty} \frac{2}{3}(\frac{5}{3}x)^n = \sum_{n=0}^{\infty} \frac{2}{3}(\frac{5}{3})^n x^n$$

(b) Find the radius of convergence for the power series in part (a).

This converges for $|\frac{5}{3}x| < 1$ that is $|x| < 3/5$. So convergence radius is $3/5$.

(c) Find the power series representation of $f(x) = \frac{1}{(1-x)^2}$.

Notice that $(\frac{1}{1-x})' = \frac{1}{(1-x)^2}$. We can differentiate the series for $1/(1-x)$ to get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=1}^{\infty} nx^{n-1}$$

(d) Find a power series representation for $f(x) = (\frac{x}{1-x})^2$.

Note that

$$(\frac{x}{1-x})^2 = x^2 \frac{1}{(1-x)^2} = x^2 \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n+1}$$

5. (15 pts) Find a Taylor series for $f(x) = x^{-2}$ about $a = 1$.

The Taylor series looks like

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

We need to compute all derivatives of $f(x)$ at $x = 1$.

$$f(x) = x^{-2}, \quad f(1) = 1$$

$$f'(x) = -2x^{-3}, \quad f'(1) = -2.$$

$$f''(x) = (-2)(-3)x^{-4}, \quad f''(x) = (-2)(-3).$$

We continue to see that $f^{(n)}(1) = (-2)(-3)\cdots(-(n+1)) = (-1)^n(n+1)!$. Therefore,

$$x^{-2} = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

6. (15 pts) Consider $f(x) = 2\cos x$.

(a) Find a 4th degree Taylor polynomial approximation $T_4(x)$ for $f(x)$ about $a = \pi/3$.

The 4th degree Taylor polynomial for $f(x)$ at $a = \pi/3$ is

$$T_4(x) = f(\pi/3)+f'(a)(x-\pi/3)+\frac{f^{(2)}(\pi/3)}{2}(x-\pi/3)^2+\frac{f^{(3)}(\pi/3)}{3!}(x-\pi/3)^3+\frac{f^{(4)}(\pi/3)}{4!}(x-\pi/3)^4$$

Then we find

$$f(x) = 2\cos x \Longrightarrow f(\pi/3) = 1.$$
$$f'(x) = -2\sin x \Longrightarrow f'(\pi/3) = -\sqrt{3}.$$
$$f^{(2)}(x) = -2\cos x \Longrightarrow f^{(2)}(\pi/3) = -1.$$
$$f^{(3)}(x) = 2\sin x \Longrightarrow f^{(3)}(\pi/3) = \sqrt{3}.$$
$$f^{(4)}(x) = 2\cos x \Longrightarrow f^{(4)}(\pi/3) = 1.$$

Therefore,

$$T_4(x) = 1 - \sqrt{3}(x - \pi/3) - \frac{1}{2}(x - \pi/3)^2 + \frac{\sqrt{3}}{6}(x - \pi/3)^3 + \frac{1}{24}(x - \pi/3)^4$$

(b) Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \simeq T_4(x)$ when x lies in the interval $0 \le x \le 2\pi/3$.

We apply Taylor's inequality to $f(x) = 2\cos x$ on $0 \le x \le 2\pi/3$.

$$|f(x) - T_4(x)| \le \frac{M}{(4+1)!}|x - \pi/3|^{4+1}$$

where M is such that $|f^{(4+1)}(x)| \le M$. We find that

$$f^{(5)}(x) = -2\sin x$$

so that

$$|f^{(5)}(x)| \le 2|\sin x| \le 2$$

for x on the interval $0 \le x \le 2\pi/3$. We can take $M = 2$. Therefore, we get

$$|f(x) - T_4(x)| \le \frac{2}{5!}|x - \pi/3|^5 \le \frac{2}{5!}(\pi/3)^5$$