Math 112z, Fall 2019
Practice Midterm 2
Solution key

Name: ______________________________

Student ID Number: ______________________________

- There are 6 pages of questions. Make sure your exam contains all these questions.
- This is a closed book, closed note, no calculator exam. You must show your work on all problems. The correct answer with no supporting work may result in no credit.
- **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- If you need more room, use the backs of the pages and indicate clearly that you have done so.
- Raise your hand if you have a question.
- Remember the **Honor Code**. Avoid suspicion of cheating by keeping your eyes on your paper and clearly showing your work on each problem!
- The problems are not ordered according to their difficulties, so please take a look at all problems and do not waste too much time on one problem. Budget your time wisely.
- You have 75 minutes to complete the exam.

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GOOD LUCK!

1. (15 pts) Find all solutions of the equation

$$(x^2+1)y' = xy$$

Solution: The equation is separable. There is one equilibrium solution $y = 0$. Now suppose $y \neq 0$ and separate the variables of the equation

$$\frac{1}{y}dy = \frac{x}{x^2+1}dx$$

We integrate to get

$$\ln|y| = \int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1) + C$$

Thus,

$$|y| = e^C \cdot \sqrt{x^2+1}$$

We define a new constant $D = \pm e^C$ and get

$$y = D\sqrt{x^2+1}$$

So all solutions (including the equilibrium solution) can be represented as

$$y = D\sqrt{x^2+1}$$

where D is an arbitrary real number.

2. (15 pts) Solve the initial value problem

$$xy' - y = x \ln x, \quad y(1) = 2.$$

Note $x > 0$ in this problem.

Solution: First solve the equation. The equation is linear. We find its standard form and apply integrating factor method. The standard form is

$$y' + (-1/x)y = \ln x$$

The integrating factor is $I(x) = 1/x$. Then we find

$$(\frac{1}{x})y = \int \frac{1}{x} \ln x dx = \frac{1}{2}(\ln x)^2 + C$$

Thus,

$$y = \frac{1}{2}x(\ln x)^2 + Cx$$

and C is an arbitrary constant. Now we use the initial condition to find C.

$$y(1) = 2 = \frac{1}{2}(\ln 1)^2 + C$$

so $C = 2$. The solution is

$$y = \frac{1}{2}x(\ln x)^2 + 2x.$$

3. (20 pts) Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and the surrounding temperature. Let $T(t)$ be the temperature of the object and T_s be the surrounding temperature. We get

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. Suppose that the temperature of the object is 200°F in the beginning and 1 minute later, it has cooled down to 190°F in a room at 70°F. Find the time when the temperature of the object becomes 150°F.

Solution: We are given that $T_s = 70$, $T(0) = 200, T(1) = 190$. So we get the equation

$$\frac{dT}{dt} = k(T - 70)$$

This is separable equation and we solve that

$$T(t) = 70 + De^{kt}$$

Now we use $T(0) = 200, T(1) = 190$ to find the two unknown parameters D, k.

$$T(0) = 70 + D = 200 \Longrightarrow D = 130.$$

Next,

$$T(1) = 70 + 130e^k = 190 \Longrightarrow k = \ln(12/13).$$

So the solution is

$$T(t) = 70 + 130e^{t\ln(12/13)}$$

Finally, we find t such that $T(t) = 150$.

$$150 = 70 + 130e^{t\ln(12/13)}$$

and get

$$t = \frac{\ln(8/13)}{\ln(12/13)}.$$

4. (15 pts) Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = (1 + 2/n)^n$.

Solution: Use the fundamental method. Consider $f(x) = (1 + 2/x)^x$ and find

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{x\ln(1+2/x)}.$$

It suffices to find

$$\lim_{x\to\infty} x\ln(1 + 2/x) = \lim_{x\to\infty} \frac{\ln(1 + 2/x)}{1/x}$$

which is ∞/∞ type indeterminate form. Apply L'Hospital's rule, we find that the limit is 2. This shows that the sequence converges and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{x\ln(1+2/x)} = e^2.$$

(b) $a_n = 2^{-n}\cos(n\pi)$.

Solution: Use squeeze theorem. From $-1 \le \cos(n\pi) \le 1$, we get

$$-2^{-n} \le a_n \le 2^{-n}$$

We know that $\lim_{n\to\infty} 2^{-n} = 0$ and the same for -2^{-n}. By squeeze theorem, the sequence converges and $\lim_{n\to\infty} a_n = 0$.

5. (15 pts) Determine whether the series converges or diverges. If it converges, find the sum.

(a) $\sum_{n=1}^{\infty} \frac{1}{1+e^{-n}}$

Solution: Note that

$$\lim_{n\to\infty} \frac{1}{1+e^{-n}} = 1$$

By divergence test, the series diverges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Solution: Use integral test. This is similar to a problem we did in class. The series diverges.

(c) $\sum_{n=1}^{\infty} \ln(1 + 1/n)$

Solution: We write

$$\sum_{n=1}^{\infty} \ln(1+1/n) = \sum_{n=1}^{\infty} \ln(\frac{n+1}{n}) = \sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$$

This is a telescoping series. The partial sum

$$s_n = \ln(n+1)$$

which diverges as $n \to \infty$. So the series diverges.

6. (20 pts) The following two problems are independent of each other.

(a) Find constant c such that

$$\sum_{n=0}^{\infty} e^{cn} = 10.$$

Solution: Treat the left hand side as a geometric series:

$$\sum_{n=0}^{\infty} e^{cn} = \sum_{n=0}^{\infty} (e^c)^n = 1 + \sum_{n=1}^{\infty} (e^c)^{n-1}(e^c) = \frac{e^c}{1-e^c}$$

Then solve $c = \ln(10/11)$ from

$$\frac{e^c}{1-e^c} = 10$$

(b) Consider the sequence $a_n = \dfrac{3^n}{n!}$. Determine whether it converges or diverges.

Solution: we show the sequence is monotone and bounded. First,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}$$

For $n \geq 2$, the sequence is decreasing. Thus,

$$0 \leq a_n \leq a_2$$

So the sequence is bounded decreasing. Thus it converges by the bounded monotone convergence theorem.

Remark: The original problem is for $a_n = (-3)^n/n!$. You can see the solution below if you are interested. We first consider $|a_n| = 3^n/n!$ and show the limit is 0. The argument above doesn't tell the limit is 0. We use squeeze theorem:

$$0 \leq a_n = \frac{3 \cdot 3 \cdot 3 \cdots 3}{1 \cdot 2 \cdot 3 \cdots n} \leq \frac{3 \cdot 3}{1 \cdot 2} \cdot \frac{3}{n} = \frac{27}{2n}$$

Because $3/n \leq 1$ if $n \geq 3$. Clearly

$$\lim_{n \to \infty} \frac{27}{2n} = 0$$

By squeeze theorem, $\lim |a_n| = 0$ and by absolute value theorem $\lim a_n = 0$.