Math 112z, Fall 2019  
Practice Midterm 2  
Solution key

Name: ____________________________________________

Student ID Number: __________________________________

- There are 6 pages of questions. Make sure your exam contains all these questions.

- This is a closed book, closed note, no calculator exam. You must show your work on all problems. The correct answer with no supporting work may result in no credit.

- **Put a box around your FINAL ANSWER for each problem and cross out any work that you don’t want to be graded.**

- If you need more room, use the backs of the pages and indicate clearly that you have done so.

- Raise your hand if you have a question.

- Remember the **Honor Code**. Avoid suspicion of cheating by keeping your eyes on your paper and clearly showing your work on each problem!

- The problems are not ordered according to their difficulties, so please take a look at all problems and do not waste too much time on one problem. Budget your time wisely.

- You have **75 minutes** to complete the exam.

<table>
<thead>
<tr>
<th>PAGE 1</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>PAGE 2</td>
<td>15</td>
</tr>
<tr>
<td>PAGE 3</td>
<td>20</td>
</tr>
<tr>
<td>PAGE 4</td>
<td>15</td>
</tr>
<tr>
<td>PAGE 5</td>
<td>15</td>
</tr>
<tr>
<td>PAGE 6</td>
<td>20</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>

GOOD LUCK!
1. (15 pts) Find all solutions of the equation

\[(x^2 + 1)y' = xy\]

**Solution:** The equation is separable. There is one equilibrium solution \(y = 0\). Now suppose \(y \neq 0\) and separate the variables of the equation

\[
\frac{1}{y} \, dy = \frac{x}{x^2 + 1} \, dx
\]

We integrate to get

\[
\ln |y| = \int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \ln(x^2 + 1) + C
\]

Thus,

\[
|y| = e^C \cdot \sqrt{x^2 + 1}
\]

We define a new constant \(D = \pm e^C\) and get

\[
y = D \sqrt{x^2 + 1}
\]

So all solutions (including the equilibrium solution) can be represented as

\[
y = D \sqrt{x^2 + 1}
\]

where \(D\) is an arbitrary real number.
2. (15 pts) Solve the initial value problem

\[ xy' - y = x \ln x, \quad y(1) = 2. \]

Note \( x > 0 \) in this problem.

**Solution:** First solve the equation. The equation is linear. We find its standard form and apply integrating factor method. The standard form is

\[ y' + \left(-\frac{1}{x}\right)y = \ln x \]

The integrating factor is \( I(x) = \frac{1}{x} \). Then we find

\[ \left(\frac{1}{x}\right)y = \int \frac{1}{x} \ln x \, dx = \frac{1}{2}(\ln x)^2 + C \]

Thus,

\[ y = \frac{1}{2} x(\ln x)^2 + Cx \]

and \( C \) is an arbitrary constant. Now we use the initial condition to find \( C \).

\[ y(1) = 2 = \frac{1}{2}(\ln 1)^2 + C \]

so \( C = 2 \). The solution is

\[ y = \frac{1}{2} x(\ln x)^2 + 2x. \]
3. (20 pts) Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and the surrounding temperature. Let $T(t)$ be the temperature of the object and $T_s$ be the surrounding temperature. We get

$$ \frac{dT}{dt} = k(T - T_s) $$

where $k$ is a constant. Suppose that the temperature of the object is $200^\circ F$ in the beginning and 1 minute later, it has cooled down to $190^\circ F$ in a room at $70^\circ F$. Find the time when the temperature of the object becomes $150^\circ F$.

**Solution:** We are given that $T_s = 70, T(0) = 200, T(1) = 190$. So we get the equation

$$ \frac{dT}{dt} = k(T - 70) $$

This is separable equation and we solve that

$$ T(t) = 70 + De^{kt} $$

Now we use $T(0) = 200, T(1) = 190$ to find the two unknown parameters $D, k$.

$$ T(0) = 70 + D = 200 \implies D = 130. $$

Next,

$$ T(1) = 70 + 130e^k = 190 \implies k = \ln(12/13). $$

So the solution is

$$ T(t) = 70 + 130e^{t\ln(12/13)} $$

Finally, we find $t$ such that $T(t) = 150$.

$$ 150 = 70 + 130e^{t\ln(12/13)} $$

and get

$$ t = \frac{\ln(8/13)}{\ln(12/13)}. $$
4. (15 pts) Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) \( a_n = (1 + 2/n)^n \).

\textit{Solution:} Use the fundamental method. Consider \( f(x) = (1 + 2/x)^x \) and find

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{x \ln(1+2/x)}.
\]

It suffices to find

\[
\lim_{x \to \infty} x \ln(1 + 2/x) = \lim_{x \to \infty} \frac{\ln(1 + 2/x)}{1/x}
\]

which is \( \infty/\infty \) type indeterminate form. Apply L'Hospital’s rule, we find that the limit is 2. This shows that the sequence converges and

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{x \ln(1+2/x)} = e^2.
\]

(b) \( a_n = 2^{-n} \cos(n \pi) \).

\textit{Solution:} Use squeeze theorem. From \(-1 \leq \cos(n \pi) \leq 1\), we get

\(-2^{-n} \leq a_n \leq 2^{-n}\)

We know that \( \lim_{n \to \infty} 2^{-n} = 0 \) and the same for \(-2^{-n}\). By squeeze theorem, the sequence converges and \( \lim_{n \to \infty} a_n = 0 \).
5. (15 pts) Determine whether the series converges or diverges. If it converges, find the sum.

(a) \( \sum_{n=1}^{\infty} \frac{1}{1 + e^{-n}} \)

Solution: Note that
\[
\lim_{n \to \infty} \frac{1}{1 + e^{-n}} = 1
\]
By divergence test, the series diverges.

(b) \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

Solution: Use integral test. This is similar to a problem we did in class. The series diverges.

(c) \( \sum_{n=1}^{\infty} \ln(1 + 1/n) \)

Solution: We write
\[
\sum_{n=1}^{\infty} \ln(1 + 1/n) = \sum_{n=1}^{\infty} \ln\left( \frac{n+1}{n} \right) = \sum_{n=1}^{\infty} (\ln(n+1) - \ln n)
\]
This is a telescoping series. The partial sum
\[
s_n = \ln(n+1)
\]
which diverges as \( n \to \infty \). So the series diverges.
6. (20 pts) The following two problems are independent of each other.

(a) Find constant $c$ such that

$$\sum_{n=0}^{\infty} e^{cn} = 10.$$ 

**Solution:** Treat the left hand side as a geometric series:

$$\sum_{n=0}^{\infty} e^{cn} = \sum_{n=0}^{\infty} (e^c)^n = 1 + \sum_{n=1}^{\infty} (e^c)^{n-1} = \frac{e^c}{1 - e^c}$$

Then solve $c = \ln(10/11)$ from

$$\frac{e^c}{1 - e^c} = 10$$

(b) Consider the sequence $a_n = \frac{3^n}{n!}$. Determine whether it converges or diverges.

**Solution:** we show the sequence is monotone and bounded. First,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \cdot \frac{3}{n+1} = 3$$

For $n \geq 2$, the sequence is decreasing. Thus,

$$0 \leq a_n \leq a_2$$

So the sequence is bounded decreasing. Thus it converges by the bounded monotone convergence theorem.

**Remark:** The original problem is for $a_n = (-3)^n/n!$. You can see the solution below if you are interested. We first consider $|a_n| = 3^n/n!$ and show the limit is 0. The argument above doesn’t tell the limit is 0. We use squeeze theorem:

$$0 \leq a_n = \frac{3 \cdot 3 \cdot 3 \cdots 3}{1 \cdot 2 \cdot 3 \cdots n} \leq \frac{3 \cdot 3 \cdot 3}{1 \cdot 2 \cdot n} = \frac{27}{2n}$$

Because $3/n \leq 1$ if $n \geq 3$. Clearly

$$\lim_{n \to \infty} \frac{27}{2n} = 0$$

By squeeze theorem, $\lim |a_n| = 0$ and by absolute value theorem $\lim a_n = 0$. 