Math 112z, Fall 2019
Practice Midterm 1
Solution Key

Name: ______________________________

Student ID Number: ______________________________

- There are 7 pages of questions. Make sure your exam contains all these questions.
- This is a closed book, closed note, no calculator exam. You must show your work on all problems. The correct answer with no supporting work may result in no credit.
- **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- If you need more room, use the backs of the pages and indicate clearly that you have done so.
- Raise your hand if you have a question.
- Remember the **Honor Code**. Avoid suspicion of cheating by keeping your eyes on your paper and clearly showing your work on each problem!
- The problems are not ordered according to their difficulties, so please take a look at all problems and do not waste too much time on one problem. Budget your time wisely.
- You have 75 minutes to complete the exam.

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GOOD LUCK!

1. (20 pts) Evaluate the following integrals

   (a) $\int x^2 \cos 5x dx$

   Solution key: Integration by parts. First let $u = x^2$, $dv = \cos 5x dx$ so $v = \frac{1}{5}\sin 5x$ and we get

$$\int x^2 \cos 5x dx = \frac{1}{5}x^2 \sin 5x - \int (\frac{1}{5}\sin 5x)(2x)dx = \frac{1}{5}x^2 \sin 5x - \frac{2}{5}\int x \sin 5x dx$$

   Then apply integration by part again.
   Answer: $\frac{1}{5}x^2\sin(5x) + \frac{2}{25}x\cos(5x) - \frac{2}{125}\sin(5x) + C$

   (b) $\int \tan^5 x dx$

   Solution key: This is a trig integral of the type $\int \tan^m x \sec^n x dx$ with m odd but there is no $\sec x$ to borrow. We can use trig identity to transform the integral as

$$\begin{aligned} \int \tan^5 x dx &= \int (\tan^2 x)^2 \tan x dx = \int (\sec^2 x - 1)^2 \tan x dx \\ &= \int \sec^4 x \tan x dx - 2\int \sec^2 \tan x dx + \int \tan x dx. \end{aligned}$$

   Then the first two are of the type $\int \tan^m x \sec^n x dx$ with n even so we can apply substitution $u = \tan x$.
   Answer: $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$

2. (15 pts) Evaluate the following integral

$$\int \frac{x^3 + 4x + 3}{x^4 + 5x^2 + 4} dx$$

Solution key: The integrand is a rational function. So apply partial fraction decomposition. The denominator can be factorized (it is quadratic in x^2) as

$$x^4 + 5x^2 + 4 = (x^2)^2 + 5(x^2) + 4 = (x^2 + 1)(x^2 + 4)$$

The partial fraction decomposition should be of the form

$$\frac{x^3 + 4x + 3}{x^4 + 5x^2 + 4} = \frac{x^3 + 4x + 3}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}$$

Then we can find $A = 1, B = 1, C = 0, D = -1$. Then the integral becomes

$$\begin{aligned} \int \frac{x^3 + 4x + 3}{x^4 + 5x^2 + 4} dx &= \int \frac{x + 1}{x^2 + 1} dx - \int \frac{1}{x^2 + 4} dx \\ &= \int \frac{x}{x^2 + 1} dx + \int \frac{1}{x^2 + 1} dx - \int \frac{1}{x^2 + 4} \end{aligned}$$

For the first integral, apply substitution $u = x^2 + 1$. For the last two integrals, we can apply the formula.

Answer: $\frac{1}{2}\ln(x^2 + 1) + \tan^{-1} x - \frac{1}{2}\tan^{-1}(\frac{x}{2}) + C$

3. (15 pts) Evaluate the integral

$$\int \frac{\sqrt{x^2-9}}{x^3} dx$$

Solution key: We apply trig substitution. Let $u = 3\sec\theta$ so $dx = 3\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$. We transform the integral to

$$\int \frac{\sqrt{x^2-9}}{x^3} dx = \int \frac{3\tan\theta}{27\sec^3\theta} 3\sec\theta\tan\theta d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta} d\theta = \frac{1}{3}\int \sin^2\theta d\theta.$$

This is a trig integral. We can evaluate it using half angle formula.

Answer: $\frac{1}{6}\cos^{-1}(3/x) - \frac{\sqrt{x^2-9}}{2x^2} + C$

4. (10 pts) Consider the curve

$$36y^2 = (x^2 - 4)^3, \quad y \geq 0$$

Find the length of the curve between $P_0(2, 0)$ and $P_1(3, \sqrt{125}/6)$

Solution Key: We first determine the parametrization of the curve. We can solve y:

$$y^2 = \frac{1}{36}(x^2 - 4)^3 \Longrightarrow y = \pm\frac{1}{6}(x^2 - 4)^{3/2}$$

and we pick $+$ because $y \geq 0$. So we get $y = f(x) = \frac{1}{6}(x^2 - 4)^{3/2}$. Then we can apply the arclength formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_2^3 \sqrt{1 + \frac{1}{4}(x^2 - 4)x^2} dx = \int_2^3 (\frac{1}{2}x^2 - 1) dx$$

We can find this easily.

Answer: 13/6.

5. (10 pts) Use the Comparison Theorem to determine whether

$$\int_0^\infty \frac{x}{x^3+10}dx$$

is convergent or divergent.

Solution key: This is an improper integral of the first type. The integrand is continuous. For $x > 0$, we get

$$0 < \frac{x}{x^3+10} < \frac{x}{x^3} < \frac{1}{x^2}.$$

However, $1/x^2$ is not continuous at $x = 0$. But we know that $\int_1^\infty \frac{1}{x^2}dx$ is convergent. Thus, we can split the original integral as

$$\int_0^\infty \frac{x}{x^3+10}dx = \int_0^1 \frac{x}{x^3+10}dx + \int_1^\infty \frac{x}{x^3+10}dx$$

The first integral is a well-defined definite integral. The second integral is convergent by comparison theorem. So the integral is convergent.

Another way to think about is that we find a better upper bound of the integrand. For example, for $x \geq 1$, we can still use

$$0 < \frac{x}{x^3+10} < \frac{1}{x^2}.$$

For $0 \leq x \leq 1$, we can take

$$0 < \frac{x}{x^3+10} < \frac{1}{10}.$$

Then we consider a piecewisely defined function $g(x)$ where

$$g(x) = 1/x^2 \quad x \geq 1, \text{ and } g(x) = 1/10 \quad 0 \leq x < 1.$$

Then

$$0 < \frac{x}{x^3+10} < g(x)$$

and

$$\int_0^\infty g(x)dx = \int_0^1 g(x)dx + \int_1^\infty g(x)dx = \int_0^1 \frac{1}{10}dx + \int_1^\infty \frac{1}{x^2}dx$$

is convergent.

6. (15 pts) Determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_{-2}^{3} \frac{1}{x^{10}} dx$

Solution key: The integrand is not continuous at $x = 0$ so we split the integral as

$$\int_{-2}^{3} \frac{1}{x^{10}} dx = \int_{-2}^{0} \frac{1}{x^{10}} dx + \int_{0}^{3} \frac{1}{x^{10}} dx$$

We can show by definition that the second improper integral is divergent.

(b) $\displaystyle\int_{0}^{\infty} \sin^2 t dt$

Solution key: This is an improper integral of the first type. We use definition

$$\int_{0}^{\infty} \sin^2 t dt = \lim_{a \to \infty} \int_{0}^{a} \sin^2 t dt$$

Evaluate the definite integral using half angle formula. Finally, we can see the limit does not exist.

7. (15 pts) Consider the integral

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx$$

where m, n are positive integers. Evaluate the integral without using product to sum formula. (Hint: The $m = n$ case is simple. For $m \neq n$, use integration by parts.)

Solution key: When $m = n$, the integral is just

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = \int_{-\pi}^{\pi} \sin(mx)\cos(mx)dx$$

We can evaluate using substitution $u = \sin(mx)$. The answer is 0.

When $m \neq n$, we apply integration by parts. First, let $u = \sin mx, dv = \cos(nx)dx$ so $v = \frac{1}{n}\sin(nx)$. We get

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0 - \int_{-\pi}^{\pi} \frac{1}{n}\sin(nx)m\cos(mx)dx$$

Then we let $u = \cos(mx)$ and $dv = \sin(nx)dx$ so that $v = -\frac{1}{n}\cos(nx)$. We apply integration by parts again to get

$$\int_{-\pi}^{\pi} \frac{1}{n}\sin(nx)m\cos(mx)dx = \frac{m}{n}\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0 + \frac{m}{n}\int_{-\pi}^{\pi} \frac{m}{n}\sin(mx)\cos(nx)dx$$

Combine the above two equations, we get

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = \frac{m^2}{n^2}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx$$

Because $m \neq n$, the integral must be 0.