VALUATIONS AND HIGHER LEVEL ORDERS IN COMMUTATIVE RINGS

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Introduction

Valuation theory is one of the main tools for studying higher level orders and the reduced theory of forms over fields, see, for example [BR]. In [MW], the theory of higher level orders and reduced forms was generalized to rings with many units and many of the results for fields carried over to this setting. While it seems desirable to extend these results further, the techniques used for rings with many units will not work for general commutative rings. At the same time, there is a general theory of valuations in commutative rings (see [LM], [M], and [G]), which in [Ma] was used to study orders and the reduced theory of quadratic forms over general commutative rings. Thus it seems natural to ask if the connections between valuations and higher level orders in fields exist in commutative rings. In this paper we use valuation theory to study the space of orders and the reduced Witt ring relative to a higher level preorder in a commutative ring. As in [Ma], we first localize our ring at a multiplicative set, without changing the space of orders, in order to make the valuation theory work better. This is a standard idea from real algebraic geometry.

Remarkably, many of the notions, methods, and results for fields carry over to this new setting. We define compatiblity between valuations and orders and preorders, and the ring $A(T)$ associated to a preorder T, which turns out to be Prüfer ring as in the field case. We define the relation of dependency on the set of valuations associated to a preorder and we use this to prove a decomposition theorem for the space of orders. We can then apply this to show that, under a certain finiteness condition, the space of orders is equivalent to the space of orders of a preordered field.

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§1. Preliminaries

Let R be a commutative ring with 1 and R^* the units of R. For any subset $S \subseteq R$, S^* denotes $S \cap R^*$. For a prime ideal $p \subseteq R$, let $R(p)$ denote the quotient field of R/p and α_p the canonical map $R \to R/p \hookrightarrow R(p)$. We will frequently use the following fact: If S is a multiplicative set in R and $k \in \mathbb{N}$, then any element of R localized at S can be written in the form as^{-k}, where $a \in R$ and $s \in S$, since $as^{-1} = (as^{k-1})s^{-k}$.

Valuations in commutative rings. Details on valuations in commutative rings can be found in [M] and [G]. Let Γ be an ordered abelian group, written additively, and set $\Gamma_\infty = \Gamma \cup \{\infty\}$, where $\alpha + \infty = \infty + \alpha = \infty$ and $\alpha < \infty$ for all $\alpha \in \Gamma$. A mapping $v : R \to \Gamma_\infty$ is a *valuation on* R if $v(0) = \infty$, $v(1) = 0$, and for all $x, y \in R$, $v(x+y) \geq \min\{v(x), v(y)\}$ and $v(xy) = v(x) + v(y)$. We always assume that Γ is the group generated by $\{v(r) \mid r \in R\}$. (If not we replace Γ by this group.) Γ is called the *value group of* v. If v is surjective, we say v is a *Manis valuation.*

Suppose $v : R \to \Gamma_\infty$ is a valuation. Then it is easy to check that $v^{-1}(\infty)$ is a prime ideal in R, called the *support of* v and denoted $\operatorname{supp}(v)$. Let $q := \operatorname{supp}(v)$, then there exists a unique valuation $\hat{v} : R(q) \to \Gamma_\infty$ with $v = \hat{v} \circ \alpha_q$. Conversely, if q is a prime ideal in R and $\hat{v} : R(q) \to \Gamma_\infty$ is a valuation, then $v := \hat{v} \circ \alpha_q$ is a valuation on R. Since $\hat{v}(x) = \infty$ iff $x = 0$, it follows that $q = \operatorname{supp}(v)$. Two valuations v and w are equivalent if $\operatorname{supp}(v) = \operatorname{supp}(w)$ and $\hat{v} = \hat{w}$. Note that if v and w are equivalent and v is Manis, then w is Manis. We identify equivalent valuations, thus there is a 1-1 correspondence between valuations v and pairs $(q, \hat{A})$, where q is a prime ideal in R and $\hat{A}$ is a valuation ring in $R(q)$. We write $v = (q, \hat{A})$, where $q = \operatorname{supp}(v)$ and $\hat{A}$ is the valuation ring of $\hat{v}$.

Given a valuation $v = (p, \hat{A})$, let $A = \alpha_p^{-1}(\hat{A})$ and $I = \alpha_p^{-1}(\hat{I})$, where $\hat{I}$ denotes the maximal ideal of $\hat{A}$. Then A is the called the *valuation ring of* v and I the *prime ideal of* A. It follows easily from the definitions that $A = \{r \in R \mid v(r) \geq 0\}$ and $I = \{r \in R \mid v(r) > 0\}$. Also note that if v is a Manis valuation, then A determines v, since in this case $I = \{r \in R \mid xr \in A \text{ for some } x \in R \setminus A\}$, see [G].

If $\Gamma = \{0\}$, then we say v is a *trivial valuation.* In this case we have $A = R$, $I = q = \operatorname{supp}(v)$ and $\hat{A} = R(q)$. Note each prime ideal in R gives rise to a trivial valuation and that trivial valuations are clearly Manis.

Suppose A is a subring of R and I is a prime ideal in A. Then (A, I) is called a *valuation pair* if given any $r \in R \setminus A$ there exists some $x \in I$ such that $xr \in A \setminus I$. We collect some facts about Manis valuations and valuation pairs:

Proposition 1.1.

(i) *Given $v = (p, \hat{A})$ a Manis valuation in R with valuation ring A and prime ideal I. Then (A, I) is a valuation pair. Conversely, given a valuation pair (A, I) then there exists a unique Manis valuation $v = (p, \hat{A})$ such that $A = \alpha_p^{-1}(\hat{A})$ and $I = \alpha_p^{-1}(\hat{I})$, where $\hat{I}$ is the maximal ideal of $\hat{A}$. In this case, $\hat{A} = \{\alpha_p(a)/\alpha_p(b) \mid a, b \in A \text{ and } b \notin I\}$ and $\hat{I} = \{\alpha_p(a)/\alpha_p(b) \mid a, b \in A \text{ and } b \notin I\}$.*

(ii) *Suppose* $v = (q, \hat{A})$ *is a Manis valuation with prime ideal* I. *Then* $q = \{r \in R \mid xr \in I$ *for all* $x \in R\}$.

(iii) *Suppose* v *and* w *are valuations with* w *Manis and both have valuation ring* A *and prime ideal* I. *Then* $supp(v) \subseteq supp(w)$.

Proof. (i) and (ii) follow from [M, Proposition 1].
(iii): Since $\mathrm{supp}(v)$ is an ideal in R and is contained in I, $\mathrm{supp}(v) \subseteq \{r \in R \mid xr \in I$ for all $x \in R\} = \mathrm{supp}(w)$, by (ii). □

Higher Level Preorders and Orders. For details on higher level orders and preorders in commutative rings, see [MW, §1].

A subset $T \subseteq R$ is a *preorder of level* n if $T + T \subseteq T$, $T \cdot T \subseteq T$, $-1 \notin T$, and $R^{2n} \subseteq T$. If F is a field, then a preorder P of level n in F is an order of level n if F^*/P^* is cyclic. In general, a preorder P of level n in R is an *order of level* n if there exists a prime ideal p in R and an order $\bar{P}$ on $R(p)$ such that $P = \alpha_p^{-1}(\bar{P})$. In this case we will write $P = (p, \bar{P})$. Note $p = P \cap -P$. In this paper, "order" will always mean an order of some level n. For a preorder T in R, let O_T denotes the set of orders P such that $T \subseteq P$. (We reserve X_T for the T-signatures of R, see §4.)

A prime ideal p in R is a *real prime* if $R(p)$ has an order, iff there exists an order P in R with $P \cap -P = p$. Given a preorder T in R of level n and a prime ideal p, let $T(p) = \{\alpha_p(t)\alpha_p(s)^{-2n} \mid t \in T$ and $s \in R \setminus p\}$. We say p is *T-compatible* if $T(p)$ is a preorder in $R(p)$. It is easy to see that p is T-compatible iff $-1 \notin T(p)$.

We fix a preorder T of level n. Let $S = 1 + T$, a multiplicative set in R, then $S^{-1}R$ is a nonzero ring. It is easy to check that $S^{-1}T$ is a preorder in $S^{-1}R$ and there is a 1-1 correspondence between O_T and $O_{S^{-1}T}$ given by $P \mapsto \{xs^{-2n} \mid x \in P$ and $s \in S\}$. Under this bijection we have $(p, \bar{P}) \leftrightarrow (p', \bar{P})$ where p' denotes the image of p in $S^{-1}R$. **For the rest of this paper we replace R by $S^{-1}R$ and T by $S^{-1}T$, i.e., we assume throughout that $1 + T \subseteq R^*$.**

Lemma 1.2.

(i) *Given* $r \in R$ *such that* $r \notin P \cap -P$ *for all* $P \in O_T$. *Then* $r \in R^*$.

(ii) $T^* = \bigcap_{P \in O_T} P^*$.

(iii) $R = T^* - T^*$.

Proof. (i): Given $P \in O_T$, if $r \notin P \cap -P$, then $r^{2n} \in P \setminus -P$. Thus, by [Be, Theorem 6], if $r \notin P \cap -P$ for all $P \in O_T$, there exist $t, t' \in T$ such that $tr^{2n} = 1 + t'$. Hence $r \in R^*$ since $1 + t' \in R^*$.
(ii): $T^* \subseteq \bigcap_{P \in O_T} P^*$ is clear. The reverse inclusion follows from [Be, Theorem 6].
(iii): By a standard argument we have $R = \Sigma R^{2n} - \Sigma R^{2n}$ since $\mathbb{Q} \subseteq R$. Hence $R = T - T = (1 + T) - (1 + T) = T^* - T^*$. □

Lemma 1.3. *If* v *is a valuation in* R *with valuation ring* A *and prime ideal* I, *then* (A, I) *is a valuation pair.*

Proof. Set $k = 2n$. Then $1 + x \in 1 + T \in R^*$ for all $x \in R^k$. Given $r \in R \setminus A$, let $\bar{r} = \alpha_p(r) \in R(p)$. Then $\bar{r} \notin \hat{A}$, hence $1 + \bar{r}^k \notin \hat{A}$, since a valuation ring in a field is integrally closed. Thus $\frac{1}{1+\bar{r}^k} \in \hat{I}$ and hence $\frac{\bar{r}^k}{1+\bar{r}^k} = 1 - \frac{1}{1+\bar{r}^k} \in \hat{A} \setminus \hat{I}$. Thus $\frac{\bar{r}^{k-1}}{1+\bar{r}^k} = \bar{r}^{-1} \cdot \frac{\bar{r}^k}{1+\bar{r}^k} \in \hat{I}$. We have $xr \in A \setminus I$, where $x = \frac{r^{k-1}}{1+r^k} \in I$. Therefore (A, I) is a valuation pair. $\square$

Definition. Given an order $P = (p, \bar{P}) \in O_T$, then by [B1, 3.4], $A(\bar{P}) = \{x \in R(p) \mid s \pm x \in \bar{P}$ for some $s \in \mathbb{Q}^+\}$ is a valuation ring in $R(p)$ with maximal ideal $I(\bar{P}) = \{x \in R(p) \mid s \pm x \in \bar{P}$ for all $s \in \mathbb{Q}^+\}$. Thus we have a valuation $(p, A(\bar{P}))$, with valuation ring $A(P) := \{r \in R \mid s \pm r \in P$ for some $s \in \mathbb{Q}^+\}$ and prime ideal $I(P) := \{r \in R \mid s \pm r \in P$ for all $s \in \mathbb{Q}^+\}$. We denote this valuation by v_P. If v_P is trivial, we say P is *archimedean*. In this case, since $A(\bar{P}) = R(p)$, $\bar{P}$ is an archimedean (level 1) order on $R(p)$. Thus archimedean orders on R correspond to $(p, \bar{P})$ where $\bar{P}$ is an archimedean order on $R(p)$.

The following useful fact about Manis valuations will be used frequently in later sections:

Lemma 1.4. *Suppose $v = (q, \hat{A})$ is a Manis valuation in R with valuation ring A and prime ideal I. If $r \in R \setminus q$, then there exists $x \in R$ such that $xr \in A \setminus I$. If $r \in T$ we can choose $x \in T^*$.*

Proof. Since $r \notin q = v^{-1}(\infty)$, we have $v(r) = \gamma$ for some γ in the value group of v. Since v is onto, there is some $x \in R$ such that $v(x) = -\gamma$. Then $v(xr) = 0$ and thus $xr \in A \setminus I$. If $r \in T$ then we have $(x^{2n}r^{2n-1})(r) = (xr)^{2n} \in A \setminus I$, hence we can replace x by $x^{2n}r^{2n-1} \in T$.

We need some results on Prüfer rings in R. For details on Prüfer rings in commutative rings, see [LM] and [G].

Definition. Suppose A is a subring of R and p a prime ideal in A. Define $A_{[p]} = \{r \in R \mid xr \in A$ for some $x \in A \setminus p\}$ and $p^\sharp = \{r \in R \mid xr \in p$ for some $x \in A \setminus p\}$. Then $A_{[p]}$ is a subring of R and $p^\sharp$ is a prime ideal in $A_{[p]}$.

Lemma 1.5. *Suppose A is a subring of R.*

(i) *Given prime ideals p, q in A, then $p \subseteq q$ implies $A_{[q]} \subseteq A_{[p]}$.*

(ii) *If $p_1, \ldots, p_k, q$ are prime ideals in A such that $\bigcap_{i=1}^k p_i \subseteq q$, then $A_{[q]} \subseteq A_{[p_1]} \cup \cdots \cup A_{[p_k]}$.*

(iii) *If v is a valuation on R with valuation ring A and prime ideal I, then $A_{[I]} = A$ and $I^\sharp = I$.*

Proof. (i) follows from $A \setminus q \subseteq A \setminus p$.
(ii) follows from the fact that $r \in A \setminus q$ implies $r \in A \setminus p_i$ for some i.
(iii): It is clear that $A \subseteq A_{[I]}$ and $I \subseteq I^\sharp$. Given $r \in A_{[I]}$, say $x \in A \setminus I$ with $xr \in A$. If $r \notin A$ then there is some $y \in I$ such that $yr \in A \setminus I$. But then $x(yr) \in (A \setminus I) \cdot (A \setminus I) = A \setminus I$ while $(xr)y \in I \cdot A \subseteq I$, a contradiction. Hence $A_{[I]} = A$. Given $r \in I^\sharp$, then $xr \in I$ for some $x \in A \setminus I$. Since $r \in A$ this implies $r \in I$ and thus $I = I^\sharp$. $\square$

Definition. We say A is a *Prüfer ring in R* if $(A_{[p]}, p^\sharp)$ is a valuation pair for all prime ideals p in A.

Proposition 1.6. *Suppose A is a Prüfer ring in R.*

(i) *If (B, J) is a valuation pair such that $A \subseteq B$, then $(B, J) = (A_{[p]}, p^\sharp)$ where $p = A \cap J$.*

(ii) *$A = \bigcap B$, the intersection over all overrings B of A such that (B, J) is a valuation pair for some prime ideal J in B.*

Proof. (i): Let $p = J \cap A$, clearly a prime ideal in A. Then $(A_{[p]}, p^\sharp)$ is a valuation pair since A is a Prüfer ring. It follows from the definitions that $A_{[p]} \subseteq B_{[J]}$ and $p^\sharp \subseteq J^\sharp$. This implies $A_{[p]} \subseteq B$ and $p^\sharp \subseteq J$ by 1.5,(iii). Suppose $a \in B$ and $a \notin A_{[p]}$, then there exists $x \in p^\sharp$ such that $ax \in A_{[p]} \setminus p^\sharp$. Since $ax \in A_{[p]}$, by definition there exists $y \in A \setminus p$ such that $yax \in A$, and $yax \notin p$ since $ax \notin p^\sharp$. Also, there exists $z \in A \setminus p$ such that $zx \in p$. Then we have $y, z, x, a \in B$ and $zx \in J$, hence $yzax \in J \cap A = p$. But $yax \in A \setminus p$ and $z \in A \setminus p$ implies $yzax \notin p$, a contradiction. Hence $B \subseteq A_{[p]}$ and thus $B = A_{[p]}$. A similar argument shows $J = p^\sharp$.
(ii): Let C be the intersection of all valuation overrings of A, then clearly $A \subseteq C$. By (i), $C = \bigcap A_{[p]}$, the intersection over all prime ideals p in A. By [G, Proposition 9], $A = \cap A_{[m]}$, where the intersection is over all maximal ideals m in A, hence $A = \cap A_{[m]} \supseteq \cap A_{[p]} = C$ and thus $A = C$. $\square$

Theorem 1.7. *Suppose A is a subring of R such that $\frac{1}{1+x} \in A$ for each $x \in \Sigma R^{2n}$. Then A is a Prüfer ring in R.*

Proof. Given $p \subseteq A$ a prime ideal. Let $\tilde{A}$ be the integral closure of A in R and $\tilde{p}$ a prime ideal in $\tilde{A}$ with $\tilde{p} \cap A = p$. Set $B := \{r \in R \mid yr \in \tilde{A}$ for some $y \in A \setminus p\}$ and $q := \{r \in R \mid yr \in \tilde{p}$ for some $y \in A \setminus p\}$. It is easy to see that B is a subring of R and q is a prime ideal in B. It follows from the definitions that $A_{[p]} \subseteq B$ and $p^\sharp \subseteq q$.

Claim 1: If $r^m \in B$ for some $m \in \mathbb{N}$, then $r \in B$.
Proof: If $r^m \in B$, then there exists some $y \in A \setminus p$ with $yr^m \in \tilde{A}$. Hence $(yr)^m \in \tilde{A}$ and thus $yr \in \tilde{A}$. It follows that $r \in B$.

Claim 2: $q \cap A_{[p]} = p^\sharp$
Proof: $p^\sharp \subseteq q \cap A_{[p]}$ is clear. If $r \in q \cap A_{[p]}$ then there exists some $y \in A \setminus p$ with $yr \in A$ and $x \in A \setminus p$ with $xr \in \tilde{p}$. Hence $xyr \in A \cap \tilde{p} = p$, and thus $r \in p^\sharp$.

Claim 3: Suppose $r \in R$ with $r^{2n} \notin A_{[p]}$. Then there is some $x \in p$ with $xr^{2n} \in B \setminus q$.
Proof: Let $x = \frac{1}{1+r^{2n}} \in A \subseteq A_{[p]}$ then $xr^{2n} = 1 - x \in A$. If $x \notin p$ then $r^{2n} \in A_{[p]}$, a contradiction. Thus $x \in p \subseteq q$. Since $xr^{2n} \in A_{[p]} \subseteq B$ and $xr^{2n} = 1 - x$, it follows that $xr^{2n} \in B \setminus q$.

Claim 4: Given $r \in R \setminus B$. Suppose $m \in \mathbb{N}$ and $x \in q$ with $xr^m \in B \setminus q$, then there exists $x' \in q$ with $x'r \in B \setminus q$.
Proof: We proceed by induction on m. If $m = 1$ let $x' = x$. Suppose $m > 1$ and $x \in q$ with $xr^m \in B \setminus q$, then $(xr)^m \in q$. Thus $xr \in B$ by claim 1, hence $xr \in q$. Since $xr^m = (xr)r^{m-1}$ we are done by induction.

Claim 5: (B, q) is a valuation pair in R.
Proof: Given $r \in R \setminus B$, then $r^{2n} \notin B$ by claim 1. Hence $r^{2n} \notin A_{[p]}$. Thus there exists $x' \in p \subseteq q$ with $x'r^{2n} \in B \setminus q$ by claim 3. Hence, by claim 4, there exists $x \in q$ with $xr \in B \setminus q$. Therefore (B, q) is a valuation pair.

By claim 5, it is enough to show $A_{[p]} = B$ and $p^\sharp = q$. Given $r \in B$ with $r^{2n} \notin A_{[p]}$, then there exists, by claim 3, $y \in p$ with $yr^{2n} \in B \setminus q$. But then $y \in q$, $r^{2n} \in B$, and $yr^{2n} \in B \setminus q$, a contradiction. Hence $r^{2n} \in A_{[p]}$ for each $r \in B$. Since $B = \Sigma B^{2n} - \Sigma B^{2n}$, it follows that $A_{[p]} = B$. Hence $p^\sharp = q$ by claim 2. Thus $(A_{[p]}, p^\sharp)$ is a valuation in R for each prime ideal p in A, and therefore A is a Prüfer ring in R. $\square$

Remark. When R is a field and $n = 1$, Theorem 1.7 is a result of Dress [D, 9]. Becker proved Theorem 1.7 for R a field and general n [B2, 3.3].

Definition. Let $A(T) = \{r \in R \mid s \pm r \in T \text{ for some } s \in \mathbb{Q}\}$.

Proposition 1.8. *$A(T)$ is a Prüfer ring in R. In particular, for any $P \in O_T$ $(A(P), I(P)) = (A(T)_{[p]}, p^\sharp)$, where $p = A(T) \cap I(P)$.*

Proof. $A(T)$ is a Prüfer ring by 1.7. The second statement then follows from 1.6,(i). $\square$

§2. Compatible valuations

One of the key notions in studying higher level orders and forms in fields is that of compatibility between orders and valuations. For a field F, a valuation ring A with maximal ideal I, and an order P on F, we say A is compatible with P if $1 + I \subseteq P$. In this case the "pushdown of P along A", the image of $P \cap A$ in the field A/I, is an order. For details, see [BR, §2]. In our case the situation is a bit more complicated since in general a given order and a given valuation will come from different residue fields of R.

Definition. Suppose $v = (q, \hat{A})$ is a valuation with valuation pair (A, I) and $P \in O_T$. We say v is *compatible with* P if $P \cap -P \subseteq q$ and $P \cap (A \setminus I) + I \subseteq P$. We denote this by $v \sim P$. We say v is *compatible with* T if v is compatible with some $P \in O_T$, written $v \sim T$. If v is compatible with all $P \in O_T$ then we say v is *fully compatible with* T, written $v \sim_f T$.

Remark. If R is a field then $P \cap (A \setminus I) + I \subseteq P$ iff $1 + I \subseteq P$. Hence our definitions agree with the usual definitions for fields, cf. [BR, §2].

Lemma 2.1.

(i) *For all $P \in O_T$, $v_P \sim P$.*
(ii) *Let v be the Manis valuation with valuation pair $(A(P), I(P))$, which exists by 1.1. Then $v \sim P$.*

Proof. (i): Suppose $P = (p, \bar{P}) \in O_T$. By [BHR, 2.7] we have $A(\bar{P}) \sim \bar{P}$. Given $x \in P \cap (A \setminus I)$ and $y \in I$, then $\alpha_p(x) \in \bar{P} \cap (A(\bar{P}) \setminus I(\bar{P}))$. Hence $\alpha_p(x) + \alpha_p(y) \in \bar{P}$ and thus $x + y \in P$. Therefore $v_P \sim P$.
(ii): By 1.1,(iii), $p = \operatorname{supp}(v_P) \subseteq \operatorname{supp}(v)$, hence $v \sim P$ follows from (i). $\square$

Proposition 2.2. *Suppose $v = (q, \hat{A})$ is a Manis valuation with valuation ring A and prime ideal I. Given $P = (p, \bar{P}) \in O_T$, then the following are equivalent:*

(i) $v \sim P$

(ii) *$P(q)$ is an order in $R(q)$, $\hat{A}$ is compatible with $P(q)$, and $\alpha_q^{-1}(P(q)) = P \cup q$.*

Proof. Assume $v \sim P$, we first show that $\alpha_q^{-1}(P(q)) = P \cup q$. It is clear that $P \cup q \subseteq \alpha_q^{-1}(P(q))$. Suppose $r \in R$ with $\alpha_q(r) \in P(q)$. Assume $r \notin q$, then there exist $x \in P$ and $s \in R \setminus q$ such that $\alpha_q(r) = \alpha_q(x)\alpha_q(s)^{-2n}$. Thus there is some $y \in q$ such that $s^{2n}r = x + y$. Since $s^{2n}r \notin q$, we have $x \notin q$. By 1.4 (applied to P), there exists $t \in P^*$ such that $tx \in A \setminus I$. Then $ts^{2n}r = tx + ty \in P \cap (A \setminus I) + I \subseteq P$. Thus $s^{2n}r \in P$ since $t \in P^*$. Since $p \subseteq q$, $\alpha_p(s) \neq 0$, hence in $R(p)$ we have $\alpha_p(s)^{2n}\alpha_p(r)$, $\alpha_q(s)^{2n} \in \bar{P} \setminus \{0\}$. Thus $\alpha_p(r) \in \bar{P}$ and so $r \in P$. Hence $\alpha_q^{-1}(P(q)) \subseteq P \cup q$.

We have shown $\alpha_q^{-1}(P(q)) = P \cup q$. It follows that $-1 \notin P(q)$, hence q is a P-compatible prime ideal. Define $\theta : R(q)^*/P(q)^* \to R(p)^*/\bar{P}^*$ by

$$\theta\big(\alpha_q(a)\alpha_q(b)^{-2n}P(q)^*\big) = \alpha_p(a)\alpha_p(b)^{-2n}\bar{P}^*.$$

Note that since $p \subseteq q$, if $\alpha_q(b) \neq 0$, then $\alpha_p(b) \neq 0$. We have $\alpha_q(a)\alpha_q(b)^{-2n} \in P(q)^*$ iff $\alpha_q(a) \in P(q)^*$ iff $a \in P$ iff $\alpha_p(a) \in \bar{P}^*$ iff $\alpha_p(a)\alpha_p(b)^{-2n} \in \bar{P}^*$. Hence θ is well-defined and 1-1. Thus $R(q)^*/P(q)^*$ is cyclic, since $R(p)^*/\bar{P}^*$ is cyclic, and hence $P(q)$ is an order.

Given $i \in \hat{I}$, say $i = \alpha_q(x)\alpha_q(a)^{-2n}$. By 1.4, we can assume $a \in A \setminus I$, hence $\hat{v}(i) = v(x)$ and thus $x \in I$. Then $1 + i = \alpha_q(a^{2n} + x)\alpha_q(a)^{-2n}$ and $a^{2n} + x \in P$ by (i), thus $1 + i \in P(q)$. Therefore $\hat{A}$ is compatible with $P(q)$.

Suppose (ii) holds, then $p \subseteq q$ follows from $-1 \notin P(q)$. Given $a \in P \cap (A \setminus I)$ and $x \in I$, then $\alpha_q(a + x) \in P(q)$ since $\hat{A}$ is compatible with $P(q)$. Thus $a + x \in P \cup q$, which, together with $a \notin I$, $x \in I$, implies $a + x \in P$. Hence $v \sim P$. □

Proposition 2.3. *Suppose $Q \supseteq T$ is a preorder in R. A Manis valuation $v = (q, \hat{A})$ is compatible with Q iff q is a Q-compatible prime ideal and $\hat{A} \sim Q(q)$ in $R(q)$.*

Proof. Suppose $v \sim T$, then there is some $P \in O_Q \subseteq O_T$ such that $v \sim P$. By 2.2, q is a P-compatible prime ideal, hence it must be Q-compatible. Then $\hat{A} \sim P(q) \in O_{Q(q)}$ and thus $\hat{A} \sim Q(q)$.

Suppose q is a Q-compatible prime ideal and $\hat{A} \sim Q(q)$, then there is some $\bar{P} \in O_{Q(q)}$ such that $\hat{A} \sim \bar{P}$. Let $P = {\alpha_q}^{-1}(\bar{P}) \in O_Q$. Then $P \cap -P = q$ and for any $x \in P \cap (A \setminus I)$ and $y \in I$ we have $\alpha_q(x + y) \in \bar{P}$, hence $x + y \in P$. Thus $v \sim P$. □

Definition. Given a valuation v with valuation ring A and prime ideal I, let $\tilde{A}$ denote the domain A/I and K_v the quotient field of $\tilde{A}$. We define the *pushdown of T along v* to be the image of $T \cap A$ in $\tilde{A}$, denoted $\tilde{T}$.

Lemma 2.4. *Suppose v is a valuation with valuation ring A and prime ideal I which is compatible with T. Then $\tilde{T}$ is a preorder in $\tilde{A}$.*

Proof. Suppose $-1 \in \tilde{T}$, then there exist $t \in T \cap A$ and $x \in I$ such that $-1 = t + x$. Pick $P \in O_T$ such that $v \sim P$, then we have $t \in T \cap (A \setminus I) \subseteq P \cap (A \setminus I)$, hence $-1 = t + x \in P$, a contradiction. Thus $-1 \notin \tilde{T}$ and it follows that $\tilde{T}$ is a preorder in $\tilde{A}$. $\square$

Lemma 2.5. *Suppose $v = (q, \hat{A})$ is a Manis valuation with valuation ring A and prime ideal I which is fully compatible with T. Then*

(i) $R \setminus q = R^*$
(ii) $1 + I \subseteq T^*$

Proof. (i): $R^* \subseteq R \setminus q$ is clear. Given $r \in R \setminus q$, since $p \subseteq q$ for all $P = (p, \bar{P}) \in O_T$, we have $r \notin P \cap -P$ for all $P \in O_T$. Thus $r \in R^*$ by 1.2,(i).
(ii): Given $x \in I$, then for any $P \in O_T$, we have $1 + x \in P$ since $(A, I) \sim P$. Also $1 + x \notin q$ since $q \subseteq I$ and thus $1 + x \in R^*$ by (i). Hence $1 + x \in \bigcap_{P \in O_T} P^* = T^*$ by 1.2,(ii). $\square$

Proposition 2.6. *Suppose $v = (q, \hat{A})$ is a Manis valuation which is fully compatible with T. Then the map $\theta : R^*/T^* \to R(q)^*/T(q)^*$ given by $\theta(rT^*) = \alpha_q(r)T(q)^*$ is an isomorphism.*

Proof. By 2.3, $T(q)$ is a preorder in $R(q)$. Given $r \in R^*$ with $\alpha_q(r) \in T(q)^*$, then by 2.2 for each $P \in O_T$ there is some $s \in R \setminus q$ such that $s^{2n} r \in P$. By 2.5,(i), $s \in R^*$, hence $r \in P$. Thus $r \in \bigcap_{P \in O_T} P^* = T^*$ by 1.2,(ii). Hence θ is 1-1. Given $\alpha_q(r)\alpha_q(s)^{-2n} \in R(q)^*$, then $r \notin q$ and hence $r \in R^*$ by 2.5,(i). Then $\theta(rT^*) = \alpha_q(r)T(q)^* = \alpha_q(r)\alpha_q(s)^{-2n}T(q)^*$. Thus θ is onto and therefore an isomorphism. $\square$

§3. Dependency Classes

For the rest of this paper, we assume that all valuations are Manis valuations. Thus we replace v_P by the Manis valuation with valuation pair $(A(P), I(P))$, which exists by 1.1. By 2.2, we still have $v_P \sim P$.

As in the field case (see [BR, §5]), we can define an equivalence relation on O_T using the valuations v_P. This allows us to "break up" T into pieces which are fully compatible with a valuation.

Definition.

(i) Suppose v_1 and v_2 are nontrivial valuations in R. For $i = 1, 2$, let Γ_i denote the value group, A_i the valuation ring, and I_i the prime ideal of v_i. Following [G], we say v_2 is *coarser* than v_1, denoted $v_2 \leq v_1$, if there is an order homorphism $f : \Gamma_1 \to \Gamma_2$ such that $v_2 = f \circ v_1$, iff (by [G, Proposition 4]) $A_1 \subseteq A_2$ and $I_2 \subseteq I_1$.
(ii) Nontrivial valuations v_1 and v_2 are *dependent valuations* if there exists a nontrivial valuation coarser than both. Otherwise, they are *independent*.

Proposition 3.1. *Suppose $v_1 = (q_1, \hat{A}_1)$ and $v_2 = (q_2, \hat{A}_2)$ are nontrivial valuations. Then $v_2 \leq v_1$ iff $q_1 = q_2$ and $\hat{A}_1 \subseteq \hat{A}_2$ in $R(q_1)$.*

Proof. Assume $v_2 \leq v_1$, say $f : \Gamma_1 \to \Gamma_2$ with $v_2 = f \circ v_1$. Since $I_2 \subseteq I_1$, it follows from 1.1,(ii) that $q_2 \subseteq q_1$. Given $x \in q_1$, suppose $x \notin q_2$, then by 1.4 there is some $y \in R$ with $yx \notin I_2$. Given any $r \in R$, since q_1 is an ideal, $ryx \in q_1 \subseteq A_1 \subseteq A_2$. We have $r(yx) \in A_2$ and $yx \notin I_2$ which implies $r \in A_2$. This shows $A_2 = R$, but we assumed not. Hence $q_1 \subseteq q_2$ and thus $q_1 = q_2$. Let $q := q_1 = q_2$. Then $v_2 = f \circ v_1$ implies $\hat{v}_2 \circ \alpha_q = f \circ \hat{v}_1 \circ \alpha_q$, hence $\hat{A}_1 \subseteq \hat{A}_2$.

If $q_1 = q_2$ and $\hat{A}_1 \subseteq \hat{A}_2$, then $\hat{I}_2 \subseteq \hat{I}_1$. It follows easily that $A_1 \subseteq A_2$ and $I_2 \subseteq I_1$. $\square$

Corollary 3.2. *Suppose v is a nontrivial valuation on R. Given valuations v_1 and v_2 such that $v_1 \leq v$ and $v_2 \leq v$, then $v_1 \leq v_2$ or $v_2 \leq v_1$.*

Proof. The corollary follows from 3.1 and the fact that in a field valuation rings containing a given valuation ring are linearly ordered by inclusion. $\square$

Definition. We define the relation of *dependency*, denoted $\sim$, on O_T as follows: Given $P, Q \in O_T$. If P is archimedean, then $P \sim Q$ iff $Q = P$. If P is nonarchimedean, then $P \sim Q$ if Q is nonarchimedean and v_P and v_Q are dependent valuations.

Lemma 3.3. *Given nonarchimedean $P, Q \in O_T$, then $P \sim Q$ iff $A(P) \cdot A(Q) \neq R$. In this case, let $A := A(P) \cdot A(Q)$, then there is a valuation v with valuation ring A which is coarser than both v_P and v_Q.*

Proof. Suppose $P \sim Q$, then by definition there exists a nontrivial valuation v with valuation ring A such that $A(P) \subseteq A$ and $A(Q) \subseteq A$. Hence $A(P) \cdot A(Q) \subseteq A \neq R$.

Suppose $A := A(P) \cdot A(Q) \neq R$. By 1.7 $A(P)$ and $A(Q)$ are Prüfer rings, hence, by [G, Proposition 13], A is the valuation ring of a (Manis) valuation v with $v \leq v_P$ and $v \leq v_Q$. Therefore, $P \sim Q$. $\square$

Lemma 3.4. *Suppose $P_1, \ldots, P_k \in O_T$ are nonarchimedean such that $P_1 \sim P_i$ for all i. Then there exists a nontrivial valuation on R which is coarser than each v_{P_i}.*

Proof. For $2 \leq i \leq k$, set $A_i = A(P_1) \cdot A(P_i)$. Then, by 3.3, for each i, $A_i \neq R$ and there exists a valuation v_i which is coarser than v_1 and v_{P_i}. Hence, by 3.2 and induction, there is some k such that v_k is coarser than each v_{P_i}. $\square$

Corollary 3.5. *The relation of dependency is an equivalence relation on O_T.*

Definition. For $P \in O_T$, let $[P]$ denote the equivalence class of P, called the *dependency class of P*.

Proposition 3.6. *Suppose there are only finitely many valuations among $\{v_P \mid P \in O_T\}$ and $P \in O_T$ is nonarchimedean. Let [P] denote the dependency class of P and set $S = \bigcap_{Q \in [P]} Q$. Then*

(i) $O_S = [P]$.
(ii) *There exists a valuation v such that $v \leq v_Q$ for each $Q \in [P]$.*

Furthermore, if $P \not\sim Q$ and v and w are the valuations in (ii) corresponding to $[P]$ and $[Q]$, then v and w are independent.

Proof. (i): Since $\{v_Q \mid Q \in [P]\}$ is finite, there exist $Q_1, \ldots, Q_m \in [P]$ such that $\{v_Q \mid Q \in [P]\} = \{v_{Q_j}\}_{j=1}^m$. We have $[P] \subseteq O_S$ by definition of S. Given $P' \in O_S$, then $P' \in O_T$ and $S \subseteq P'$. Since $\bigcap_{Q\in[P]} Q \subseteq P'$, we have $\bigcap_{Q\in[P]} I(Q) \subseteq I(P')$. Hence $\bigcap_{j=1}^m I(Q_j) = \bigcap_{Q\in[P]} I(Q) \subseteq I(P')$. We want to show $Q_1 \sim P'$. By 3.4 there exists a nontrivial valuation v with valuation ring A such that $A(Q_i) \subseteq A$ for all i. Let $p = I(P') \cap A(T)$ and, for each i, let $p_i = I(Q_i) \cap A(T)$. Then $A(Q_i) = A(T)_{p_i}$ for each i and $A(P') = A(T)_p$ by 1.6. Thus, since $\bigcap p_i \subseteq p$, we have $A(P') \subseteq A(Q_1) \cup \cdots \cup A(Q_m)$ by 1.5,(ii). Hence $A(Q_1) \cdot A(P') \subseteq A \neq R$ and thus $Q_1 \sim P'$ by 3.3. Hence $P' \in [P]$ and therefore $O_S = [P]$.
(ii) follows from 3.4.

Suppose $P \not\sim Q$ and v and w are the valuations of (ii) corresponding to $[P]$ and $[Q]$. If v and w are dependent, then there exists a nontrivial valuation coarser than both, hence coarser than v_P and v_Q. But this implies $P \sim Q$, a contradiction. Thus v and w are independent valuations. □

Definition. Following [Ma2], we define a *V-topology* on R to be a triple (F, α, τ) where F is a field, $\alpha : R \to F$ a ring homomorphism such that F is the field of fractions of $\alpha(R)$, and τ a V-topology on F. For details, see [Ma2]. A V-topology (F, α, τ) is *archimedean* if τ is archimedean on F. It is *coarse* if $\alpha(R)$ is τ-unbounded.

An approximation theorem for V-topologies on rings is proven in [Ma2]. As in the field and skew field cases we can apply this to the valuations we have constructed which correspond to our dependency classes.

We assume that there are only finitely many valuations among $\{v_P \mid P \in O_T\}$. Then there are only finitely many dependency classes, say $[P_1], \ldots, [P_k]$. For each i, there is a V-topology (R_i, α_i, τ_i) defined as follows: If P_i is nonarchimedean, we have a valuation $v_i = (p_i, \hat{A}_i)$ corresponding to $[P_i]$ defined in 3.6. In this case, set $R_i = R(p_i)$, $\alpha_i = \alpha_{p_i}$, and let τ_i be the V-topology on R_i induced by $\hat{A}_i$. If P_i is archimedean, set $p_i = P_i \cap -P_i$, $R_i = R(p_i)$, $\alpha_i = \alpha_{p_i}$ and let τ_i be the (archimedean) V-topology induced by $\bar{P}_i$. By remarks in [Ma2], each of these V-topologies is coarse. Also, they are all distinct: In the nonarchimedean case this follows from the independence of the v_i's. In the archimedean case this follows from the fact the if P_1 and P_2 are archimedean orders on a field F, then the V-topologies induced by P_1 and P_2 are equal iff $P_1 = P_2$, see [BR, §4]. Finally, note that archimedean and nonarchimedean V-topologies are never equal.

Theorem 3.7. *Suppose there are only finitely many valuations among $\{v_P \mid P \in O_T\}$ and only finitely many archimedean orderings in O_T. Let $[P_1], \ldots, [P_k]$ be the dependency classes of O_T and for each i set $T_i = \bigcap_{P\in[P_i]} P$. Also, let (R_i, α_i, τ_i) be the V-topology defined above and let $S_i = T_i(p_i)$, by 2.3 a preorder in R_i. Then the canonical map*

$$\theta : R^*/T^* \to {R_1}^*/{S_1}^* \times \cdots \times R_k^*/{S_k}^*$$

is an isomorphism.

Proof. By 1.2,(ii) and 3.6,(i), we have

$$(*)\qquad T^* = \bigcap_{P \in O_T} P^* = \bigcap_{i=1}^{k} \Big(\bigcap_{P \in [P_i]} P^* \Big).$$

Given $r \in R^*$ such that $\alpha_i(r) \in S_i^*$ for all i. Then for each i we have $\alpha_i(r) \in \bar{P}$ for all $\bar{P} \in [P_i]$. Hence $r \in P$ for all $P \in [P_i]$ and all i and thus $r \in T^*$ by $(*)$. Hence θ is 1-1.

By the remarks above on the V-topologies (R_i, α_i, τ_i), we can apply [Ma2, 2.3] to our situation if we show that for each i, S_i^* is a τ_i-neighborhood of 1. Given $\bar{P} \in O_{S_i}$, let $P = \alpha_i^{-1}(\bar{P})$, i.e., $P = (p_i, \bar{P}) \in O_T$. By construction and 3.6, $P \in O_{T_i}$, hence $v_i \sim P$. Thus, by 2.2, $\hat{v}_i \sim \bar{P}$. Hence we have shown that $\hat{v}_i \sim_f S_i$ and thus $1 + \hat{I}_i \subseteq S_i^*$. It follows that S_i^* is a τ_i-neighborhood of 1. Thus, by [Ma2, 2.3], given $y = (r_1 T_1{}^*, \dots, r_k T_k{}^*) \in \prod R_i{}^*/S_i{}^*$, there is some $r \in R \setminus \cup \ker \alpha_i$ such that $\alpha_i(r) S_i^* = r_i S_i^*$ for all i. By 1.2,(i), $r \in R^*$, hence θ is onto. Therefore θ is an isomorphism. $\square$

§4. T-Forms and the Reduced Witt Ring

We define signatures, T-forms and the reduced Witt ring of T as in [MW].

For any abelian group G, let $G^\vee$ denote $\mathrm{Hom}(G, \mu)$, where μ denotes the complex roots of unity.

If F is a field and Q a preorder in F then a Q-signature is any $\chi \in (F^*)^\vee$ such that $Q^* \subseteq \ker \chi$ and $\ker \chi$ is additively closed. Note that if χ is a Q-signature then $\ker \chi \cup \{0\} \in O_Q$. A *$T$-signature in R* is a character $\sigma \in (R^*)^\vee$ such that there exists a T-compatible prime ideal p and a $T(p)$-signature χ with $\sigma = \chi \circ \alpha_p \mid_{R^*}$, where $\mid_{R^*}$ denotes restriction to R^*. In this case we have $P = {\alpha_p}^{-1}(\ker \chi \cup \{0\}) \in O_T$ and $P^* = \ker \sigma$. Conversely, given $P = (p, \bar{P}) \in O_T$ then there is a $T(p)$-signature χ with $\bar{P}^* = \ker \chi$. Hence there is a T-signature σ, defined by $\sigma = \chi \circ \alpha_p \mid_{R^*}$, such that $\ker \sigma = P^*$. We write X_T to denote the set of T-signatures.

An r-dimensional form over T is an r-tuple $\rho = \langle a_1, \dots, a_r \rangle$, where $a_i \in R^*$. The sum and product of forms are defined in the usual way: For ρ as above and $\tau = \langle b_1, \dots, b_k \rangle$,

$$\rho \oplus \tau = \langle a_1, \dots, a_r, b_1, \dots, b_k \rangle$$

and

$$\rho \otimes \tau = \langle a_1 b_1, \dots, a_1 b_k, \dots, a_r b_1, \dots, a_r b_k \rangle.$$

If $\rho = \langle a_1, \dots a_r \rangle$ and σ is a T-signature, we define $\sigma(\rho) = \Sigma_{i=1}^r \sigma(a_i)$. Two forms ρ and τ are *T-equivalent*, denoted $\rho \sim \tau$, if $\sigma(\rho) = \sigma(\tau)$ for all T-signatures σ. If in addition ρ and τ have the same dimension, they are *T-isometric*, denoted $\rho \cong \tau$. The *Witt ring of* T, denoted $W_T(R)$, consists of T-equivalence classes of forms with operations induced by $\oplus$ and $\otimes$.

Definition.

(i) We say a form $\rho = \langle a_1, \dots a_r\rangle$ is *isotropic* if there exist $t_1, \dots t_r \in T^* \cup \{0\}$, not all 0, such that $a_1t_1 + \cdots + a_rt_r = 0$. Otherwise, ρ is *anisotropic*.

(ii) The *represented set of* ρ, denoted $D_T(\rho)$, is $Ta_1 + \cdots + Ta_r$.

Lemma 4.1. *Suppose $u \in D_T(\rho)^*$, then there exist $t_1, \dots, t_r \in T^*$ such that $u = a_1t_1 + \cdots + a_rt_r$.*

Proof. We have $u = \Sigma a_i s_i$ where $s_i \in T$. By 1.3,(iii) there exist $s, t \in T^*$ such that $s - t = u^{-1}(a_1 + \cdots + a_r)$. Then $su = tu + a_1 + \cdots + a_r = \Sigma a_i(1 + ts_i)$. Since $1 + ts_i \in T^*$, we are done with $t_i = s^{-1}(1 + ts_i)$. $\square$

Corollary 4.2. *(i) Suppose $\rho = \langle a_1, \dots, a_r\rangle$ is isotropic. Then there exist $t_1, \dots, t_r \in T^*$ such that $t_1a_1 + \cdots + t_ra_r = 0$.*
(ii) Suppose $\rho = \langle 1, a_2, \dots, a_r\rangle$ and $-1 \in D_T(\rho)$. Then ρ is isotropic.

Proof. (i): Wlog we can assume $s_1a_1 + \cdots + s_ra_r = 0$, where $s_1 \in T^*$. Then apply 4.1 to the form $\langle a_2, \dots, a_r\rangle$ with $u = -s_1a_1$.
(ii): By 4.1 there exist $t_1, \dots, t_r \in T^*$ such that $-1 = t_1 + a_2t_2 + \cdots + a_rt_r$, hence $(1 + t_1) + a_2t_2 + \cdots + a_rt_2 = 0$. $\square$

Proposition 4.3. *Suppose $\rho = \langle a_1, \dots, a_r\rangle$ is a form, and $b \in R^*$. Then $b \in D_T(\rho)^*$ iff $\alpha_p(b) \in D_{T(p)}(\alpha_p(\rho))$ for all T-compatible primes p, where $\alpha_p(\rho) = \langle \alpha_p(a_1), \dots, \alpha_p(a_r)\rangle$.*

Proof. Suppose $b \in D_T(\rho)^*$, then given a T-compatible prime p we have $b \notin p$. Hence $\alpha_p(b) \in D_{T(p)}(\alpha_p(\rho))$.

Now suppose $\alpha_p(b) \in D_{T(p)}(\alpha_p(\rho))$ for all T-compatible primes p. Since $b \in D_T(\rho)^*$ iff $-1 \in D_T(-b^{-1}\rho)^*$, we can assume $b = -1$. Suppose first that $-1 \in D_T(\langle 1\rangle \oplus \rho)$, then $-1 = t + c$, where $t \in T$ and $c \in D_T(\rho)$. Hence $-1 = (1+t)^{-1}c \in D_T(\rho)$. Thus it is enough to show $-1 \in D_T(\langle 1\rangle \oplus \rho)$. If not, then $D_T(\langle 1\rangle \oplus \rho)$ is a T-module, hence by [MW, 1.6] there exists a $D_T(\langle 1\rangle \oplus \rho)$-compatible prime p. But then we have $-1 \notin D_{T(p)}(\alpha_p(\rho))$, a contradiction. Therefore $-1 \in D_T(\rho)$. $\square$

Theorem 4.4. *Suppose ρ and τ are T-forms such that $\rho \sim \tau$ and dim $\rho <$ dim τ. Then τ is isotropic.*

Proof. Suppose $\tau = \langle b_1, \dots, b_k\rangle$. Then ${b_1}^{-1}\tau$ isotropic implies τ isotropic, hence wlog we can assume $b_1 = 1$. For each T-compatible prime p, dim $\alpha_p(\rho) <$ dim $\alpha_p(\tau)$, hence $\alpha_p(\tau)$ is $T(p)$-isotropic by [BR, 4.9]. This implies $-1 \in D_{T(p)}(\alpha_p(\tau))$ for all T-compatible primes p, thus $-1 \in D_T(\tau)$ by 4.3. Hence τ is isotropic by 4.2,(i). $\square$

Corollary 4.5. *$\rho \cong \tau$ implies $D_T(\rho) = D_T(\tau)$.*

Proof. Suppose $\rho = \langle a_1, \dots, a_r\rangle$ and $\tau = \langle b_1, \dots, b_r\rangle$. Let $f = \tau \oplus \langle -a_r\rangle$ and $g = \langle a_1, \dots, a_{r-1}\rangle$. Then $f \sim g$, hence f is isotropic by 4.4. Then by 4.2(ii), there exist $t, t_1, \dots, t_r \in T^*$ such that $\sum t_ib_i - ta_r = 0$. Since $t \in T^*$ we have $\alpha_p(t) \neq 0$ for all T-compatible primes p. Hence $\alpha_p(a_r) \in D_{T(p)}(\alpha_p(\tau))$ for all T-compatible primes p and thus $a_r \in D_T(\tau)$ by 4.3.

A similar argument shows $a_i \in D_T(\tau)$ for all i, hence $D_T(\rho) \subseteq D_T(\tau)$. The same proof shows $D_T(\rho) \subseteq D_T(\tau)$ and we are done. $\square$

Remark. Theorem 4.4 and Corollary 4.5 are proven for rings with many units in [MW, 3.5].

Spaces of Signatures Spaces of signatures (hereafter SOS) provide an abstract setting for studying the reduced theory of higher level forms over fields. For details and terminology see [Mu] and [MM]. The advantage of this abstract approach is that once we prove we have a SOS then much of the theory for fields generalizes immediately to our setting. In [MW] it is shown that a preordered ring with many units gives rise to a SOS. We cannot prove this in general in our setting, but using the results of §3 we prove it for preorders T such that there are only finitely many valuations among $\{v_P \mid P \in O_T\}$.

We generalize some ideas from the theory of SOS's:

Definition. A *signature pair* is a pair (X, G) where G is an abelian group of finite even exponent and X is a subset of $G^\vee$. Two signature pairs (X_1, G_1) and (X_2, G_2) are *equivalent* if there is an isomorphism $\alpha : G_1 \to G_2$ such that $\alpha^\vee(X_2) = X_1$, where $\alpha^\vee$ is the dual isomorphism.

Given signature pairs $\{(X_i, G_i)\}_{i=1}^k$, let $G = G_1 \times \cdots \times G_k$ and let $X = X_1 \dot\cup \cdots \dot\cup X_k$, where X_i is identified with its image in $G^\vee$, and $\dot\cup$ denotes disjoint union. Then (X, G) is a signature pair, called the *direct sum* of the (X_i, G_i)'s. We write $(X, G) = \bigoplus_{i=1}^k (X_i, G_i)$.

Remark.

(i) A SOS is a signature pair which satisfies certain axioms, see [Mu], [MM].
(ii) Given $\sigma \in X_T$, we identify σ with its image in $(R^*/T^*)^\vee$ and thus $(X_T, R^*/T^*)$ is a signature pair. If R is a field then $(X_T, R^*/T^*)$ is a SOS by [Mu, 1.10].
(iii) If a signature pair is equivalent to a SOS, then it is also a SOS.

Proposition 4.6. *The direct sum of finitely many SOS's is a SOS.*

Proof. [Mu, 2.6]. $\square$

Proposition 4.7. *Suppose $v = (q, \hat{A})$ is a valuation in R which is fully compatible with T. Then $(X_T, R^*/T^*)$ is equivalent to $(X_{T(q)}, R(q)^*/T(q)^*)$. In particular, $(X_T, R^*/T^*)$ is a SOS.*

Proof. The mapping $\theta : R^*/T^* \to R(q)^*/T(q)^*$ given by $\theta(rT^*) = \alpha_q(r)T(q)^*$ is an isomorphism by 2.6. Given $\sigma \in X_T$, we can define $\bar\sigma : R(q)^*/T(q)^* \to \mu$ by $\bar\sigma := \sigma \circ \theta^{-1}$, then $\bar\sigma \in (R(q)^*/T(q)^*)^\vee$. Since $\sigma \in X_T$ there is some $P = (p, \bar{P}) \in O_T$ such that $\sigma = \chi \circ \alpha_p \mid_{R^*}$ where χ is a $T(p)$-signature with $\ker \chi = \bar{P}$.

Given $x_1, x_2 \in \ker \bar\sigma$, then there exists $r_1, r_2 \in R^*$ such that $x_i T(q)^* = \alpha_q(r_i)T(q)^*$. Since $x_i \in \ker \bar\sigma$, we have $r_i \in \ker \sigma$, hence $r_1, r_2 \in P^*$. Then $r_1 + r_2 \in P$, thus $\alpha_q(r_1 + r_2) \in \alpha_q^{-1}(P(q)) = P \cup q$ by 2.2. Suppose $r_1 + r_2 \in q$, then $0 \neq -\alpha_q(r_1) = \alpha_q(r_2) \in P(q)$ and also $\alpha_q(r_1) \in P(q)$, a contradiction. Thus we must have $r_1 + r_2 \notin q$ and hence $r_1 + r_2 \in R^*$ by 2.5. Then $r_1 + r_2 \in P^*$

and so $\bar{\sigma}(x_1 + x_2) = \sigma(r_1 + r_2) = 1$ and thus $\ker \bar{\sigma}$ is additively closed. Hence $\bar{\sigma}$ is a $T(q)$-signature and clearly $\theta^\vee(\bar{\sigma}) = \sigma$. Thus $\theta^\vee(X_{T(q)}) \supseteq X_T$. It is clear that $\theta^\vee(X_{T(q)}) \subseteq X_T$ and therefore $(X_{T(q)}, R(q)^*/T(q)^*)$ and $(X_T, R^*/T^*)$ are equivalent. □

We would like to combine 4.7 and 3.7 to conclude that in the situation of 3.7 $(X_T, R^*/T^*)$ is a SOS. We cannot apply 4.7 directly, however, since it only applies to T, not to the T_i of 3.7. (The point is that we do not necessarily have $1 + T_i \subseteq R^*$.)

Theorem 4.8. *Suppose there are only finitely many valuations among $\{v_P \mid P \in O_T\}$ and only finitely many archimedean orders on R. Then $(X_T, R^*/T^*)$ is a SOS.*

Proof. Let T_i, p_i, R_i, and S_i be as in Theorem 3.7 and let $v_i = (p_i, \hat{A}_i)$ be the valuations defined in 3.6. For each i, define $\tilde{R}_i := (1 + T_i)^{-1}R$ and $Q_i := (1 + T_i)^{-1}T_i$, a preorder in $\tilde{R}_i$. Fix i and pick $P \in X_{T_i}$. Then $v_i \sim P$, hence by 2.2, $P(p_i)$ is an order in R_i. It follows that $(1 + T_i) \cap p_i = \emptyset$, hence we can define $\tilde{p}_i := (1 + T_i)^{-1}p_i$, a prime ideal in $\tilde{R}_i$ such that $\tilde{R}_i(\tilde{p}_i) = R_i$. Now we define, for each i, a valuation $w_i := (\tilde{p}_i, \hat{A}_i)$. We want to show that $w_i \sim_f Q_i$. There is a 1-1 correspondence between O_{T_i} and O_{Q_i} given by $P \leftrightarrow (1 + T_i)^{-1}P$. Then given $\tilde{P} = (1 + T_i)^{-1}P \in O_{Q_i}$, it follows from the definitions that $\tilde{P}(\tilde{p}_i) = P(p_i)$ (in R_i). Since $v_i \sim P$, by 2.2 we have $P(p_i)$ is an order in R_i, thus, applying 2.2 again, $w_i \sim \tilde{P}$. Hence $w_i \sim_f Q_i$. Let $\tilde{\alpha}_i$ be the canonical map $\tilde{R}_i \to R_i$.

Let $\theta : R/T^* \to \tilde{R}_1^*/Q_1^* \times \cdots \times \tilde{R}_k^*/Q_k^*$ be the canonical map. By 3.7 the canonical map $R^*/T^* \to R_1^*/S_1^* \times \cdots \times R_k^*/S_k^*$ is an isomorphism and by 2.6, for each i, the map $\tilde{R}_i^*/Q_i^* \to R_i^*/S_i^*$ given by $xQ_i^* \mapsto \tilde{\alpha}_i(x)S_i^*$ is an isomorphism. It follows easily that θ is an isomorphism.

Given $\sigma \in X_T$, there is some $P = (p, \bar{P}) \in O_T$ and some $\chi_{\bar{P}} \in X_{T(p)}$ with $\ker \chi_{\bar{P}} = \bar{P}$ such that $\sigma = \chi_{\bar{P}} \circ \alpha_p|_{R^*}$. Then $P \in O_{T_i}$ for some i. Now define $\tilde{P} := (1 + T_i)^{-1}P$ and $\tilde{p} := (1 + T_i)^{-1}p$. Since $(1 + T_i) \cap p = \emptyset$, $\tilde{p}$ is a prime ideal in $\tilde{R}_i$ and it follows from the definitions that $\tilde{R}_i(\tilde{p}) = R(p)$ and $\tilde{P}(\tilde{p}) = P(p) = \bar{P}$. Thus we have $\tilde{P} = (\tilde{p}, \bar{P}) \in O_{Q_i}$ and we can define $\tilde{\sigma} := \chi_{\bar{P}} \circ \alpha_{\tilde{p}}|_{\tilde{R}_i^*} \in X_{Q_i}$. Then $\tilde{\sigma} \circ \theta = \sigma$. Hence $X_T \subseteq \theta^\vee(X_{Q_i} \dot{\cup} \cdots \dot{\cup} X_{Q_k})$. The reverse inclusion is clear and thus $(X_T, R^*/T^*)$ is equivalent to $\bigoplus_{i=1}^k (X_{Q_i}, \tilde{R}_i^*/{Q_i}^*)$, which is a SOS by 4.6 and 4.7. Therefore $(X_T, R^*/T^*)$ is a SOS. □

Corollary 4.9. *Suppose T satisfies the conditions of 4.8. Then there exists a field K and a preorder $Q \subseteq K$ such that $(X_T, R^*/T^*)$ and $(X_Q, K^*/Q^*)$ are equivalent SOS's. In particular, $W_T(R)$ is isomorphic to $W_Q(K)$.*

Proof. This follows from 4.8 and [P, 2.8]. □

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