A quantitative Pólya's Theorem with corner zeros

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Abstract

Pólya's Theorem says that if p is a homogeneous polynomial in n variables which is positive on the standard n-simplex, and F is the sum of the variables, then for a sufficiently large exponent N, $F^N * p$ has positive coefficients. Pólya's Theorem has had many applications in both pure and applied mathematics; for example it provides a certificate for the positivity of p on the simplex. The authors have previously given an explicit bound on N, determined by the data of p; namely, the degree, the size of the coefficients and the minimum value of p on the simplex. In this paper, we extend this quantitative Pólya's Theorem to non-negative polynomials which are allowed to have simple zeros at the corners of the simplex.

1 Introduction

Fix a positive integer n and let $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_n]$. We use the following polynomial notation: Given $\alpha \in \mathbb{N}^n$, say $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote by X^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. For $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, we similarly write u^α for $u_1^{\alpha_1} \dots u_n^{\alpha_n} \in \mathbb{R}$. We denote the standard n-simplex $\{(x_1, \dots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$ by Δ_n.

Pólya's Theorem says that if a form (homogeneous polynomial) $p \in \mathbb{R}[X]$ is positive on Δ_n, then for sufficiently large N all the coefficients of

$$(x_1 + \cdots + x_n)^N * p(x_1, \dots, x_n)$$

are positive. This elegant and beautiful result has many applications, both in pure and applied mathematics.

Pólya's theorem appeared in 1928 [10] (in German) and is also in *Inequalities* by Hardy, Littlewood, and Pólya [8] (in English). It is interesting to note that they realized the algorithmic nature of the result; in the book they write

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> The theorem gives a systematic process for deciding whether a given form F is strictly positive for positive x. We multiply repeatedly by $\sum x_i$ and, if the form is positive, we shall sooner or later obtain a form with positive coefficients.

Pólya's proof is elementary: for p positive on Δ_n of degree d, he constructs a sequence of real polynomials p_ϵ in n variables such that p_ϵ converges uniformly to p on Δ_n as $\epsilon \to 0$ and so that for $|\beta| = N + d$, the coefficient of X^β in $(\sum x_i)^N * p$ is a positive multiple of $p_\epsilon(\frac{\beta}{N+d})$ for $\epsilon = \frac{1}{N+d}$. Then, since $p_\epsilon > 0$ for sufficiently small ϵ, it follows that for sufficiently large N, each coefficient of X^β is positive.

By analyzing Pólya's proof, the authors in [11] were able to give an explicit bound on the N needed in the theorem. If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Suppose $f \in \mathbb{R}[X]$ is homogeneous of degree d, then write

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$.

Theorem 1. *Suppose that $f \in \mathbb{R}[X]$ is given as above. If*

$$N > \frac{d(d-1)}{2} \frac{L}{\lambda} - d,$$

then $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ has positive coefficients.

We describe a few applications of Pólya's Theorem and this bound. In 1940, Habicht [5] used Pólya's Theorem to give a direct proof of a special case of Hilbert's 17th Problem; namely, he used it to prove that a positive definite form is a sum of squares of rational functions. More recently, M. Schweighofer [12] used Pólya's Theorem to give an algorithmic proof of Schmüdgen's Positivstellensatz, which says that if the basic closed semialgebraic set $K = \{g_1 \geq 0, \ldots, g_k \geq 0\}$ is compact and $f > 0$ on K, then f is in the preorder generated by the g_i's. This can be used to give an algorithm for optimization of polynomials on compact semialgebraic sets; see [13] for details. Using the bound for Pólya's Theorem, Schweighofer obtained complexity bounds for Schmüdgen's Positivstellensatz [14].

Pólya's Theorem has been used in the study of copositive programming. Let $\mathbb{S}^n$ denote the $n \times n$ symmetric matrices over $\mathbb{R}$ and define the copositive cone

$$C_n = \{M \in \mathbb{S}^n \mid Y^T M Y \geq 0 \text{ for all } Y \in \mathbb{R}^n_+\}.$$

Copositive programming is optimization over C_n. By Pólya's Theorem, the truncated cones

$$C_n^r := \{M \in \mathbb{S}^n \mid \left(\textstyle\sum_i x_i\right)^r * X^T M X\}$$

have non-negative coefficients and will converge to C_n and using linear programming, membership in C_n^r can be determined numerically. De Klerk and Pasechnik [4] use this fact, along with the bound for Pólya's Theorem, to give results on approximating the stability number of a graph.

Motzkin and Strauss [9] partially generalized the theorem to power sequences in several variables and Catlin and D'Angelo [1, 2] generalized the theorem to polynomials in several complex variables. Handelman [6, 7] has studied a related question, namely, for which pairs (q, f) of polynomials does there exist $N \in \mathbb{N}$ so that $q^N * f$ has nonnegative coefficients? (See also de Angelis and Tuncel [3].) Pólya's Theorem and the generalization described in this paper (without the bound) can be deduced from Handelman's work. It also follows (again, without the bound) from recent work of Schweighofer [15].

In this paper we discuss an extension of Pólya's Theorem to the case where the form p has zeros on Δ_n. By methods similar to those used in [11] to prove Theorem 1, we give a bound on the N needed in the case where p is positive on Δ_n except for possible zeros at the "corners" of the simplex; the bound is in terms of information about the coefficients of p and the minimum of p on Δ_n away from the zeros.

Remark 1. By "positive coefficients" we mean that every coefficient that is non-zero is positive. If a form p of degree d is strictly positive on Δ_n, then Pólya's Theorem shows that there is N so that every monomial in $(x_1 + \cdots + x_n)^N p$ has a positive coefficient. However this will not be possible if p has zeros on Δ_n, e.g., if $p(1, 0, \ldots, 0) = 0$ then the coefficient of x_1^{N+d} in $(x_1 + \cdots + x_n)^N p$ will always be zero.

2 Pólya's Theorem for forms non-negative on the simplex

Let $P_{n,d}(\Delta_n)$ denote the closed cone of degree d forms in n variables which are non-negative on Δ_n and let $Po(n, d)$ be the degree d forms in n variables which satisfy Pólya's Theorem; i.e., $p \in Po(n, d)$ if there is some N such that every monomial in $(x_1 + \cdots + x_n)^N p$ has a positive coefficient. For ease of exposition, denote the form $x_1 + \cdots + x_n$ by F.

Given $p \in \mathbb{R}[X]$ of degree d, write

$$p = \sum_{|\alpha| \leq d} a_\alpha X^\alpha;$$

we let supp(p) denote $\{\alpha \mid a_\alpha \neq 0\}$. We write $e_1, \ldots, e_n$ for the vertices of Δ_n, i.e., $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$.

Suppose $p \in P_{n,d}(\Delta_n)$ and p has a zero on the interior of Δ_n, say $p(u) = 0$ for $u = (u_1, \ldots, u_n) \in \Delta_n$ with $u_i > 0$ for all i. Then it is not too hard to see that $p \notin Po(n, d)$. For every $N \in \mathbb{N}$, if $F^N p = \sum c_\alpha X^\alpha$, then $0 = \sum_\alpha c_\alpha u^\alpha$

and $u^\alpha > 0$ for each α, hence at least one c_α must be negative. On the other hand, $p = x_1 \cdots x_n$ is trivially in $Po(n,d)$ and has $p(u) = 0$ for every u on the boundary of Δ_n. Also, if p vanishes at an interior point of a face of Δ_n, then p vanishes everywhere on that face and has a monomial factor, hence it makes sense to restrict our attention to zeros on faces of co-dimension at least 2.

We note also that it is possible to have $p \in Po(n,d)$, but $p \notin Po(n+1,d)$ when p is considered as a form in $n+1$ variables. If $p = x^2 - xy + y^2$, then $(x+y)p = x^3 + y^3$ and hence $p \in Po(2,2)$. However, for every N, the coefficient of $x^1y^1z^N$ in $(x+y+z)^N(x^2 - xy + y^2)$ is -1, thus $p \notin Po(3,2)$.

Example 1. The following forms are non-negative on Δ_3 with zeros only at the unit vectors:

$$f = xz^3 + yz^3 + x^2y^2 - xyz^2,$$
$$g = x^2y + y^2z + z^2x - xyz.$$

We claim $f \notin Po(3,3)$, but $g \in Po(3,3)$. Consider the coefficient of $x^{N+1}yz^2$ in $F^N * f$. There is no contribution from $F^N xz^3$ or $F^N yz^3$ because the power of z is too large and there is no contribution from $F^n x^2y^2$ because the power of y is too large. Hence the only contribution comes from $F^N(-xyz^2)$ and thus the coefficient will always be -1. On the other hand, it is easy to compute that F^3g has only positive coefficients. This example shows that the location of the zeros of $p \in P_{n,d}(\Delta_n)$ is not enough to determine whether p is in $Po(n,d)$ or not.

Definition 1. The form $p \in P_{n,d}(\Delta_n)$ has a *simple zero* at the unit vector e_j if the coefficient of x_j^d in p is zero, but the coefficient of $x_j^{d-1}x_i$ is non-zero (and necessarily positive) for each $i \neq j$. In other words, $\mathrm{supp}(p)$ contains $(d-1)\cdot e_j + e_i$ for $i \neq j$, but not $d \cdot e_j$. Geometrically, this means that when p is restricted to lines through e_j and another point in Δ_n, it has only a simple zero at e_j.

For $r \in \mathbb{R}$, $0 < r < 1$ and $j = 1, \ldots, n$, let $\Delta_n(j,r)$ denote the simplex with vertices $\{e_j\} \cup \{e_j + r(e_i - e_j) \mid i \neq j\}$. For example, $D_3(2,r)$ is the triangle with vertices $\{(0,1,0), (r, 1-r, 0), (0, 1-r, r)\}$. Hence $\Delta_n(j,r)$ is the scaled simplex $r \cdot \Delta_n$ translated by $(1-r)e_j$.

Lemma 1. *If $p \in P_{n,d}(\Delta_n)$ has a simple zero at e_j, then there exist $s, r > 0$ such that*

$$p(u_1, \ldots, u_n) \geq s(u_1 + \cdots + u_{j-1} + u_{j+1} + \cdots + u_n)$$

for all $u = (u_1, \ldots, u_n) \in \Delta_n(j,r)$. More precisely, let C be the sum of the absolute value of the coefficients of p and let $v = \min_{i \neq j} \frac{\partial p}{\partial x_i}(e_j)$, then we can take

$$r = \frac{v}{v + 2C}, \quad s = \frac{v}{2}\left(\frac{2C}{v + 2C}\right)^{d-1}.$$

Proof. For $i \neq j$, let $v_i = \frac{\partial p}{\partial x_i}(e_j)$, which is the coefficient of $x_j^{d-1}x_i$. By assumption, $v_i > 0$, so $v > 0$.

Given $t = (t_1, \ldots, t_{j-1}, 1, t_j, \ldots, t_n) \in \mathbb{R}^n$ with $t_i \geq 0$, let $z = \sum t_i$ and consider the value of $p(t)$. Suppose we have a monomial $x_1^{a_1} \ldots x_n^{a_n}$ in p which is not one of the monomials $x_j^{d-1}x_i$, then $t_1^{a_1} \ldots t_n^{a_n} \leq z^{a_1+\cdots+a_n} \leq z^2$. It follows that

$$p(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_n) \geq \textstyle\sum_{i\neq j} v_i t_i - Cz^2 \geq vz - Cz^2. \tag{1}$$

In particular, $p(t) > 0$ for sufficiently small z.

Now let $r = \frac{v}{v+2C}$. Suppose $u \in \Delta_n(j, r)$; it follows by homogeneity that

$$p(u) = u_j^d p(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_n)$$

where $t_i := u_i/u_j$ for $i \neq j$. Given $u \in \Delta_n(j, r)$, then $\sum u_i = 1$, and $u_j \geq 1 - r$, so we can write $u_j = 1 - r + \epsilon$ with $\epsilon \geq 0$. It follows that $\sum_{i\neq j} u_i = r - \epsilon$. Then

$$\sum_{i\neq j} t_i = \frac{1}{u_j} \sum_{i\neq j} u_i = \frac{r - \epsilon}{1 - r + \epsilon} \leq \frac{r}{1 - r}.$$

Thus we have in this case $z \leq \frac{r}{1-r} = \frac{v}{2C}$. Then from (1), we have

$$p(u) \geq u_j^d \cdot \sum_{i\neq j} t_i \cdot \left(v - C \cdot \frac{v}{2C}\right) =$$

$$(u_j^{d-1})(\frac{v}{2})(u_1 + \cdots + u_{j-1} + u_j + \cdots + u_n).$$

Since $u_j^{d-1} \geq (1 - r)^{d-1}$, we can take $s = \frac{v}{2}(1 - r)^{d-1} = \frac{v}{2}(\frac{2C}{v+2C})^{d-1}$. □

Our goal is to find a quantitative version of Pólya's Theorem which applies to polynomials which are positive on Δ_n, except for some simple zeros at the vertices. We begin with the case of a single simple zero, which we take at e_1, without loss of generality.

We fix some notation. Suppose $p \in P_{n,d}(\Delta_n)$ is positive on Δ_n except for a simple zero at e_1. Let $s, r > 0$ be as in Lemma 1 and define the following constants associated to p. Let K be the closure of Δ_n with the corner $\Delta_n(1, r)$ removed and let λ be the the minimum of p on K. Define M to be the size of the largest coefficient of p, i.e., $M := \max\{|a_\alpha| \mid \alpha \in \mathrm{supp}(p)\}$ and set $L = L(p)$, as in Theorem 1. Finally, for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ define

$$c(\beta) := \prod_{\beta_i \geq 2} \frac{\beta_i(\beta_i-1)}{2} \text{ and } c := \sum_{\alpha\in\mathrm{supp}(p)} c(\alpha).$$

Proposition 1. *Suppose p is positive on Δ_n except for a simple zero at e_1. With the notation as above, if*

$$N > \max\left(\frac{d(d-1)}{2}\frac{L}{\lambda} - d, \frac{cM}{s} - d\right),$$

then $F^N * p$ has positive coefficients. Thus, if $p \in P_{n,d}(\Delta_n)$ is positive, except for a single simple zero, then $p \in Po(n,d)$.

Proof. We proceed as in [10] and [11]. For positive $t \in \mathbb{R}$, $m \in \mathbb{N}$ and a single real variable y, define

$$(y)_t^m := y(y-t)\cdots(y-(m-1)t) = \prod_{i=0}^{m-1}(y-it).$$

and

$$p_t(u_1,\dots,u_n) := \sum_{\alpha\in\operatorname{supp}(p)} a_\alpha (x_i)_t^{\alpha_1}\dots(x_n)_t^{\alpha_n}.$$

For $|\beta| = N+d$, denote the coefficient of X^β in $F^N * p$ by A_β. For ease of exposition, set $t = \frac{1}{N+d}$. Then we have

$$A_\beta = \frac{N!(N+d)^d}{\beta_1!\cdots\beta_n!} p_t(\beta_1 t,\dots,\beta_n t), \tag{2}$$

Thus to show $A_\beta > 0$, we need $p_t(\beta_1 t,\dots,\beta_n t) > 0$. Let $z = (\beta_1 t,\dots,\beta_n t) \in \Delta_n$. If $z \in K$, then the proof of Theorem 1 in [11] shows that for $N > \frac{d(d-1)}{2}\frac{L}{\lambda} - d$, $p_t(z) > 0$ and hence $A_\beta > 0$.

Now suppose $z \in \Delta_n(1,r)$ and let $N = \frac{cM}{s} - d$. Our goal is to show that $p_t(z) > 0$. As in [11], we write

$$p_t(z) = p(z) - \sum_{\alpha\in\operatorname{supp}(p)} a_\alpha \left(z^\alpha - (z_1)_t^{\alpha_1}\dots(z_n)_t^{\alpha_n}\right) \tag{3}$$

We have the bound

$$p(z) \geq s(z_2+\cdots+z_n), \tag{4}$$

and so consider the summation in (3). As in the proof of Theorem 1 in [11], it is easy to see that for $0 \leq y \leq 1$ and $k \geq 2$,

$$(y)_t^k \geq y^k - \tfrac{k(k-1)}{2} t y^{k-1}; \tag{5}$$

this also holds for $k = 1$ trivially. Then, using (5), we have

$$\left| \sum_{\alpha \in \text{supp}(p)} a_\alpha \left(z^\alpha - (\prod_{i=1}^{n} (z_i)_t^{\alpha_i}) \right) \right| <$$

$$M \left| \sum_{\alpha \in \text{supp}(p)} \left(z^\alpha - \prod_{i=1}^{n} (z_i^{\alpha_i} - \tfrac{\alpha_i(\alpha_i - 1)}{2} \cdot t \cdot z_i^{\alpha_i - 1} \right) \right| =$$

$$M \left| \sum_{\alpha \in \text{supp}(p)} \left[z^\alpha - z^\alpha \prod_{i=1}^{n} \left(1 - \tfrac{\alpha_i(\alpha_i - 1)}{2} \cdot t \cdot \frac{1}{z_i} \right) \right] \right| =$$

$$M \left| \sum_{\alpha \in \text{supp}(p)} \left[z^\alpha \cdot \left(\prod_{\alpha_i \geq 2} \tfrac{\alpha_i(\alpha_i - 1)}{2} \cdot t \cdot \frac{1}{z_i} \right) \right] \right|$$

Now, because p has a simple zero at e_1, for every $\alpha \in \text{supp}(p)$ and every i with $\alpha_i \geq 2$, the monomial z^α / z_i contains at least one of $\{z_2, \ldots, z_n\}$. It follows that

$$M \left| \sum_{\alpha \in \text{supp}(p)} \left(z^\alpha \cdot \prod_{\alpha_i \geq 2} \tfrac{\alpha_i(\alpha_i - 1)}{2} \cdot t \cdot \frac{1}{z_i} \right) \right| \leq$$

$$M \sum_{\alpha \in \text{supp}(p)} c(\alpha) \cdot t \cdot (z_2 + \cdots + z_n),$$

recalling that $c(\alpha) = \prod_{\alpha_i \geq 2} \frac{\alpha_i(\alpha_i - 1)}{2}$.

Combining this with (3) and (4), we have

$$p_t(z) > \left(s - \frac{M}{N + d} \sum_{\alpha \in \text{supp}(p)} c(\alpha) \right) (z_2 + \cdots + z_n) >$$

$$\left(s - \frac{Mc}{N + d} \right) (z_2 + \cdots + z_n).$$

Since $z_2 + \cdots + z_n > 0$ and $s - \frac{Mc}{N+d} > 0$, we conclude that $p_t(z) > 0$. □

Remark 2. We note that the bound in Proposition 1 does not depend on n, the number of variables. Also, observe that as $r \to 0$, $\lambda \to 0$ because p has a zero at e_1. On the other hand, s is bounded and thus the choice of r is more important to the bound from the main part of Δ_n than the bound from the corner.

Corollary 1. *Suppose $p \in P_{n,d}(\Delta_n)$ is positive on Δ_n except for simple zeros at unit vectors $e_{j_1}, \ldots, e_{j_k}$. Then $p \in Po(n, d)$ and there is a bound for N so that $F^N p$ has only positive coefficients similar to the bound in Proposition 1.*

Note that in the Corollary, the simplex K is replaced by the unit simplex with k corners snipped off and s will be replaced by the minimum of the s's obtained by applying Lemma 1 to each simple zero.

Example 2. For $0 < \alpha < 1$, let

$$p_\alpha(x, y, z) := x(y-z)^2 + y(x-z)^2 + z(x-y)^2 + \alpha xyz.$$

Note that the first three terms give a form non-negative on Δ_3 with zeros at the unit vectors and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, thus p_α is psd on Δ_3 with zeros at the unit vectors and is symmetric in $\{x, y, z\}$. We will compute the bound from the proposition, directly compute the minimum N so that $F^N * p$ has positive coefficients, and compare the two. We are interested in the behavior as $\alpha \to 0$.

We start by computing the bound on the corners. In this case, we have $d = 3$, $v = 1$ and $C = 12 - \alpha$, hence the constants from Lemma 1 are

$$r = \frac{1}{25 - 2\alpha}, \quad s = \frac{1}{2}\left(\frac{24 - 2\alpha}{25 - 2\alpha}\right)^2.$$

Then in Proposition 1, $c = 6$ and $M = 6 - \alpha$. Thus

$$\frac{cM}{s} - d = 12(6-\alpha)^2\left(\frac{25 - 2\alpha}{24 - 2\alpha}\right)^2.$$

Recall that K is Δ_3 minus the three corners $D_3(j, r)$. It is a straightforward calculus exercise to compute the minimum of p on K: the interior extreme values occur at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and all three permutations of $(\frac{1}{9-\alpha}, \frac{1}{9-\alpha}, \frac{7-\alpha}{9-\alpha})$. Since $\alpha < 1$, the smallest value of p_α occurs at the centroid and equals $\frac{\alpha}{27}$. Finding the minimum on the boundary of K involves two line segments, one from $(r, 1-r, 0)$ to $(1-r, r, 0)$, and the other from $(1-r, r, 0)$ to $(1-r, 0, r)$. In the first case, $z = 0$ and the minimum is clearly $r(1-r)$. In the second case, an easy calculation shows that the minimum occurs at $(1-r, \frac{r}{2}, \frac{r}{2})$; the exact value is $r - 3r^2 + \frac{r^2}{4}(\alpha(1-r) + 9r) > r - 3r^2$. If we decrease r then the bound on the corners will still hold and if we assume r small enough, say $r \le \frac{\alpha}{18}$, then the minimum value of p on K is $\lambda = \frac{\alpha}{27}$. The remaining constants are $L = \min\{1, \frac{6-\alpha}{6}\} = 1$ and $d = 3$ and so one bound is equal to $\frac{81}{\alpha} - 3$.

Putting this together, if

$$N > \max\left\{\frac{81}{\alpha} - 3, 12(6-\alpha)\left(\frac{25 - 2\alpha}{24 - 2\alpha}\right)^2\right\},$$

then $F^N * p$ has positive coefficients. The second is clearly bounded away from 0 as $\alpha \to 0$ and hence we have that N is asymptotically $\frac{81}{\alpha}$.

Finally, we compute N directly. We claim that for $N \ge \frac{18}{\alpha} - 3$, $(x+y+z)^N p_\alpha$ has non-negative coefficients and that this bound is sharp when $\alpha = 6w^{-1}$ for some integer w.

For $a+b+c = N+3$, the coefficient of $\frac{N!}{a!b!c!}x^a y^b z^c$ in $(x+y+z)^N p_\alpha$ is easily seen to be $fg - 3h - 2g - (6-\alpha)f$, where $f = a+b+c$, $g = ab+ac+bc$, and $h = abc$. But this equals

$$(f-2)g - (9-\alpha)h = (N+1)g - (9-\alpha)h = h\left((N+1)\frac{g}{h} - (9-\alpha)\right).$$

Now, $g/h = 1/a + 1/b + 1/c$, and it's easy to show that if $a, b, c \geq 0$ and $a+b+c = N+3$, then the minimum occurs when $a = b = c = \frac{N+3}{3}$. That is, $g/h \geq 9/(N+3)$ and equality holds if 3 divides N. We have

$$\frac{9(N+1)}{N+3} - (9-\alpha) = \alpha - \frac{18}{N+3},$$

and thus if $N \geq \frac{18}{\alpha} - 3$, all coefficients are non-negative. If $\frac{18}{\alpha} - 3$ is a multiple of 3, i.e., if $w = \frac{6}{\alpha} - 1$ is an integer, then for $N = 3w - 3$ the coefficient of $x^w y^w z^w$ will be 0, hence in this case N is best possible.

As $\alpha \to 0$, the bound from the theorem will be $\frac{81}{\alpha} - 3$ which has the same order of growth as the true bound $\frac{18}{\alpha} - 3$.

Remark 3. The technique used for forms with zeros at the corners should extend to forms in $Po(n,d)$ with zeros on the boundary and yield a quantitative Pólya's Theorem in this case. Also, using Schweighofer's construction [12], this should have applications to representations of polynomials nonnegative on compact sets and to optimization. These topics will be the subject of future work.

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