A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra

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1 Introduction

Fix a positive integer n and let $\mathbf{R}[X] := \mathbf{R}[x_1, \ldots, x_n]$. We write Δ_n for the simplex $\{(x_1, \ldots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$.

Pólya's Theorem ([**?**], [**?**, pp.57-59]) says that if $f \in \mathbf{R}[X]$ is homogeneous and positive on Δ_n, then for sufficiently large N all the coefficients of

$$(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$$

are positive. In this note, we give an explicit bound for N and give an application to some special representations of polynomials positive on polyhedra. In particular, we give a bound for the degree of a representation of a polynomial positive on a convex polyhedron as a positive linear combination of products of the linear polynomials defining the polyhedron.

We use the following multinomial notation: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, let X^α denote $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Let us fix homogeneous $f \in \mathbf{R}[X]$ of degree d,

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$.