Polynomials That Are Positive on an Interval

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1   Introduction

Suppose that $p \in \mathbb{R}[x]$ is a real polynomial in a single real variable. If $p(x) \geq 0$ for all $x \in \mathbb{R}$, then an easy consequence of the Fundamental Theorem of Algebra is that p can be written as a sum of two squares of polynomials. It is natural to wonder what one can say if $p(x) \geq 0$ or $p(x) > 0$ for x in a fixed interval.

There are several such representations which resonate with more recent work in real algebraic geometry. It has long been known that if $p(x) > 0$ for $x \in (-1, 1)$, then p can be written as a positive linear combination of polynomials $(1-x)^i(1+x)^j$ for suitable integers i and j (Bernstein); however, it might be necessary for $i + j$ to exceed the degree of p. And if $p(x) \geq 0$ for $x \in [-1, 1]$, then one can write $p(x) = f(x) + (1 - x^2)g(x)$, where $f(x), g(x) \geq 0$ for all x (Fekete). It has also long been known that if $p(x) \geq 0$ for $x \in [0, \infty)$, then p can be written in the form $f_1 + xf_2$, where each f_i is a sum of two squares (Pólya-Szegö). The proofs of these results are elementary, and are included in this paper.

This question can be viewed from a more abstract algebraic perspective. Recently, K. Schmüdgen [20] has proved a remarkable theorem which can be viewed as a broad generalization of these representations to positive functions. Roughly speaking, if a compact set S in $\mathbb{R}^n$ is defined by finitely many polynomial inequalities, then any polynomial which is strictly positive on S can be written in terms of the defining polynomials for S and sums of squares (of polynomials). The proof of this result in [20] is neither elementary nor constructive. We now present the requisite definitions.

Given non-constant polynomials $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_m]$, define $S(f_1, \ldots, f_n)$ to be the *basic closed semi-algebraic set* generated by the f_i's, i.e.,

$$S(f_1, \ldots, f_n) = \{\alpha \in \mathbb{R}^m \mid f_i(\alpha) \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Let Σ denote the set of *sums of squares* $\sum_j f_j^2$, with $f_j \in \mathbb{R}[x_1, \ldots, x_m]$. For any $I \subset \{1, ..., n\}$, let $f_I(x) = \prod_{i \in I} f_i(x)$, with the usual understanding that $f_\emptyset(x) = 1$.

Then we define $P(f_1, \ldots, f_n)$, the *preorder* generated by the f_i's, by

$$P(f_1, \ldots, f_n) = \left\{ \sum_{I \subset \{1,\ldots,n\}} s_I(x) f_I(x) \mid s_I \in \Sigma \right\}. \tag{1}$$

Note that since Σ is closed under multiplication and $f_I f_{I'} = f_{I \cap I'}^2 f_{I \triangle I'}$, the pre-order $P(f_1, \ldots, f_n)$ is also closed under multiplication.

For a set $A \subseteq \mathbb{R}^m$, let $\mathrm{Psd}(A)$ (resp. $\mathrm{Pd}(A)$) denote the set of polynomials $p \in \mathbb{R}[x_1, \ldots, x_m]$ so that $p(a_1, \ldots, a_m) \geq 0$ (resp. $p(a_1, \ldots, a_m) > 0$) for every $(a_1, \ldots, a_m) \in A$. Such polynomials are said to be *positive semidefinite (psd)* on A (resp. *positive definite (pd)* on A). These sets are also closed under multiplication.

Let $P = P(f_1, \ldots, f_n)$ and $S = S(f_1, \ldots, f_n)$, then clearly, P is contained in $\mathrm{Psd}(S)$. Schmüdgen's Theorem says that if S is compact then a stronger statement is true:

$$\mathrm{Pd}(S) \subseteq P \subseteq \mathrm{Psd}(S).$$

In other words, if $g > 0$ on compact $S(f_1, \ldots, f_n)$, then g is in the preorder generated by the f_i's. Schmüdgen's Theorem is somewhat simpler in one variable. As noted earlier, $f \in \Sigma$ iff $f \in \mathrm{Psd}(\mathbb{R})$; that is, iff $S(f) = \mathbb{R}$. (The situation is more complicated for polynomials in more than one variable; see [17].) Thus, to give a simple example, one consequence of Schmüdgen's theorem is that if $p(x) > 0$ for $x \in [-1, 1]$, then one can write $p(x) = f(x) + (1 - x^2)^3 g(x)$, where $f(x), g(x) \geq 0$ for all x. This paper contains a constructive proof of this result, with degree bounds for f and g which depend on the degree of p and the location of its roots.

In section two of this paper, we show that the study of $\mathrm{Psd}(I)$ and $\mathrm{Pd}(I)$ for real intervals I essentially reduces to two cases: $I = [-1, 1]$ and $I = [0, \infty)$. We review the literature on this problem, which goes back to Hermite, and discuss work of Goursat, Bernstein, Hausdorff, Pòlya and Szegö, Fekete, Lukàcs, Karlin and Shapley and Karlin and Studden. (There is not much literature on $\mathrm{Psd}(A)$ for $A \subset \mathbb{R}^2$, and we will not discuss the subject in this paper, but see [14].)

In section three, we combine recent results of de Loera and Santos with work of Goursat, Pòlya and Szegö to give a constructive proof that, if p is in $\mathrm{Pd}([-1, 1])$, then for a computable value of m, there exist $d_k \geq 0$ so that

$$p(x) = \sum_{k=0}^{m} d_k (1 - x)^k (1 + x)^{m-k}.$$

(Without the information on m, this theorem is due to Bernstein.) Other computations of m have been made by Erdélyi and Erdélyi and Szabados.

In section four, we give an elementary and constructive proof of Schmüdgen's Theorem in one special case. Let h be a given polynomial for which $S(h) = [-1, 1]$. If $p \in \mathrm{Pd}([-1, 1])$, we give a constructive proof of the existence of $s_0, s_1 \in \Sigma$ so that $p = s_0 + h s_1$. This includes an *a priori* bound on the degrees of s_0 and s_1, based on

h, the degree of p, and the smallest absolute value of the roots of p. In the special case $h(x) = 1 - x^2$, Lukács proved a stronger theorem: $\text{Psd}([-1, 1]) = P(1 - x^2)$. We shall give a necessary and sufficient condition on h so that $\text{Psd}([-1, 1]) = P(h)$. Our proof relies on a non-constructive result of Scheiderer, and so is not constructive.

In section five, we turn our attention to the non-compact interval $[0, \infty)$, to which Schmüdgen's Theorem does not apply. As noted earlier, Pòlya and Szegö proved that, if $p \in \text{Psd}([0, \infty))$, then there exist $s_i \in \Sigma$ so that $p(x) = s_0(x) + x s_1(x)$. In this section, we prove that this is essentially the only case in which Schmüdgen's conclusion holds for $[0, \infty)$: If $S(f_1, \ldots, f_n) = [0, \infty)$ and $\text{Pd}([0, \infty)) \subset P(f_1, \ldots, f_n)$, then $f_j(x) = cx$ for some j.

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2   Background and Historical Remarks

Suppose $I \subseteq \mathbb{R}$ is an interval. How can one describe $\text{Psd}(I)$ and $\text{Pd}(I)$? At first glance, this question might seem to involve many cases, depending on whether I is open or half-open or closed; finite, half-infinite or infinite. In fact, there are really only two cases.

First observe that, by continuity, $\text{Psd}(I) = \text{Psd}(\bar{I})$; $\text{Pd}(I)$ contains $\text{Pd}(\bar{I})$, with the complement consisting of those polynomials which are positive on I but vanish at one or the other of its endpoints. If $a \in \bar{I} \setminus I$ is a left-hand endpoint and $p \in \text{Pd}(I) \setminus \text{Pd}(\bar{I})$, then there exists an integer k so that $p(x) = (x - a)^k q(x)$, where $q \in \text{Pd}(I \cup \{a\})$. A similar consideration applies if $b \in \bar{I} \setminus I$ is a right-hand endpoint. Thus, it suffices to consider $\text{Psd}(I)$ and $\text{Pd}(I)$ for closed intervals I. Furthermore, if $p \in \text{Psd}(I) \setminus \text{Pd}(I)$, then p has (finitely many) zeroes in I. If $p(t) = 0$ and t is interior to I, then $p(x) = (x - t)^{2k} q(t)$ for some even integer $2k$, and $q \in \text{Psd}(I)$. If a (resp. b) is a left-hand (resp. right-hand) endpoint of I and $p(a) = 0$ (resp. $p(b) = 0$), then $p(x) = (x - a)^k q(x)$ (resp. $p(x) = (b - x)^k q(x)$) for some $q \in \text{Psd}(I)$. In any event, p can only have a finite number of zeroes, so $p \in \text{Psd}(I)$ for the closed interval $I = [a, b]$ if and only if

$$p(x) = (x - a)^{k_0} \left(\prod_{j=1}^{r} (x - t_j)^{2k_j} \right) (b - x)^{k_{r+1}} q(x) \tag{2}$$

for some non-negative integers k_j, where $t_j \in (a, b)$ and $q \in \text{Pd}(I)$. (If I is half-infinite or infinite, then this formula is modified accordingly.)

If $I = (-\infty, \infty)$, then it is classically known that $\text{Psd}(I)$ consists of the sums of two squares of polynomials and $\text{Pd}(I)$ consists of the sums of two squares of

3

polynomials which have no common real zeros. If I is half-infinite, then $I = [a, \infty)$ (resp. $(-\infty, b]$), and if $p \in \mathrm{Pd}(I)$ and $f(x) = p(x - a)$ (resp. $f(x) = p(b - x)$), then $f \in \mathrm{Pd}([0, \infty))$. Finally, if $I = [a, b]$, $p \in \mathrm{Pd}(I)$ and

$$f(x) = p\left(\tfrac{(b-a)x + (b+a)}{2}\right),$$

then $f \in \mathrm{Pd}([-1, 1])$. In other words, there are essentially only two cases: $I = [0, \infty)$ and $I = [-1, 1]$.

(It might seem more natural to identify $[0, 1]$ as the archetypal finite interval, and the early theorems of Bernstein and Hausdorff were cast in this way. The interval $[-1, 1]$ was preferred by analysts wanting to know which polynomials p have the property that the trigonometric polynomial $p(\cos\theta)$ takes non-negative values.)

It turns out that $\mathrm{Pd}([0, \infty))$ and $\mathrm{Pd}([-1, 1])$ are closely related to each other. Given a polynomial f of exact degree m, we define the (m-th degree) *Goursat transform* of f, $\tilde{f}$, by:

$$\tilde{f}(x) = (1 + x)^m f\left(\tfrac{1-x}{1+x}\right). \tag{3}$$

It is worth noting that the Goursat transform is nearly its own inverse:

$$\tilde{\tilde{f}}(x) = (1 + x)^m \tilde{f}\left(\tfrac{1-x}{1+x}\right) = (1 + x)^m (1 + \tfrac{1-x}{1+x})^m f\left(\frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}}\right) = 2^m f(x). \tag{4}$$

If $\deg f = m$ and $\deg \tilde{f} = m - k < m$, then the $(m - k)$-th degree Goursat transform of $\tilde{f}$ is already a polynomial, hence (4) implies that $(1 + x)^k | f(x)$.

Lemma 1 (Goursat's Lemma). *Suppose f is a polynomial of degree m. Then $f \in Pd([-1, 1])$ if and only if $\tilde{f} \in Pd([0, \infty))$ and $\deg f = m$; $f \in Psd([-1, 1])$ if and only if $\tilde{f} \in Psd([0, \infty))$ and $\deg \tilde{f} \leq m$.*

Proof. If $x \in (-1, 1]$, then $y = \tfrac{1-x}{1+x} \in [0, \infty)$. Since $(1 + x)^m > 0$, we see from (3) that $f(x) > 0$ (resp. $f(x) \geq 0$) if and only if $\tilde{f}(y) > 0$ (resp. $\tilde{f}(y) \geq 0$.) Write $f(x) = \sum_{k=0}^m a_k x^k$; the coefficient of x^m in $\tilde{f}(x)$ is $\sum_{k=0}^m a_k (1 - x)^k (1 + x)^{m-k} = \sum_{k=0}^m (-1)^k a_k = f(-1)$.

It follows immediately that $f \in \mathrm{Pd}([-1, 1])$ if and only if $\tilde{f} \in \mathrm{Pd}([0, \infty))$ and $\deg f = m$. If we weaken the hypothesis from "positive" to "non-negative", then the same conclusions carry over, with the loss of information about the degree of $\tilde{f}$. (Note that $f(-1) \geq 0$ by continuity, in any case.) $\qquad\square$

This subject appears to have been inaugurated [10] in 1894 by the 71-year old French mathematician Charles Hermite, in the first volume of the French problems journal *Interméd. des math.* Let

$$\mathcal{P}_d := \left\{ \sum_{i+j \leq d} c_{ij}(1 - x)^i (1 + x)^j \mid c_{ij} \geq 0 \right\}. \tag{5}$$

Hermite asked: if $p \in \mathrm{Pd}([-1, 1])$ has degree d, must it belong to $\mathcal{P}_d$? This question was quickly answered in the negative, by E. Goursat [7], J. Sadier [18] and J. Franel [6], in several different ways. A later solution appeared in Pólya-Szegö [16, VI 48]. We present Goursat's proof.

Suppose $p \in \mathcal{P}_d$ and $p(x) = \sum_{i+j \leq d} c_{ij}(1-x)^i(1+x)^j$, with $c_{ij} \geq 0$. Then

$$\tilde{p}(x) = \sum_{i+j \leq d} 2^{i+j} c_{ij} x^i (1+x)^{d-i-j},$$

so that the coefficients of $\tilde{p}$ are nonnegative. For $\epsilon > 0$, let $p_\epsilon(x) = x^2 + \epsilon$. We have

$$\tilde{p}_\epsilon(x) = \epsilon(1+x)^2 + (1-x)^2 = (1+\epsilon) - (2 - 2\epsilon)x + (1+\epsilon)x^2.$$

Clearly, if $\epsilon > 0$, then $p \in \mathrm{Pd}([-1, 1])$. But $p \in \mathcal{P}_2$ if and only if the coefficients of $\tilde{p}_\epsilon$ are nonnegative, and this is true only for $\epsilon \geq 1$. Thus, for $0 < \epsilon < 1$, p_ϵ provides a negative answer to Hermite's question.

However, if $p \in \mathrm{Pd}([-1, 1])$, then it is true that $p \in \mathcal{P}_m$ for sufficiently large m. This was proved by Bernstein [1] in 1915, although Pólya-Szegö attributes this result to Hausdorff [9, pp. 98-99] in 1921, as part of his solution of the classical moment problem on $[0, 1]$. Two proofs of this theorem are given in [16]. One uses Goursat's transform, combined with another theorem of Pólya's [15]. We will give a computational version of the latter proof in the next section. Finally, it is worth noting that if $0 \neq p \in \mathrm{Psd}([-1, 1])$, and $p(u) = 0$ for $u \in (-1, 1)$, then upon setting $x = u$ in the equation $p(x) = \sum_{i,j} c_{ij}(1-x)^i(1+x)^j$, with $c_{ij} \geq 0$, we conclude that $c_{ij} = 0$ for all (i, j), a contradiction. It then follows easily that $\cup_m \mathcal{P}_m = \mathrm{Pd}((-1, 1))$.

Two other results found in [16] give degree information absent in Schmüdgen's Theorem. If $p(x) \in \mathrm{Psd}([-1, 1])$ and p has degree n, then

$$p(x) = \big(f(x)\big)^2 + (1 - x^2)\big(g(x)\big)^2$$

for some polynomials f and g of degree at most n and $n - 1$ respectively. (This result ([16, VI 46]) is attributed to M. Fekete, but no additional bibliographic details are given.) Under the same hypotheses, p can be written as

$$p(x) = \big(f_1(x)\big)^2 + (1 - x)\big(f_2(x)\big)^2 + (1 + x)\big(f_3(x)\big)^2 + (1 - x^2)\big(f_4(x)\big)^2$$

so that each summand has degree $\leq n$. (This result ([16, VI 47]) is attributed to F. Lukàcs, again with no details.) Further, if n is even, then $f_2 = f_3 = 0$, and if n is odd, then $f_1 = f_4 = 0$.

Karlin and Shapley [11, p. 35] gave an even more precise representation. Suppose $p(x) \in \mathrm{Pd}([-1, 1])$ has even degree $n = 2m$, then

$$p(x) = \alpha \prod_{j=1}^{m}(x - x_{2j-1})^2 + \beta(1 - x^2) \prod_{j=1}^{m-1}(x - x_{2j})^2.$$

If p has odd degree $n = 2m + 1$, then

$$p(x) = \alpha(1 + x)\prod_{j=1}^{m}(x - x_{2j})^2 + \beta(1 - x)\prod_{j=1}^{m-1}(x - x_{2j-1})^2.$$

These representations are unique under the additional condition that $-1 < x_1 < \cdots < x_{n-1} < 1$. Karlin and Studden [12, p. 169] give a similarly interlaced representation for polynomials in $\mathrm{Pd}([0, \infty))$.

We present now a short proof of the representation result [16, VI 45] for $[0, \infty)$.

Proposition 2 (Pólya-Szegö). *If $p \in Psd([0, \infty))$, then $p \in P(x)$. More specifically, there exist $f, g \in \Sigma$ so that $p(x) = f(x) + xg(x)$, and $\deg f, \deg xg \leq \deg p$.*

Proof. First observe that if $p_i = f_i + xg_i$, with $\deg f_i, \deg xg_i \leq \deg p_i$, then

$$p := p_1 p_2 = (f_1 f_2 + x^2 g_1 g_2) + x(f_1 g_2 + f_2 g_1) := f + xg,$$

where $\deg f, \deg xg \leq \deg p$. Thus, it suffices to write $p \in \mathrm{Psd}([0, \infty))$ as a product of factors, each of which satisfies the desired condition.

Now factor p over $\mathbb{R}[x]$. Any positive roots appear to an even degree, hence the linear factors of p will either appear to an even degree, or will be a product of terms $x + x_0$, with $x_0 \geq 0$. The irreducible quadratic factors of p are positive definite. Since any psd factor q is already in Σ, it can be written as $q + x \cdot 0$; the linear factor $x + x_0$ can be written as $x_0 + x \cdot 1$. In view of the first paragraph, this completes the proof. $\qquad\square$

Proposition 2 implies a stronger conclusion than Schmüdgen's Theorem for $P(1 - x^2)$ and $P(1 - x, 1 + x)$.

Corollary 3. $Psd([-1, 1]) = P(1 - x^2) = P(1 - x, 1 + x)$.

Proof. Recall that if $p \in P(1 - x^2)$, then $p(x) = f(x) + (1 - x^2)g(x)$ for $f, g \in \Sigma = \mathrm{Psd}(\mathbb{R})$, so $p \in \mathrm{Psd}([-1, 1])$. A similar argument applies to $P(1 - x, 1 + x)$.

To prove the converse, suppose $p \in \mathrm{Psd}([-1, 1])$ and $\deg p = m$. By Goursat's Lemma and the last Proposition 2, there exist $f_i, g_i \in \mathbb{R}[x]$, $r_i = \deg f_i \leq m/2$ and $s_i = \deg g_i \leq m/2$ so that

$$\tilde{p}(x) = \sum_{i=1}^{2} f_i^2(x) + x \sum_{i=1}^{2} g_i^2(x).$$

Now perform another Goursat transform of degree m (c.f. (4)):

$$2^m p(x) = \sum_{i=1}^{2}(1 + x)^{m-2r_i}\tilde{f}_i^2(x) + (1 - x)\sum_{i=1}^{2}(1 + x)^{m-1-2s_i}\tilde{g}_i^2(x).$$

If m is even, we can absorb the extra factors of $1 + x$ to obtain:

$$2^m p(x) = \sum_{i=1}^{2} \left((1+x)^{m/2-r_i} \tilde{f}_i(x) \right)^2 + (1 - x^2) \sum_{i=1}^{2} \left((1+x)^{m/2-1-s_i} \tilde{g}_i(x) \right)^2,$$

so $p \in P(1 - x^2)$. If m is odd, then we obtain a similar, if longer, expression:

$$2^m p(x) =$$

$$(1 + x) \sum_{i=1}^{2} \left((1+x)^{(m-1)/2-r_i} \tilde{f}_i(x) \right)^2 + (1 - x) \sum_{i=1}^{2} \left((1+x)^{(m-1)/2-s_i} \tilde{g}_i(x) \right)^2,$$

hence $p \in P(1 - x, 1 + x)$

Finally, observe that $P(f) = P(g)$ if and only if $f \in P(g)$ and $g \in P(f)$. We are done if we can show that $1 \pm x \in P(1 - x^2)$ and $1 - x^2 \in P(1 - x, 1 + x)$. The latter is immediate from $(1 - x^2) = (1 - x)(1 + x)$, and the identity

$$1 \pm x = \frac{(1 \pm x)^2}{2} + \frac{1}{2}(1 - x^2) \in P(1 - x^2).$$

$\square$

Note that

$$1 - x^{2k} = (1 - x^2)(1 + x^2 + \cdots + x^{2k-2})$$

and

$$1 - x^2 = \frac{(k - 1) - kx^2 + x^{2k}}{k} + \frac{1}{k}\left(1 - x^{2k} \right),$$

where $(k - 1) - kx^2 + x^{2k}$ is psd by the arithmetic-geometric inequality. Thus, $P(1 - x^2) = P(1 - x^{2k})$ for all positive integers k. However, Stengle [21] has shown that $1 - x^2 \notin P((1 - x^2)^3)$, so $S(f) = S(g)$ does not imply that $P(f) = P(g)$. (See also Corollary 11 below.)

3 Computing the Bernstein Degree

Suppose $f \in \text{Pd}([-1, 1])$ has degree m. Define $r(f)$ to be the smallest integer m so that $f \in \mathcal{P}_m$; $r(f)$ has been called the *Bernstein degree* of f (by DeVore and Lorentz [3]) and the *Lorentz degree* of f (by Borwein and Erdélyi [2]). Our first task in this section is to compute $r(f)$.

Pòlya proved in 1928 [15] that if $p \in \text{Pd}(\mathbb{R}_+^m)$, then for sufficiently large d, $(1 + \sum_{j=1}^{m} x_j^2)^d p(x_1, \ldots, x_m)$ has positive coefficients. In particular, if $g \in \text{Pd}([0, \infty))$, then for sufficiently large d, $(1 + x)^d g(x)$ has non-negative coefficients. Let $\tilde{r}(g)$ denote the smallest integer d for which $(1 + x)^d g(x)$ has non-negative coefficients. Recently, De Loera and Santos [13, p. 232] have made an algorithmic analysis of [15]; there are several unfortunate typos in [13], but the statement below is correct. We use this to give an upper bound for $\tilde{r}(g)$.

Proposition 4 ([13]). *Suppose $q(x_1, \ldots, x_m)$ is a real homogeneous polynomial of degree d, the coefficients of q are bounded above in absolute value by L and $q(u) \geq \lambda > 0$ for $u \in \Delta = \{(u_1, \ldots, u_m) \mid u_j \geq 0, \sum_j u_j = 1\}$. If*

$$r \geq \frac{Lmd^2}{\lambda} + md,$$

then $(\sum_j x_j)^r q(x_1, \ldots, x_m)$ has positive coefficients.

This proposition has an immediate interpretation for (inhomogeneous) polynomials of one variable:

Corollary 5. *Suppose $g(x) = \sum_{j=0}^{d} b_j x^j \in Pd([0, \infty))$, $|b_j| \leq L$ and*

$$\lambda = \min \left\{ \sum_{j=0}^{d} b_j t^j (1-t)^{d-j} \mid t \in [0, 1] \right\}.$$

Then

$$\tilde{r}(g) \leq 2d + \left\lceil \frac{2d^2 L}{\lambda} \right\rceil.$$

Proof. Let $q(x, y) = \sum_{j=0}^{d} b_j x^j y^{d-j}$, and apply Proposition 4 to q, noting that $m = 2$. For later reference, observe that $\lambda = \inf\{(1-t)^d g(\frac{t}{1-t}) \mid t \in [0, 1)\}$. $\qquad\square$

Theorem 6. *Suppose $f(x) \in Pd([-1, 1])$ has degree n. and let $\tilde{f}(x) = \sum_j e_j x^j$. Let λ denote the minimum of $f(x)$ for $x \in [-1, 1]$ and let L denote $\max\{|e_j|\}$. Then*

$$r(f) = n + \tilde{r}(\tilde{f}) \leq 3n + \left\lceil \frac{2n^2 L}{\lambda} \right\rceil.$$

Proof. First suppose $m = \tilde{r}(\tilde{f})$, so that

$$(1+x)^{m+n} f\left(\tfrac{1-x}{1+x}\right) = (1+x)^m \tilde{f}(x) = \sum_{k=0}^{m+n} b_k x^k,$$

with $b_k \geq 0$. Apply the Goursat transform (of degree $m + n$) to both sides above, to obtain

$$2^{m+n} f(x) = \sum_{k=0}^{m+n} b_k (1-x)^k (1+x)^{m+n-k}.$$

Thus, $r(f) \leq m + n = \tilde{r}(\tilde{f}) + n$. This proof of Bernstein's theorem is in [16].

To prove the converse, we first observe that a representation

$$f(x) = \sum_{i+j \leq d} c_{ij} (1-x)^i (1+x)^j$$

8

with non-negative c_{ij} can always be homogenized:

$$f(x) = \sum_{i+j\leq r} c_{ij}(1-x)^i(1+x)^j\left(\tfrac{1-x}{2} + \tfrac{1+x}{2}\right)^{r-(i+j)}$$

$$= \sum_{i+j\leq r} c_{ij}(1-x)^i(1+x)^j\frac{1}{2^{r-(i+j)}}\sum_{\ell=0}^{r-i-j}\binom{r-(i+j)}{\ell}(1-x)^\ell(1+x)^{r-(i+j+\ell)}$$

$$:= \sum_{k=0}^{r} d_k(1-x)^k(1+x)^{r-k},$$

and if $c_{ij} \geq 0$ for all i, j, then $d_k \geq 0$ for all k as well. Thus, $r(f)$ is always achieved by a homogeneous representation.

If we have such a representation for f with $d_k \geq 0$, then, upon taking the Goursat transform of degree $r \geq n$, we get

$$(1+x)^r f\left(\tfrac{1-x}{1+x}\right) = (1+x)^{r-n}\tilde{f}(x) = 2^r\sum_{k=0}^{r} d_k x^k. \tag{6}$$

Thus, $\tilde{r}(\tilde{f}) \leq r(f) - n$, and so $r(f) = \tilde{r}(\tilde{f}) + n$.

Finally, by Corollary 5, $\tilde{r}(\tilde{f}) - n \leq \lceil\frac{2Ln^2}{\lambda}\rceil + 2n$, where λ is the infimum of $(1-t)^d\tilde{f}(\frac{t}{1-t})$ for $t \in [0,1)$. However,

$$(1-t)^d\tilde{f}\left(\tfrac{t}{1-t}\right) = (1-t)^d(1+\tfrac{t}{1-t})^d f\left(\frac{1-\frac{t}{1-t}}{1+\frac{t}{1-t}}\right) = f(1-2t),$$

so that λ is precisely the minimum of f on $[-1,1]$.

$\square$

Remark. Erdélyi [4] and Erdélyi and Szabados [5] have given detailed computations of $r(f)$ for quadratic polynomials. For example, if the (complex) roots of the definite quadratic f lie on the ellipse $x^2 + \frac{y^2}{a^2} = 1$, then $r(f) \in [a^{-2}, 1+2a^{-2}]$, and these bounds are essentially achieved. These results can be found in [2, pp.83–89].

Example. We compute $r(p_\epsilon)$ for $\epsilon > 0$, where $p_\epsilon(x) = \epsilon + x^2 \in \mathrm{Pd}([-1,1])$. Recall that
$$\tilde{p}_\epsilon(x) = \epsilon(1+x)^2 + (1-x)^2 = (1+\epsilon) - (2-2\epsilon)x + (1+\epsilon)x^2,$$
so that $r(p_\epsilon) = 2$ if $\epsilon \geq 1$. Henceforth, assume $\epsilon < 1$, and that $\epsilon^{-1} \in (2k-1, 2k+1]$ for some positive integer k. We shall show that $r(p_\epsilon) = 2k + 1$, the least odd integer $\geq \epsilon^{-1}$. (For $\epsilon \leq 1/3$, compare with Theorem 6: We have $n = 2$, $\lambda = \epsilon$, and $\ell = \max\{1+\epsilon, 2-2\epsilon\}$, which yields an upper bound of $3\cdot2+\lceil2\cdot2^2\cdot\frac{2-2\epsilon}{\epsilon}\rceil = \lceil\frac{16}{\epsilon}\rceil-10$.) Let $g_\delta(x) = 1 - (2-\delta)x + x^2$, so that $\tilde{p}_\epsilon = (1+\epsilon)g_\delta$, where $\delta = \frac{4\epsilon}{1+\epsilon}$. We have $\delta \in [\frac{2}{k+1}, \frac{2}{k})$, and $\tilde{r}(\tilde{p}_\epsilon) = \tilde{r}(g_\delta)$.

9

By the binomial theorem,

$$(1+x)^m g_\delta(x) = \sum_{j=0}^{m+2} \left(\binom{m}{j} - (2-\delta)\binom{m}{j-1} + \binom{m}{j-2} \right) x^j;$$

a further calculation shows that

$$(1+x)^m g_\delta(x) = \sum_{j=0}^{m+2} \frac{m!}{j!(m+2-j)!} \left((m+2-2j)^2 - (m+2) + j(m+2-j)\delta \right) x^j.$$

Thus $\tilde{r}(g_\delta)$ is the smallest m so that

$$(m+2-2j)^2 - (m+2) + j(m+2-j)\delta \geq 0 \qquad \text{for all } 0 \leq j \leq m+2. \qquad (7)$$

Suppose (7) holds for even $m = 2s$. The inequality for $j = s+1$ implies that $\delta \geq \frac{2}{s+1}$, so $s \geq k$. And the algebraic identity

$$(2s+2-2j)^2 - (2s+2) + j(2s+2-j)\delta = \frac{2(2s+1)(s+1-j)^2}{s+1} + j(2s+2-j)(\delta - \frac{2}{s+1})$$

shows that (7) holds when $\delta \geq \frac{2}{s+1}$. Similarly, if (7) holds for odd $m = 2s-1$, then $j = s$ implies $\delta \geq \frac{2}{s+1}$, so $s \geq k$, and the identity

$$(2s+1-2j)^2 - (2s+1) + j(2s+1-j)\delta = \frac{2(2s+1)(s-(j-1))(s-j)}{s+1} + j(2s+1-j)(\delta - \frac{2}{s+1})$$

shows that (7) holds when $\delta \geq \frac{2}{s+1}$, since j is an integer-valued variable.

Since we wished to find the *smallest* m so that (7) holds, for given δ, we conclude that $\tilde{r}(g_\delta) = 2k-1$, and so $r(p_\epsilon) = \tilde{r}(\tilde{p}_\epsilon) + 2 = \tilde{r}(g_\delta) + 2 = 2k+1$. The computation in this example is quite similar to [8, pp. 59–60].

4   Schmüdgen's Theorem for $[-1, 1]$

In this section, we give a constructive proof of Schmüdgen's Theorem in the special case that there is a single polynomial h such that $S(h) = [-1, 1]$. That is, if $[-1, 1] = \{x : h(x) \geq 0\}$, and $p(x) > 0$ for $x \in [-1, 1]$, then we construct psd real polynomials s and t so that

$$p = s + t\,h. \qquad (8)$$

(As noted earlier, in one variable Σ consists of the psd polynomials, so it suffices to show that s and t are non-negative on $\mathbb{R}$.)

We begin with some simple remarks. First, if $S(h) = [-1, 1]$, then $h(x)$ changes sign only at $x = \pm 1$, and can have zeroes with even degree only in $(-1, 1)$. Thus $h(x) = (1-x)^c (1+x)^d q(x)$, where $q(\pm 1) \neq 0$ and c and d are both odd. Furthermore, $q(x) \geq 0$ on $(-1, 1)$ and $q(x) > 0$ for $|x| \geq 1$. It follows from (1) that, for

10

any polynomial ϕ, $P(h) \supset P(\phi^2 h)$. Thus, since $c - d$ is even, we can multiply h by an even power of $1 \pm x$ and assume that $c = d =: r$ is odd.

Fix $h(x) = (1 - x^2)^r q(x)$ so that $S(h) = [-1, 1]$. Since q is a polynomial in one variable, $q(x) \not\to 0$ as $x \to \infty$, so there exists α so that $q(x) \geq \alpha > 0$ for $x \in (-\infty, 1] \cup [1, \infty)$. Since q is continuous, there exists β so that $0 \leq q(x) \leq \beta$ for $x \in [-1, 1]$.

The main technical result is the following:

Theorem 7. *Suppose $\epsilon > 0$, and $h(x) = (1 - x^2)^r q(x)$, where $q(x) \geq \alpha > 0$ for $x \in [-1, 1]$ and $0 \leq q(x) \leq \beta$ for $x \in (-\infty, 1] \cup [1, \infty)$. Let*

$$A = \frac{1}{r 2^r \epsilon^{r-1} \alpha}, \qquad and \qquad m = \left\lfloor \frac{r}{2\epsilon} \left(\frac{\beta}{r\alpha} \right)^{\frac{1}{r}} \right\rfloor,$$

and set $F(x) = 1 + \epsilon + x - A x^{2m} h(x)$. Then $F(x)$ is psd.

Remark. This theorem implies that $F \in \Sigma$, hence $1 + \epsilon + x = F + A x^{2m} h \in P(h)$. Let $\bar{h}(x) = h(-x)$, then, with the same values for α, β, A and M, we have $1 + \epsilon + x \in P(\bar{h})$; upon taking $x \to -x$, we see that $1 + \epsilon - x \in P(h)$. Thus,

$$1 + \epsilon \pm x = F_\pm(x) + G_\pm(x) h(x),$$

where $F_\pm, G_\pm \in \Sigma$, and $\deg F_\pm, \deg G_\pm h$ are bounded above by

$$\left\lfloor \frac{r}{\epsilon} \left(\frac{\beta}{r\alpha} \right)^{\frac{1}{r}} \right\rfloor + \deg h \leq \frac{r}{\epsilon} \left(\frac{\beta}{r\alpha} \right)^{\frac{1}{r}} + \deg h.$$

Proof. We want to show that $F(x) \geq 0$ for all $x \in \mathbb{R}$. Observe that $1 + \epsilon + x \geq 0$ for $x \geq -1 - \epsilon$ and $-A x^{2m} h(x) \geq 0$ for $|x| \geq 1$. Thus we need only consider $x \in (-\infty, -1 - \epsilon) \cup (-1, 1)$.

First, write $x \in (-\infty, -1 - \epsilon)$ as $x = -1 - \epsilon - y$, where $y > 0$. We must show that

$$F(x) = -y - A(1 + \epsilon + y)^{2m} \left(1 - (1 + \epsilon + y)^2 \right)^r q(-1 - \epsilon - y) \geq 0.$$

We have the estimates $(1 + \epsilon + y)^{2m} \geq 1$ and $q(-1 - \epsilon - y) \geq \alpha$, and since r is odd,

$$\left(1 - (1 + \epsilon + y)^2 \right)^r = (2 + \epsilon + y)^r (\epsilon + y)^r \geq 2^r (r \epsilon^{r-1} y).$$

(The last inequality follows from $\epsilon, y \geq 0$, and the selection of one term from the binomial expansion of $(\epsilon + y)^r$.) Thus it suffices to show that

$$-y + A 2^r r \epsilon^{r-1} y \alpha = y (A \alpha 2^r r \epsilon^{r-1} - 1) \geq 0,$$

and this follows directly from the definition of A.

11

Now suppose $x \in (-1, 1)$. An easy calculus exercise shows that if a and b are positive and $0 \le t \le 1$, then $t^a(1-t)^b \le \frac{a^a b^b}{(a+b)^{a+b}}$. Using this argument with $t = x^2$, we have

$$F(x) = 1 + \epsilon + x - A(x^2)^m(1-x^2)^r q(x) \ge \epsilon - A\frac{m^m r^r}{(m+r)^{m+r}}\beta.$$

But $\frac{m}{m+r} < 1$, and since $m + r > \frac{r}{2\epsilon}\left(\frac{\beta}{r\alpha}\right)^{1/r}$, we have

$$\epsilon - A\frac{m^m r^r}{(m+r)^{m+r}}\beta > \epsilon - A\beta\left(\frac{r}{m+r}\right)^r = \epsilon\left(1 - \frac{\beta}{r\alpha}\left(\frac{r}{2\epsilon(m+r)}\right)^r\right) \ge 0.$$

$\square$

Remark. For $A \in \mathbb{R}^+$ and $m \in \mathbb{N}$, let $F(m, A; x) := 1 + \epsilon + x - Ax^{2m}h(x)$. We have not attempted to be precise in the calculation of m, and as one might suspect, for particular examples $F(m, A; x)$ will be psd for a smaller value of m than the one asserted above. Note for example that the argument of the last theorem works with the value of m reduced roughly by r, provided ϵ is small enough that this integer is non-negative.

Consider $h(x) = 1 - x^2$, for which, as we've seen, no additional machinery is required to prove Schmüdgen's Theorem. The following identity shows that $m = 0$ will actually work: if $a > 1$, then $a \pm x = \frac{a}{2}(x \pm \frac{1}{a})^2 + \frac{a^2-1}{2a} + \frac{a}{2}(1-x^2)$. If we take $h(x) = (1-x^2)^3$, then $r = 3$ and $q(x) = 1$, so $\alpha = \beta = 1$, and the bound for m is $\frac{3^{2/3}}{2\epsilon} - 3 \approx \frac{1.04}{\epsilon} - 3$. More precise results can be computed in this specific case. It can be shown, for example, that the minimum of $x - (1-x^2)^3$ on $\mathbb{R}$ is approximately -1.11987, hence, for example, with $\epsilon = .12$, $F(0, 1; x) = 1.12 + x - (1-x^2)^3$ is psd. On the other hand, set $\epsilon = .01$ and consider $F(0, A; x) = 1.01 + x - A(1-x^2)^3$. If $F(0, A; 0) \ge 0$, then $1.01 \ge A$, but if $F(0, A; -1.02) \ge 0$, then $A(.0404)^3 \ge .01$, hence $A > 150$. Thus, $F(0, A, x)$ is not psd for any A. (This does *not* imply that $1.01 + x \notin P((1-x^2)^3)$, merely that there is no expression of this simple form.) We can repeat this argument for $m = 1$: $F(1, A; -.6) \ge 0$ implies $A \le 4.3445...$ and $F(1, A; -.1.02) \ge 0$ implies $A \ge 145.764...$ These arguments can be repeated to show that if $1.01 + x - Ax^{2m}$ is psd, then $m \ge 11$, and a numerical argument shows that for all x,

$$F(11, 140; x) = 1.01 + x - 140x^{22}(1-x^2)^3 \ge .0002 > 0.$$

The following quantitative special case of Schmüdgen's Theorem contains degree information unavailable in the original. Observe that, if p is positive on $[-1, 1]$, then by continuity, it is positive on $[-1 - \epsilon, 1 + \epsilon]$ for some $\epsilon > 0$.

Corollary 8. *Suppose $h(x) = (1 - x^2)^r q(x)$, where $0 \le q(x) \le \beta$ for $|x| \le 1$ and $q(x) \ge \alpha > 0$ for $|x| \ge 1$. Suppose $p(x) \in Pd([-1, 1])$ has degree n, and p has s real roots u_j, with $|u_j| \ge u > 1$. Then there exist polynomials $F, G \in \Sigma$, so that*

$$p = F + G\,h,$$

12

where the degrees of F and G are bounded above by

$$s\left(\frac{r}{2(u-1)}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{r}} + \deg h\right) + n - s.$$

Proof. Factor p over $\mathbb{R}[x]$ into linear factors $x-u_j$ and irreducible quadratic factors $q_k \in \Sigma$. Then $|u_j| \geq u > 1$. Observe that $\pm(x - u_j) = |u_j| - sgn(u_j)x$, with the sign chosen so that the factor is positive on $[-1, 1]$; thus,

$$p(x) = \prod_{j=1}^{s}\Big(|u_j| - sgn(u_j)x\Big)\prod_{k=1}^{\frac{n-s}{2}} q_k(x).$$

Now use Theorem 6 to write $|u_j| - sgn(u_j)x = F_j + hG_j$, with the degrees $\leq \frac{r}{2(|u_j|-1)}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{r}} + \deg h \leq \frac{r}{2(u-1)}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{r}} + \deg h$, and substitute above. $\qquad\square$

There is no degree dependence in Fekete's Theorem on the location of the roots of p. On the other hand, Stengle has shown [21, p. 170] that there is a constant C such that given $s_i \in \Sigma$ with

$$1 - x^2 + \epsilon = s_0(x) + s_1(x)(1 - x^2)^3,$$

then $\deg s_i \geq C\epsilon^{-1/2}$. He also [21, p. 171] constructed such a representation in which $\deg s_i \leq C\epsilon^{-1/2}|\log \epsilon|$. This implies that if $r \geq 3$, then there is no bound for the degrees of F and G which depends solely on the degree of p and information about h. It also suggests that a better construction might reduce the exponent on $u - 1$ from 1 to $\frac{1}{2}$.

Corollary 8 states that $\mathrm{Pd}([-1, 1]) \subseteq P(h)$. In view of the results of Fekete and Lukacs, as well as Corollary 3, it is natural to wonder when we have the stronger result that $\mathrm{Psd}([-1, 1]) = P(h)$. We are able to answer that question completely: this happens if and only if $h(x) = (1-x^2)q(x)$, where q is as before, and $q(\pm 1) > 0$. Our proof relies on a very recent result of Scheiderer [19, 4.8].

Lemma 9 (Scheiderer). *Suppose $f, g \in \mathbb{R}[x]$ satisfy the following conditions:*

1. *f and g are relatively prime,*

2. *$\{x \mid f(x) \leq 0\} \cap \{x \mid g(x) \leq 0\} = \emptyset$, and*

3. *$S(f, g)$ is bounded.*

Then there exist $s, t \in \Sigma$ such that $sf + tg = 1$.

Proposition 10. *Suppose $h \in \mathbb{R}[x]$ such that $S(h) = [-1, 1]$ and h has a root of multiplicity a at $x = 1$ and a root of multiplicity b at $x = -1$. Then $(1 - x)^a$ and $(1 + x)^b$ are in $P(h)$.*

13

Proof. We give the proof for $(1-x)^a$; the proof for $(1+x)^b$ is similar. Write $h(x) = (1-x)^a q(x)$, and note that a must be odd. We want to apply Lemma 9 to the polynomials $(1-x)^a$ and $q(x)$. They are clearly relatively prime. We have $\{x \mid (1-x)^a \leq 0\} = [1, \infty)$, and $q(x) > 0$ on $[1, \infty)$ by its definition. Hence conditions (2) and (3) of the lemma hold. Thus there exist $s, t \in \Sigma$ so that $1 = s(x)(1-x)^a + t(x)q(x)$. Multiplying both sides by $(1-x)^a$ yields $(1-x)^a = s(x)(1-x)^{2a} + t(x)h(x) \in P(h)$. $\qquad\square$

Remark. The proof of Lemma 9 is not constructive, hence this does not yield a constructive method for finding an explicit representation of $(1-x)^a$ and $(1+x)^b$ in the preorder.

Corollary 11. *Suppose $S(h) = [-1, 1]$ and $p \in Psd([-1, 1])$. Then $p \in P(h)$ if and only if for $y = \pm 1$, the order of p at y is either even or at least as big as the order of h at y. In particular, $P(h) = Psd([-1, 1])$ if and only if the order of h at both ± 1 is 1.*

Proof. Write $h(x) = (1-x)^a(1+x)^b q(x)$, where $q(x) > 0$ for $|x| \geq 1$ and $q(x) \geq 0$ for $|x| < 1$. Every $p \in \mathrm{Psd}([-1, 1])$ can be written as

$$p(x) = (1+x)^c \left(\prod_{j=1}^{r} (x - x_j)^{2k_j} \right) (1-x)^d q(x),$$

where $|x_j| < 1$ and $q \in \mathrm{Pd}([-1, 1])$. If c is even or $c - a \geq 0$ is even and if d is even or $d - b \geq 0$ is even, then by Corollary 8 and Proposition 10, p is a product of factors from $P(h)$, hence $p \in P(h)$.

Now suppose $c < a$ is odd or $d < b$ is odd. The proofs are similar, and we give the first. Suppose we could write $p = s + th$, with $s, t \in \Sigma$. Then $(1+x)^c | p$ and $(1+x)^a | h$, hence $(1+x)^c | s$. But s is psd, so any linear factors appear to an even degree. Thus, $(1+x)^{c+1} | s$, and $c + 1 < a$ implies that $(1+x)^{c+1} | p$, which contradicts the definition of c. $\qquad\square$

There are several other directions in which these results could be generalized. The most obvious is to allow $\{f_1, \ldots, f_n\}$ with $n \geq 2$ so that $S(f_1, \ldots, f_n) = [-1, 1]$. (As noted earlier, any compact interval might as well be $[-1, 1]$, but Schmüdgen's Theorem also applies when the semialgebraic set is a union of closed intervals.) Stengle proved [21] that if $S(f_1, \ldots, f_n) = [-1, 1]$, then there exists a single h in $P(f_1, \ldots, f_n)$ so that $S(h) = [-1, 1]$. His proof requires various non-constructive Stellensätze, and we have been unable to find a constructive proof.

Nevertheless, we can present here some remarks towards a constructive proof. Suppose $S(f_1, \ldots, f_n) = [-1, 1]$. Assume each f_i has a factor of $1 - x^2$. (If not, replace f_i by $(1 - x^2)^2 f_i \in P(f_1, \ldots, f_n)$), so that $f_i = (1 - x^2)g_i$.) There is no real α so that $g_i(\alpha) < 0$ for all i. Our goal is to find psd $p_1, \ldots, p_k$ such that $p_1 g_1 + \cdots + p_k g_k$ is positive definite. Setting $h = \sum_k p_k f_k = (1 - x^2) \sum p_k g_k$, we then have $S(h) = [-1, 1]$, and we can apply Theorem 7.

Example. Suppose $f_1 = (1 - x^2)(x + a)$ and $f_2 = b^2 - x^2$, where $1 < b < a$. Then $S(f_1, f_2) = \{\alpha : f_1(\alpha) \geq 0\} \cap \{\alpha : f_2(\alpha) \geq 0\} = \big((-\infty, -a) \cup [-1, 1]\big) \cap [-b, b] = [-1, 1]$. An algebraic identity shows that

$$
\begin{aligned}
h(x) &= 2b(b^2 - 1)f_1(x) + (1 - x^2)^2 f_2(x) \\
&= 2b(b^2 - 1)(1 - x^2)(x + a) + (1 - x^2)^2(b^2 - x^2) \\
&= (1 - x^2)\big(2b(b^2 - 1)(x + b + (a - b)) + (1 - x^2)(b^2 - x^2)\big) \\
&= (1 - x^2)\big((x + b)^2\big((x - b)^2 + b^2 - 1\big) + 2b(b^2 - 1)(a - b)\big).
\end{aligned}
$$

Since $(x + b)^2\big((x - b)^2 + b^2 - 1\big) + 2b(b^2 - 1)(a - b) \geq 2b(b^2 - 1)(a - b) > 0$, we have $S(h) = [-1, 1]$. In other words, we have used the above construction with $g_1 = x + a$, $g_2 = (1 - x^2)(b^2 - x^2)$, $p_1 = 2b(b^2 - 1)$, and $p_2 = 1$.

It is difficult to apply Theorem 7 directly to h in this case, because of the computation of α and β. For concreteness, set $a = 3$ and $b = 2$, so that $2b(b^2 - 1) = 12$, $h(x) = (1 - x^2)(x^4 - 5x^2 + 12x + 40)$ and $r = 1$. It is routine to verify that $q(x) = x^4 - 5x^2 + 12x + 40$ is increasing on $[-1, 1]$, hence $\beta = h(1) = 48$, and $q(x) = 12 + (x + 2)^2(x^2 - 4x + 7)$, hence $\alpha = 12$. If we set $\epsilon = 1$, then Theorem 7 tells us that, with $A = \frac{1}{12}$ and $m = 1$,

$$
H(x) := 2 + x - \tfrac{1}{12}x^2(1 - x^2)(x^4 - 5x^2 + 12x + 40)
$$

is psd. (This can easily be confirmed by graphing it.) That is, we can explicitly write $2 + x \in \mathrm{Pd}([-1, 1])$ as an element of $P(f_1, f_2)$:

$$
2 + x = H(x) + \tfrac{1}{12}x^2 h(x) = H(x) + 12x^2 f_1(x) + \tfrac{1}{12}x^2(1 - x^2)^2 f_2(x).
$$

5   The semi-infinite interval

A simple example shows that there exists h so that $S(h) = [0, \infty)$, but $\mathrm{Pd}([0, \infty))$ is not contained in $P(h)$.

Example. Observe that $S(x^3) = [0, \infty)$, and that $1 + x \in \mathrm{Pd}([0, \infty))$. But if f $1 + x \in P(x^3)$, then there would exist $g, h \in \Sigma = \mathrm{Psd}(\mathbb{R})$ so that

$$
1 + x = g(x) + x^3 h(x).
$$

Observe that the degrees of g and h are even. Hence, the degrees of $g(x)$ and $x^3 h(x)$; namely, $2m$ and $2n + 3$, are different, and hence the degree of their sum is $\max\{2m, 2n + 3\}$, which must equal 1, a clear contradiction.

This example generalizes considerably. We first need a familiar folk-lemma.

Lemma 12. *Suppose $-\infty < a < b < \infty$ are given. Then for every positive integer n, there exists $C(a, b, n)$ so that, if $|\sum_{j=0}^{n} a_j x^j| \leq M$ for $x \in [a, b]$, then $|a_j| \leq MC(a, b, n)$.*

Proof. Let $p(x) = \sum_{j=0}^{n} a_j x^j$ and choose nodes $a = x_0 < x_1 < \cdots < x_n = b$. The following system of $n+1$ equations in the $n+1$ unknowns $\{a_j\}$:

$$p(x_k) = \sum_{j=0}^{n} x_k^j a_j.$$

has Vandermonde determinant $\prod_{0 \le j < k \le n}(x_j - x_k) \ne 0$, hence there exist $c_{j,k}$ so that

$$a_j = \sum_{k=0}^{n} c_{j,k} p(x_k).$$

The assertion is now immediate. $\qquad\square$

Remark. We note the following useful property of $P(f_1, \ldots, f_n)$ in the case where $S(f_1, \ldots, f_n) = [0, \infty)$: Each f_i and each $s \in \Sigma$ has positive leading coefficient. Thus, there can be no cancellation in the highest degree terms. In particular, if $p = \sum f_I s_I$, then for each I, $\deg p \ge \deg s_I f_I$.

Theorem 13. *Suppose $[0, \infty) = S(f_1, \ldots, f_n)$ and suppose $\{p_m\} \subset P(f_1, \ldots, f_n)$ satisfy $\deg p_m = \deg p$ and $p_m \to p$ coefficientwise. Then $p \in P(f_1, \ldots, f_n)$.*

Proof. Write

$$p_m(x) = s_{0,m}(x) + \sum_{I \subset \{1,\ldots,n\}} s_{I,m}(x) \prod_{i \in I} f_i(x), \tag{9}$$

with $s_{0,m}, s_{I,m} \in \Sigma$, and any summand which is zero has been deleted from the sum. For each m, $\deg p = \deg p_m = \max\{\deg s_{I,m} + \sum_{i \in I} \deg f_i\}$, hence there is a uniform bound on the degrees of $\deg s_{I,m}$. Each $f_i(x)$ is a polynomial which is non-negative on $[0, \infty)$, hence there exists an interval $(0, \lambda_i)$ on which $f_i(x) > 0$. Let $\lambda = \min_i \lambda_i$. Then there exists $\epsilon > 0$ so that $f_i(x) \ge \epsilon$ for $x \in [\frac{\lambda}{3}, \frac{2\lambda}{3}] := [a, b]$. Since $p_m \to p$, there exists T so that $p_m(x) \le T$ for all m and $x \in [a, b]$. It follows from (9) that, for every subset I, and $x \in [a, b]$,

$$T \ge p_m(x) \ge s_{I,m}(x) \prod_{i \in I} f_i(x) \ge \epsilon^{|I|} s_{I,m}(x).$$

Thus, $s_{I,m}(x) \le \epsilon^{-|I|} T$ for $x \in [a, b]$, and $\deg s_{I,m} \le \deg p$. It follows from Lemma 12 that there is a uniform upper bound to the coefficients of $s_{I,m}(x)$. Thus there exists a convergent subsequence $\{m_r\}$ so that, for each I, $s_{I,m_r}(x) \to s_I(x)$ coefficientwise. We conclude from (9) that

$$p(x) = s_0(x) + \sum_{I \subset \{1,\ldots,n\}} s_I(x) \prod_{i \in I} f_i(x).$$

$\qquad\square$

Corollary 14. *Suppose $S(f_1, \ldots, f_n) = [0, \infty)$ and $\epsilon + x \in P(f_1, \ldots, f_n)$ for every $\epsilon > 0$. Then $f_j(x) = cx$ for some j and positive c.*

Proof. In view of Theorem 13, it suffices to show that $x \in P(f_1, \ldots, f_n)$ implies that some f_j is a multiple of x. Suppose otherwise, and write

$$x = s_\emptyset(x) + \sum_{I \subset \{1,\ldots,n\}} s_I(x) \prod_{i \in I} f_i(x).$$

Each summand has degree ≤ 1. Since $s_I \in \Sigma$, it must be a non-negative constant: $s_I(x) = \sigma_I$. Further, if $\sigma_I > 0$, then $\deg \prod_{i \in I} f_i(x) \leq 1$, so I is a singleton, $I = \{j\}$, where $f_j(x) = a_j x + b_j$ is linear. Since $f_j(x) \geq 0$ on $[0, \infty)$, we have $a_j, b_j \geq 0$, and $b_j > 0$ by hypothesis. After a suitable relabeling, we obtain the identity

$$x = \sigma_0 + \sum_{j=1}^{r} \sigma_j (a_j x + b_j).$$

By setting $x = 0$, we conclude that $\sigma_0 = \sigma_1 = \cdots = \sigma_r = 0$, a contradiction. $\qquad \square$

Example. Let $p(x) = x + x^3$; clearly, $S(p) = [0, \infty)$. We show that, for every odd integer m, and ϵ sufficiently small, $x^m + \epsilon \notin P(p)$. It suffices to show by induction on odd m that $x^m \notin P(p)$. Suppose otherwise. There would exist $g, h \in \Sigma$ so that

$$x^m = (x + x^3)g(x) + h(x).$$

Observe that $m = \max\{3 + \deg g, \deg h\}$, where $\deg g, \deg h$ are even. This is impossible for $m = 1$. Suppose we have shown this impossible for $m - 2$, where $m \geq 3$. We have $h(0) = 0$, and so $x | h(x)$, and since $h \in \Sigma$, it follows that $h(x) = x^2 \bar{h}(x)$ for some $\bar{h} \in \Sigma$. Thus,

$$x^{m-1} = (1 + x^2)g(x) + x\bar{h}(x).$$

But this implies that $g(0) = 0$, so $x | g(x)$ and so $g(x) = x^2 \bar{g}(x)$, with $\bar{g} \in \Sigma$, hence

$$x^{m-2} = (x + x^3)\bar{g}(x) + \bar{h}(x),$$

which contradicts the induction hypothesis.

Added in proof: Very recently, we have discovered that there is a gap in the proof of [13, Theorem 1.1]: The procedure given to find the maximum of the linear form $\sum x_i$ on the region G is reduced to finding the maximum on another region G', however the containment $G \subseteq G'$ is wrong. For example, with $F(x, y) = x^2 - 1.9xy + y^2$, the region G is $\{(x, y) \mid x^2 + y^2 - 1.9(x + 2)(y + 2) \leq 0\}$ while G' is $\{(x, y) \mid x^2 - 1.9xy + y^2 - 2(1.9x + 1.9y) \leq 0\}$. However, we have been able to find an alternate proof of the bound in [13] for the two variable case, in fact we can improve the bound slightly.

Theorem 15. *Suppose $f(x, y)$ is a homogeneous polynomial of degree d whose coefficients are bounded above by L and suppose $p(x, y) \geq \lambda > 0$ for $x, y \geq 0, x + y = 1$. If*

$$r + d \geq \sigma d^2 \frac{L}{\lambda},$$

where σ is such that $2\sigma > e^{1/\sigma}$ (for example, if $\sigma = 2$), then the coefficients of $(x + y)^r f(x, y)$ are positive.

Substituting this theorem for Proposition 4, we obtain our results. The theorem above, and its generalization to n variables, will be the subject of a forthcoming paper.

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