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Sums of powers in rings and the real holomorphy ring

Jutta and Martin Kneser in Dankbarkeit gewidmet

By *Eberhard Becker* at Dortmund and *Victoria Powers* at Atlanta

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Introduction

The real holomorphy ring of a formally real field K is by definition the intersection of all valuation rings of K with a formally real residue class field. It is used extensively in the study of formally real fields, especially for quadratic forms and sums of $2n$-th powers, and in real algebraic geometry, see for example [Sch], [B1]–[B4], [KS], III, § 12. Several authors have begun a theory of the real holomorphy ring of a commutative ring, for instance K. G. Valente [V], N. Schwartz [Sch], M. Marshall [M], and M. Prechtel [P]. Also, M. Knebusch and C. Scheiderer introduce in their book [KS] a relative real holomorphy ring (although they do not call it this).

Following these examples we define the real holomorphy ring of a commutative ring R with 1 as the subring $H(R)$ of elements which are globally finite on $\operatorname{Sper} R$. With this definition we can carry much of the field theory over to $H(R)$. We obtain particularly complete results in the case where each element in $1 + \Sigma R^2$ is a unit in R: We show that $H(R)$ is a Prüfer ring in R, and establish a correspondence between the ideal – and unit – theory of $H(R)$ and the units of R which are sums of $2n$-th powers. This correspondence allows us to study quantitative problems on sums of $2n$-th powers in R.

When the elements of $1 + \Sigma R^2$ are not units our results are not as complete, but there are still new and interesting phenomena: We show in § 1 that we have a representation $H(R) \to C(X, \mathbb{R})$, where X is a suitable compact topological space. In general, one cannot interpret X using places and valuation theory as in the case where $1 + \Sigma R^2 \subseteqq R^*$. In the general case, the sums of $2n$-th powers are replaced by the elements of R which are non-negative with respect to all orderings of level n in R.

Special attention is given to affine algebras A and their quotient algebras A_S over suitable fields k. For example, if $k = \mathbb{R}$, the holomorphy ring $H(A)$ is investigated by appealing to the geometric properties of A. Setting $H^{n+1}(A) = H(H^n(A))$, $H^1(A) = H(A)$ we prove $H^{d+1}(A) = H^d(A)$ if $\dim A = d$.

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§ 1. The real holomorphy ring

Throughout the paper, R denotes a commutative ring with 1. The real spectrum of R, denoted $\operatorname{Sper} R$, is the set of (prime) orderings of R, i.e. the subsets $P \subseteqq R$ that satisfy the following properties (see [B4], [BCR], Ch. 7, [KS], III, [L], § 4):

$$P + P \subseteqq P, \quad P \cdot P \subseteqq P, \quad P \cup -P = R, \quad P \cap -P \in \operatorname{Spec} R.$$

Alternatively, the elements of $\operatorname{Sper} R$ can be defined as pairs $(\wp, \bar{P})$, where $\wp \in \operatorname{Spec} R$ and $\bar{P}$ is an ordering of $k(\wp) :=$ the quotient field of $R/\wp$. The identification is given by

$$(\wp, \bar{P}) \leftrightarrow P := \{a \in R \mid a + \wp \in \bar{P}\},$$

in which case $\wp = P \cap -P =: \operatorname{supp}(P)$. We write $\operatorname{Min} \operatorname{Sper} R$ (respectively $\operatorname{Max} \operatorname{Sper} R$) for the elements of $\operatorname{Sper} R$ minimal (resp. maximal) with respect to inclusion. Of course, if K is a field then $\operatorname{Min} \operatorname{Sper} K = \operatorname{Max} \operatorname{Sper} K = \operatorname{Sper} K$. If $P \subseteqq Q$ we say Q is a *specialization* of P and P is a *generalization* of Q.

An element $a \in R$ induces a mapping

$$\operatorname{Sper} R \to \coprod_{P \in \operatorname{Sper} R} k(\operatorname{supp}(P)),$$

where $P \mapsto a + \operatorname{supp}(P) =: a(P)$.

We write $a(P) > 0$ (resp. $a(P) \geqq 0$, $a(P) = 0$), if $P = (\wp, \bar{P})$ satisfies $a + \wp >_{\bar{P}} 0$ (resp. $a + \wp \geqq_{\bar{P}} 0$, $a + \wp = 0$). As is well-known, the family $\{D(a)\}_{a \in R}$ with

$$D(a) = \{P \in \operatorname{Sper} R \mid a(P) > 0\}$$

defines a subbasis of a compact topology of $\operatorname{Sper} R$. Note: Here, and throughout the paper, the term "compact" does not imply Hausdorff.

Definition. Given $X \subseteqq \operatorname{Sper} R$.

(i) We say $a \in R$ is *bounded on* X if there exists some $k \in \mathbb{N}$ such that $(k - a)(P) \geqq 0$ and $(k + a)(P) \geqq 0$ for all $P \in X$.

(ii) Set $H(X) := \{a \in R \mid a \text{ bounded on } X\}$.

(iii) The *real holomorphy ring of* R is $H(\operatorname{Sper} R)$, denoted $H(R)$.

Remark. A more general notion of the relative real holomorphy ring of an extension of rings $f\colon B \to R$ is defined in [Sch], 7.3, [P], 1.1 and [KS], III, § 11. Note that in our case $B = \mathbb{Z}$. Most of the basic results on the real holomorphy ring can be found in [KS], III, §11.

Definition. Suppose $P \in \operatorname{Sper} R$. If a is bounded on $\{P\}$, then we say a is *bounded with respect to P.*

One can show directly that $H(X)$ is a ring or deduce it from Corollary 1.2.

Proposition 1.1. *Given* $X \subseteq \operatorname{Sper} R$.

(i) *Suppose X is compact. Then* $a \in R$ *is bounded on X iff a is bounded with respect to each* $P \in X$.

(ii) *If* $\bar{X}$ *is the closure of X, then* $H(\bar{X}) = H(X)$.

Proof. (i) We prove the nontrivial direction. Suppose a is bounded with respect to each $P \in X$. Then, given P, there exists $r_P \in \mathbb{N}$ such that $(r_P \pm a)(P) \geqq 0$. Replacing r_P by $r_P + 1$, we can assume $(r_P \pm a)(P) > 0$. Thus $(r_P^2 - a^2)(P) > 0$ and hence $X \subseteq \bigcup_P D(r_P^2 - a^2)$. Since X is compact we have $X \subseteq \bigcup_{i=1}^{m} D(r_i^2 - a^2)$ for some $r_1, \ldots, r_m \in \mathbb{N}$. Let $r = \max\{r_i\}$, then clearly $X \subseteq D(r^2 - a^2)$. Hence, given any P, since $(r^2 - a^2)(P) > 0$ and $r > 0$ it follows that $(r - a)(P) > 0$ and $(r + a)(P) > 0$.

(ii) $H(\bar{X}) \subseteq H(X)$ is clear. Given $a \in H(X)$ and $Q \in \bar{X}$. Pick $n \in \mathbb{N}$ such that $(n \pm a)(P) > 0$ for all $P \in X$. Suppose $(n + a)(Q) < 0$, then $(-n - a)(Q) > 0$ and so $Q \in D(-n - a)$. But then $D(-n - a) \cap X \neq \emptyset$ which contradicts $(n + a)(P) > 0$ for all $P \in X$. Thus $(n + a)(Q) \geqq 0$, and similarly $(n - a)(Q) \geqq 0$. Hence a is bounded with respect to Q and so a is bounded on $\bar{X}$ by (i). □

Definition. For $P \in \operatorname{Sper} R$ we set:

(i) $P^+ := P \backslash (-P) = \{a \in R \mid a(P) > 0\}$,

(ii) $A(P) = \{a \in R \mid a$ is bounded with respect to $P\}$.

Since $A(\bar{P})$ is a ring (in fact, a valuation ring), it follows from the identity

$$A(P) = \{a \in R \mid a(P) \in A(\bar{P})\}$$

that $A(P)$ is a ring.

For $\wp \in \operatorname{Spec} R$, let $\pi_\wp$ denote the canonical map $R \to k(\wp)$. As asual, we say $\wp$ is *real* if $k(\wp)$ is formally real.

Corollary 1.2. (i) $H(R) = \bigcap_{P \in \operatorname{Sper} R} A(P)$,

(ii) $H(R) = \bigcap_{\substack{\wp \in \operatorname{Spec} R \\ \wp \text{ real}}} \pi_\wp^{-1} H(k(\wp))$,

(iii) *$H(R)$ is integrally closed in R.*

Proof. (i) follows from 1.1, since $\operatorname{Sper} R$ is compact. Then

$$H(R) = \bigcap_{\wp \text{ real}} \left(\bigcap_{\bar{P} \in \operatorname{Sper} k(\wp)} \pi_\wp^{-1}(A(\bar{P})) \right) = \bigcap_{\wp \text{ real}} \pi_\wp^{-1} H(k(\wp)),$$

which proves (ii). It is well-known that (iii) holds for fields, and thus it holds generally by (ii). □

Definition. (i) $T(R) := \bigcap_{P \in \operatorname{Sper} R} P$, the set of *totally positive* elements of R,

(ii) $T^+(R) := \bigcap_{P \in \operatorname{Sper} R} P^+$, the set of *strictly totally positive* elements of R.

For a further description of $H(R)$, we note the following characterization of elements in $T(R)$ (resp. $T^+(R)$), essentially due to Stengle, see [St], [B4], § 4, [KS], III, § 9, [L], § 7. Clearly $\Sigma R^2 \subseteq T(R)$.

Proposition 1.3. *Given $a \in R$. Then*

(i) *$a \in T(R)$ iff there exists $k \in \mathbb{N}$, $t, t' \in \Sigma R^2$ with $at = a^{2k} + t'$,*

(ii) *$a \in T^+(R)$ iff there exists $t, t' \in \Sigma R^2$ with $at = 1 + t'$.*

Remark. In (ii), t can be replaced by $1 + s$ for some $s \in \Sigma R^2$ as follows: Given $at = 1 + t'$, $t, t' \in \Sigma R^2$, then $a^2 t = a(1 + t')$ and thus

$$a(1 + t + t') = at + a(1 + t') = 1 + (t' + a^2 t),$$

and $s = t' + a^2 t$ does the job.

From the definition of the real holomorphy ring and the proof of 1.1 it follows that

$$\begin{aligned} a \in H(R) &\Leftrightarrow \text{there exists } k \in \mathbb{N} \text{ with } k \pm a \in T(R) \\ &\Leftrightarrow \text{there exists } k \in \mathbb{N} \text{ with } k \pm a \in T^+(R). \end{aligned}$$

The following proposition follows immediately from 1.3:

Proposition 1.4. $H(R) = \{a \in R \mid \exists k, l \in \mathbb{N}$ *and* $t, t' \in \Sigma R^2$ *with*

$$(k^2 - a^2)t = (k^2 - a^2)^{2l} + t'\}$$

$$= \{a \in R \mid \exists k, l \in \mathbb{N} \text{ and } t, t' \in \Sigma R^2 \text{ with } (k^2 - a^2)t = 1 + t'\}.$$

Note that in the above equations, we can replace the condition on $k^2 - a^2$ by the same conditions on $k - a$ and $k + a$ separately, see the proof of 1.1.

The description of $H(R)$ before 1.4 shows that $H(R) \cap T(R)$ and $H(R) \cap T^+(R)$ are archimedean partial orderings of $H(R)$. Further, using 1.3 we get:

Corollary 1.5. (i) *If $\varphi : R \to S$ is a ring homomorphism, then*

$$\varphi(H(R)) \subseteqq H(S), \quad \varphi(T(R)) \subseteqq T(S) \quad \text{and} \quad \varphi(T^+(R)) \subseteqq T^+(S).$$

(ii) *The mappings $R \mapsto (H(R), H(R) \cap T(R))$ and $R \mapsto (H(R), H(R) \cap T^+(R))$ are covariant functors from the category of commutative rings to the category of archimedean partially ordered rings.*

The holomorphy ring $H(K)$ of a field K admits a topological representation $H(K) \to C(M, \mathbb{R})$, where M is the space of real places of K, see [S], [B3], §1, [KS], III, §12. In general, there is a corresponding representation for $H(R)$, however the representation space cannot necessarily be interpreted as a space of "places". Such an interpretation is possible in the case where the elements $1 + \Sigma R^2$ are units in R, see §4.

The representation of $H(R)$ is a consequence of the Representation Theorem of Kadison-Dubois. We use the version in [BS]. As stated above,

$$Q := H(R) \cap T(R) \quad \text{and} \quad Q^+ := (H(R) \cap T^+(R)) \cup \{0\}$$

are archimedean partial orderings of $H(R)$. According to [BS], the topological space

$$X := \operatorname{Hom}((H(R), Q^+), (\mathbb{R}, \mathbb{R}_+))$$

is non-empty and compact. For the representation

$$\Phi : H(R) \to C(X, \mathbb{R}), \quad a \mapsto (\varphi \mapsto \varphi(a))$$

we apply the results from [BS], see also [B3], 1.1:

Proposition 1.6. (i) $\Phi^{-1}(C_+(X, \mathbb{R})) = \{a \in H(R) \mid$ *for each* $n \in \mathbb{N}$,

$$(1 + na) \in Q^+\} = Q.$$

(ii) $\ker \Phi$ *is a radically closed ideal of* $H(R)$.

(iii) $\mathbb{Q} \cdot \Phi(H(R))$ *is dense in* $C(X, \mathbb{R})$.

Remark. Instead of the partial ordering Q^+ we could have used Q in the above. However this would not yield a different representation.

We now take a look at some examples.

Examples 1.7. (i) Suppose K is a formally real field in which all orderings are archimedean. Then clearly $H(K) = K$. The identity $H(R) = R$ also holds if R is an integral domain, $R/\mathbb{Z}$ an integral extension. In this case, $\operatorname{Sper} R = \{(0, P) \mid P \in \operatorname{Sper} K\}$, where K is the quotient field of R. Since K is an algebraic extension of $\mathbb{Q}$, K has only archimedean orderings, thus $H(R) = R$ and $T(R) = R \cap \Sigma K^2 = T^+(R) \cup \{0\}$.

(ii) For any ring R and its polynomial ring we have

$$H(R[x_1, \dots, x_n]) = H(R).$$

This can be shown using the surjection $\operatorname{Sper} R[x_1, \dots, x_n] \to \operatorname{Sper} R$, see e.g. [V], 5.2. In particular, we have

$$H(R[x_1, \dots, x_n]) = R$$

for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

(iii) Let X be a topological space and $R = C(X, \mathbb{R})$. Then $T(R) = C_+(X, \mathbb{R})$ and hence (cf. [M], 4.1)

$$H\big(C(X, \mathbb{R})\big) = \{f: X \to \mathbb{R} \mid f \text{ is bounded}\} =: C^b(X, \mathbb{R}).$$

We need some notations from Algebraic Geometry, which will provide a rich source of examples. In this paper, an affine variety V over $\mathbb{R}$ is a reduced affine scheme over $\mathbb{R}$ of finite type, i.e., $V = \operatorname{Spec} A$, where A is a reduced affine $\mathbb{R}$-algebra. As usual set $\mathbb{R}[V] := A$. Define $V(\mathbb{R}) := \operatorname{Hom}_{\mathbb{R}}(A, \mathbb{R})$, the set of real points of V. There is a natural embedding

$$V(\mathbb{R}) \hookrightarrow \operatorname{Sper} A$$

given by $x \mapsto \{f \mid f(x) \geqq 0\}$. The topology of $\operatorname{Sper} A$ induces the subspace topology on $V(\mathbb{R})$, which is contained in the compact Hausdorff space $\operatorname{Max}\operatorname{Sper} A$. By the Artin-Lang Homomorphism Theorem (see, e.g., [BCR], 4.1.12), $V(\mathbb{R})$ is dense in $\operatorname{Max}\operatorname{Sper} A$ and $\operatorname{Sper} A$. If we have a representation of A, say $A = \mathbb{R}[X]/\mathfrak{A}$, then the evaluation map gives a natural homeomorphism between $\{\alpha \in \mathbb{R}^n \mid f(\alpha) = 0 \text{ for all } f \in \mathfrak{A}\}$ and $V(\mathbb{R})$.

An ideal $\mathfrak{A}$ of R is *real* if $\Sigma a_i^2 \in \mathfrak{A}$ implies $a_i \in \mathfrak{A}$ for all i. R itself is called real if the zero ideal is real. Note that an ideal $\mathfrak{A}$ is real iff $R/\mathfrak{A}$ is real and that a prime ideal $\wp$ is real (in the present sense) iff the residue field $k(\wp)$ is formally real. A variety V is real if $V(\mathbb{R})$ is dense in V. By the Artin-Lang Theorem, V is real iff $\mathbb{R}[V]$ is a real ring.

Proposition 1.8. *Let V be a real affine variety over $\mathbb{R}$, then $H(\mathbb{R}[V]) = H\big(V(\mathbb{R})\big)$.*

Proof. By 1.1, $H\big(V(\mathbb{R})\big) = H\big(\overline{V(\mathbb{R})}\big)$. By density, $\overline{V(\mathbb{R})} = \operatorname{Sper} \mathbb{R}[V]$. Hence $H(\mathbb{R}[V]) = H\big(V(\mathbb{R})\big)$. □

Example 1.9. Let V be an affine variety over $\mathbb{R}$ and let $R = \mathbb{R}[V]_S$, where S is a semigroup of functions that do not vanish on $V(\mathbb{R})$. Then $s(P) \neq 0$ for all $P \in \operatorname{Sper} R$ and $s \in S$, hence the inclusion $\mathbb{R}[V] \to R$ induces a homeomorphism

$$\operatorname{Sper} R \xrightarrow{\text{res}} \operatorname{Sper} \mathbb{R}[V].$$

Then, by 1.8, $H(R) = \{f \in R \mid f \text{ bounded on } V(\mathbb{R})\}$.

Specializing to $R=\mathbb{R}[x_1,\dots,x_n]$, we get $H(R)=\mathbb{R}$, as in 1.7. Localizing at

$$S=\{(1+\Sigma x_i^2)^k \mid k\in\mathbb{N}\},$$

it follows that:

$$H(\mathbb{R}[x_1,\dots,x_n]_S)=\left\{\frac{f(x_1,\dots,x_n)}{(1+\Sigma x_i^2)^k}\,\middle|\,\deg f\leqq 2k\right\}.$$

We list some cases in which we can determine explicitly the representation space X of 1.6:

Examples 1.10. (i) If K is a formally real field, then $X=M(K)$, the space of real places of K, see [B1], 2.17 and [BG2] for a recent survey.

(ii) Suppose $T(R)$ is an archimedean partial ordering of R. Then $H(R)=R$, hence, by 1.5, every homomorphism $\varphi: R\to\mathbb{R}$ (relative to $T(R)$ and $\mathbb{R}_+$) is order preserving. The natural mapping $\operatorname{Hom}(R,\mathbb{R})\to\operatorname{Sper}R$, given by $\varphi\mapsto\varphi^{-1}(\mathbb{R}_+)$, is injective. We have

$$X=\operatorname{Hom}(R,\mathbb{R})\xrightarrow{\ \sim\ }\operatorname{Max}\operatorname{Sper}R,\quad \text{cf. [BG2], 2.9.}$$

Specializing to R satisfying the conditions of example 1.7 (i) above, then the homomorphism $\varphi: R\to\mathbb{R}$ must be injective. Let K be the quotient field of R, then we get

$$X=\operatorname{Hom}(R,\mathbb{R})\cong\operatorname{Hom}(K,\mathbb{R})\cong M(K)$$

(as topological spaces).

(iii) Let Y be a Hausdorff topological space and $R=C(Y,\mathbb{R})$. We saw that

$$H(R)=C^b(Y,\mathbb{R}),$$

and clearly $T(R)=C_+(Y,\mathbb{R})$ and $H(R)\cap T(R)=C_+^b(Y,\mathbb{R})=\{f: Y\to\mathbb{R}\mid f=g^2$ for some $g\in C^b(Y,\mathbb{R})\}$. Thus $X=\operatorname{Hom}(C^b(Y,\mathbb{R}),\mathbb{R})\cong\operatorname{Max}C^b(Y,\mathbb{R})$ and hence $X=\beta Y$, the Stone-Čech compactification of Y.

(iv) Let $R=\mathbb{R}[T]_{1+T^2}$. Then it is easy to see that

$$H(R)=\left\{\frac{f}{(1+T^2)^k}\mid f\in\mathbb{R}[T] \text{ and } \deg f\leqq 2k\right\}=R\cap V_\infty,$$

where V_∞ is the valuation ring of the degree valuation. Let $\mathcal{M}_\infty$ be the maximal ideal of V_∞. Similarly, we see that $\operatorname{Spec}H(R)=\{(0),\mathcal{M}_\infty\cap H(R),\ \mathcal{M}_p\cap H(R),\ p$ an irreducible polynomial in $\mathbb{R}[T], p\neq 1+T^2\}$, where $\mathcal{M}_p=p\mathbb{R}[T]_{(p)}$ is the maximal ideal of the p-adic valuation ring. The localization $H(R)_\wp$, $\wp\in\operatorname{Spec}H(R)\setminus(0)$ is a discrete, trivial on $\mathbb{R}$, valuation ring. Hence $H(R)$ is a Prüfer ring of $\mathbb{R}(T)$ cf. e.g. [G]. Thus we get in a natural way that

$$\begin{aligned}X=\operatorname{Hom}(H(R),\mathbb{R})&\cong\{\mathcal{M}_\infty\cap H(R),\ \mathcal{M}_p\cap H(R) \text{ for } p=T-a, a\in\mathbb{R}\}\\&\cong M(\mathbb{R}(T))\cong\mathbb{P}^1(\mathbb{R}),\end{aligned}$$

see [S].

(v) We reconsider example 1.9. We will use the language of schemes from Algebraic Geometry; for details and results see [H], II. Let R be as in 1.9. Using homogenization of polynomials we have $H(R) = \left\{ \frac{f(x_0, \ldots, x_n)}{(\Sigma_0^n x_i^2)^k} \,\middle|\, f \text{ is homogeneous and } \deg f = 2k \right\}$. In the projective n-space over $\mathbb{R}$, $\mathbb{P}^n_{\mathbb{R}} := \operatorname{Proj} \mathbb{R}[x_0, \ldots, x_n]$, we consider the open affine subvariety $D_+(u)$, where $u = \Sigma_0^n x_i^2$. Then $H(R)$ is the coordinate ring of $D_+(u)$, i.e.,

$$H(R) = \Gamma(D_+(u)),$$

by [H], II, 2.5. This means that $H(R)$ is an affine $\mathbb{R}$-algebra with $\mathbb{P}^n(\mathbb{R})$ as its space of real points, more precisely, we have the natural homeomorphism $\mathbb{P}^n(\mathbb{R}) \to \operatorname{Hom}(H(R), \mathbb{R})$ given by $x \mapsto e_x$, where $e_x\left(\frac{f}{u^k}\right) = \frac{f(x_0, \ldots, x_n)}{u^k}$, with $x = [x_0 : \cdots : x_n]$. We claim that every homomorphism $H(R) \to \mathbb{R}$ is order preserving relative to T and $\mathbb{R}_+$, i.e., $X = \mathbb{P}^n(\mathbb{R})$ is the representation space. This follows from the above identification of $H(R)$ with $\Gamma(D_+(u))$, since this implies $T = \left\{ \frac{f(x_0, \ldots, x_n)}{u^k} \,\middle|\, f \text{ homogeneous and } f \geqq 0 \text{ on } \mathbb{R}^{n+1} \right\}$.

We finish this section by developing some machinery we will need later on when computing $H(R)$ for various types of rings. We are interested in reduction results, i.e., results which will allow us to reduce our calculation of $H(R)$ to simpler rings.

Lemma 1.11. *Suppose* $R = \varinjlim R_i$, *then* $H(R) = \varinjlim H(R_i)$.

Proof. The characterization of $H(R)$ given in 1.4 shows that the natural map $\varinjlim H(R_i) \to H(R)$ is a surjection. □

Given an element $f \in \mathbb{R}$, we have $R_f := \left\{ \frac{a}{f^k} \,\middle|\, k \in \mathbb{N} \right\}$. Then the natural map $R \to R_f$ induces an embedding

$$\operatorname{Sper} R_f \hookrightarrow \operatorname{Sper} R$$

with open image $\{P \mid f(P) \neq 0\}$. We allow f to be nilpotent, in which case $R_f = 0$ and $\operatorname{Sper} R_f = \emptyset$.

If $\mathfrak{A}$ is an ideal in R then $R \to R/\mathfrak{A}$ induces a mapping

$$\operatorname{Sper} R/\mathfrak{A} \hookrightarrow \operatorname{Sper} R$$

with closed image $\{P \mid f(P) = 0 \text{ for all } f \in \mathfrak{A}\}$. We often identify $\operatorname{Sper} R_f$ and $\operatorname{Sper} R/\mathfrak{A}$ with their images in $\operatorname{Sper} R$.

Remark. The expressions $H(\operatorname{Sper} R_f)$ and $H(\operatorname{Sper} R/\mathfrak{A})$ are now somewhat ambiguous depending on whether we view $\operatorname{Sper} R_f$ (resp. $\operatorname{Sper} R/\mathfrak{A}$) as a subset of $\operatorname{Sper} R$ or not. However this is not a problem since $r \in R$ is bounded on $\operatorname{Sper} R_f$ (resp. $\operatorname{Sper} R/\mathfrak{A}$) considered as a subset of $\operatorname{Sper} R$ iff $\frac{r}{1} \in R_f$ (resp. $r + \mathfrak{A} \in R/\mathfrak{A}$) is bounded on $\operatorname{Sper} R_f$ (resp. $\operatorname{Sper} R/\mathfrak{A}$).

The following is now clear:

Lemma 1.12. *Given an ideal* $\mathfrak{A} \subseteq R$, *then there is a natural decomposition*

$$\operatorname{Sper} R = \left(\bigcup_{f \in \mathfrak{A}} \operatorname{Sper} R_f \right) \cup \operatorname{Sper} R/\mathfrak{A}.$$

Definition. Given an ideal $\mathfrak{A}$ in R, the *real radical* of $\mathfrak{A}$, $\sqrt[\mathrm{re}]{\mathfrak{A}}$, is $\{r \in R \mid r^{2m} + \sigma \in \mathfrak{A}$ for some $m \in \mathbb{N}$ and $\sigma \in \Sigma R^2\}$. The *real radical of* R is $\sqrt[\mathrm{re}]{0}$.

Lemma 1.13. *Let* $\mathfrak{A} = \sqrt[\mathrm{re}]{0}$, *then* $H(R) = \{r \in R \mid \bar{r} \in H(R/\mathfrak{A})\}$, *where* $\bar{r}$ *denotes* $r + \mathfrak{A} \in R/\mathfrak{A}$.

Proof. By [L], 6.5, $\operatorname{Sper} R_f = \emptyset$ for all $f \in \mathfrak{A}$. The lemma now follows using the decomposition in 1.12. □

Lemma 1.13 allows us to reduce to the case where R is real. For the rest of this section we assume *R is a real ring*. Then R is a reduced ring and the minimal prime ideals are real [L], 2.9. Given $\wp$ a minimal prime ideal. Then R reduced and $\wp$ minimal implies $k(\wp) = R_\wp$ and the mapping $R \to R_\wp$ induces a mapping

$$(*) \qquad \operatorname{Sper} R_\wp \to \operatorname{Sper} R.$$

Definition. $P \in \operatorname{Sper} R$ is a *central point* if there exists a minimal prime ideal $\wp$ in R and Q in the image of $(*)$ such that P is a specialization of Q, i.e., $Q \subseteq P$. We write $\operatorname{Sper}_c R$ for the set of central points of $\operatorname{Sper} R$, and set $\operatorname{Max}\operatorname{Sper}_c R = \operatorname{Sper}_c R \cap \operatorname{Max}\operatorname{Sper} R$.

Proposition 1.14. *Suppose R has only finitely many minimal prime ideals. Then* $\operatorname{Sper}_c R$ *and* $\operatorname{Max}\operatorname{Sper}_c R$ *are compact.*

Proof. Combining the specialization map $\operatorname{Sper} R \to \operatorname{Max}\operatorname{Sper} R$ with $(*)$ we have, for any minimal prime $\wp$, a mapping

$$\phi_\wp : \operatorname{Sper} R_\wp \to \operatorname{Sper} R \to \operatorname{Max}\operatorname{Sper} R.$$

By [L], 4.7, $\operatorname{Sper} R_\wp$ and $\operatorname{Max}\operatorname{Sper} R$ are compact and $\phi_\wp$ is continuous. Hence $\operatorname{im}\phi_\wp$ the image of $\phi_\wp$, is a compact subspace of $\operatorname{Max}\operatorname{Sper} R$. Note that $\operatorname{im}\phi_\wp \subseteq \operatorname{Sper}_c R$ by definition. It follows that $\operatorname{Max}\operatorname{Sper}_c R$ is compact.

Assume $\operatorname{Sper}_c R \subseteq \bigcup U_i$, where U_i are open sets. Then $\operatorname{Max}\operatorname{Sper}_c R \subseteq \bigcup U_i$, thus there exist $U_1, \ldots, U_r$ such that $\operatorname{Max}\operatorname{Sper}_c R \subseteq U_1 \cup \cdots \cup U_r$. Since each U_i is open, any generalization of $P \in U_i$ is also in U_i. Thus $\operatorname{Sper}_c R \subseteq \{$generalizations of points in $\operatorname{Max}\operatorname{Sper}_c R\} \subseteq U_1 \cup \cdots \cup U_r$ and hence $\operatorname{Sper}_c R$ is compact. □

Proposition 1.15. *Suppose R has only finitely many minimal prime ideals. Then* $R = H(\operatorname{Sper}_c R)$ *iff* $R/\wp \subseteq H(R_\wp)$ *for all minimal prime ideals $\wp$ in R.*

Proof. This follows from the definition of $\mathrm{Sper}_c R$ plus the fact that if Q specializes P, then $f \geqq 0$ on P implies $f \geqq 0$ on Q. □

Definition. (i) The *regular locus* of R, denoted $\mathrm{Reg}\, R$, is the set of prime ideals $\wp$ such that $R_\wp$ is a regular local ring. The *singular locus* of R, denoted $\mathrm{Sing}\, R$, is

$$\{\wp \in \mathrm{Spec\,R} \mid \wp \notin \mathrm{Reg}\, R\}.$$

(ii) $\text{Reg-Sper}\, R := \{P = (\wp, \bar{P}) \in \mathrm{Sper}\, R \mid \wp \in \mathrm{Reg}\, R\}$ and

$$\text{Sing-Sper}\, R := \{P = (\wp, \bar{P}) \in \mathrm{Sper}\, R \mid \wp \in \mathrm{Sing}\, R\}.$$

Proposition 1.16. *Assume R is noetherian. Then* $\text{Reg-Sper}\, R \subseteqq \mathrm{Sper}_c R$.

Proof. Given $P = (\wp, \bar{P}) \in \text{Reg-Sper}\, R$. Given a prime ideal $\wp' \subseteqq \wp$, then

$$R_{\wp'} = (R_\wp)_{(\wp' R_\wp)}$$

and so $\wp'$ is also regular. In particular, let $\wp'$ be the unique minimal prime ideal contained in $\wp$. Then $\wp' R_\wp = 0$, since $R_\wp$ is an integral domain, and hence $R_{\wp'}$ is the quotient field of $R_\wp$. Since $R_\wp$ is regular the natural map $R_\wp \to (R_\wp)/(\wp R_\wp) = k(\wp)$ extends to a place $\lambda: R_{\wp'} \to k(\wp) \cup \infty$, see e.g. [B1]. Let $\bar{Q}$ be a pullback of $\bar{P}$ to $R_{\wp'}$, then P is a specialization of $Q = (\wp', \bar{Q})$, hence $P \in \mathrm{Sper}_c R$. □

§ 2. Integral domains and affine algebras

In this section we take a closer look at integral domains and affine algebras. We fix a real *integral domain* A with quotient field F and set $V = \mathrm{Spec}\, A$. A key question in this case is when does $H(A) = A \cap H(F)$? We give a partial answer to this question.

Note that the inclusion map $i: A \hookrightarrow F$ induces a map

$$\hat{\imath}: \mathrm{Sper}\, F \to \mathrm{Min\ Sper}\, A$$

given by $P \mapsto P \cap A$.

Remark. From the definition given in §1 we see that $P \in \mathrm{Sper}\, A$ is a central point if there exists $Q \in \mathrm{Sper}\, F$ such that $P \supseteqq Q \cap A$, i.e., $\mathrm{Sper}_c A = \{$specializations of elements in image $\hat{\imath}\}$.

Proposition 2.1. $H(\mathrm{Sper}_c A) = H(\mathrm{Max\ Sper}_c A) = A \cap H(F)$.

Proof. Given $f \in A$ then for $n \in \mathbb{N}$, $(n \pm \bar{f})(\bar{P}) > 0$ for all $\bar{P} \in \mathrm{Sper}\, F$ iff $(n \pm f)(P) > 0$ for all $P \in \mathrm{im}\, \hat{\imath}$. Thus $H(\mathrm{Sper}_c A) = A \cap H(F)$.

Clearly $H(\mathrm{Sper}_c A) \subseteqq H(\mathrm{Max\,Sper}_c A)$. Suppose for some $n \in \mathbb{N}$, we have

$$(n \pm f)(Q) > 0$$

for all $Q \in \mathrm{Max\,Sper}_c A$. Given $P \in \mathrm{Sper}_c A$, let $Q \in \mathrm{Max\,Sper}\,A$ be a specialization of P. Then $Q \in \mathrm{Max\,Sper}_c A$, hence $(n \pm f)(P) > 0$. Thus $H(\mathrm{Max\,Sper}_c A) \subseteqq H(\mathrm{Sper}_c A)$. □

Proposition 2.2. *$\bar{P}$ is archimedean for all $P = (\wp, \bar{P}) \in \mathrm{Max\,Sper}_c A$ iff $A \subseteqq H(F)$.*

Proof. This follows from 1.14, 1.1(i) and 2.1. □

Proposition 2.3. *Suppose A is noetherian, $\mathrm{Sing}\,A$ is closed and $A \subseteqq H(F)$. Then the quotient field of $H(A)$ is F and either $H(A) = A$ or $H(A)$ is not noetherian.*

Proof. Let $\mathfrak{A} = \bigcap \wp$, where the intersection is over all $\wp \in \mathrm{Sing}\,A$, and set $\mathfrak{B} = \sqrt[\mathrm{re}]{\mathfrak{A}}$. Then $\mathfrak{B} \neq (0)$ since $(0) \in \mathrm{Reg}\,A$. We have $\mathrm{Sper}\,A = \mathrm{Sper}_c A \cup \mathrm{Sper}\,A/\mathfrak{B}$. By 2.1, $A \subseteqq H(\mathrm{Sper}_c A)$, hence

$$H(A) = \{a \in A \mid \bar{a} \in H(A/\mathfrak{B})\}.$$

Given $b \in \mathfrak{B} \backslash (0)$ and any $a \in A$, then $b, ab \in H(A)$, hence $a \in \mathrm{quot}(H(A))$. Thus $\mathrm{quot}(H(A)) = F$.

Suppose $H(A)$ is noetherian, then $\mathfrak{B}$ is finitely generated as an $H(A)$-ideal, say $\mathfrak{B} = (b_1, \ldots, b_r)$. Pick any $a \in A$. Then $ab_i = \Sigma \alpha_{i,j} b_j$, where $\alpha_{i,j} \in H(A)$. Thus

$$(*) \qquad \Sigma(-\alpha_{i,j} + a\delta_{i,j}) b_j = 0 .$$

Let ϕ be the characteristic polynomial of $(\alpha_{i,j})$, then applying Cramer's rule to $(*)$ we get $\phi(a) b_j = 0$ for all j. Since some $b_j \neq 0$ we get $\phi(a) = 0$, i.e., a is integral over $H(A)$. Hence $a \in H(A)$ by 1.2 and therefore $H(A) = A$. □

If A is a k-algebra, let $\mathrm{tr}(A \mid k)$ denote the transcendence degree of A over k.

Proposition 2.4. *Suppose A contains a totally-archimedean field k. Then if*

$$\mathrm{tr}(F \mid k) \leqq 1,$$

we have $H(A) = A \cap H(F)$.

Proof. A has dimension $\leqq 1$. Note that if $\wp$ is a prime ideal in A with height d and $\mathrm{tr}(A/\wp \mid k) = e$, then $\mathrm{tr}(A \mid k) \geqq d + e$ ([Ku], II, 3.6). Hence if $\wp \neq \{0\}$ is a prime ideal in A, then $\wp$ is maximal and $A/\wp$ is algebraic over k. Thus $A/\wp \subseteqq H(A/\wp)$ and so every $a \in A$ is bounded with respect to $P = (\wp, \bar{P})$ if $\wp \neq \{0\}$. This implies $H(A) = H(\mathrm{Sper}_c A)$ and we are done by 2.1. □

For the rest of this section we assume that A is an *affine integral domain* over $\mathbb{R}$ and $F = \mathrm{quot}(A)$ is formally real. Let V be the associated real affine variety, as defined in §1.

Definition. Set $V_{\text{reg}}(\mathbb{R}) = V(\mathbb{R}) \cap \text{Reg-Sper}\,A$, the *regular real points* of V,

$$V_{\text{sing}}(\mathbb{R}) = V(\mathbb{R}) \cap \text{Sing-Sper}\,A\,,$$

the *singular real points* of V, and $V_c(\mathbb{R}) = V(\mathbb{R}) \cap \text{Sper}_c A$, the *central real points* of V.

Proposition 2.5. (i) $\text{Sper}_c A \subseteqq \overline{V_{\text{reg}}(\mathbb{R})}$, $V_c(\mathbb{R}) = \overline{V_{\text{reg}}(\mathbb{R})} \cap V(\mathbb{R})$.

(ii) $H(V_c(\mathbb{R})) = A \cap H(F)$.

Proof. The first statement in (i) is a result by Dubois and Efroymson rephrased in terms of the real spectrum. The proof follows from the Artin-Lang homomorphism theorem and is given in [B4], 3.2. We deduce $V_c(\mathbb{R}) = V(\mathbb{R}) \cap \text{Max Sper}_c A \subseteqq V(\mathbb{R}) \cap \overline{V_{\text{reg}}(\mathbb{R})}$. On the other hand, $\text{Max Sper}_c A$, being compact, is a closed subspace of $\text{Max Sper}\,A$. Hence $V(\mathbb{R}) \cap \text{Max Sper}_c A$ is closed in $V(\mathbb{R})$ and $V_{\text{reg}}(\mathbb{R}) \subseteqq V_c(\mathbb{R})$. This, together with 1.16, yields $\overline{V_{\text{reg}}(\mathbb{R})} \cap V(\mathbb{R}) \subseteqq V_c(\mathbb{R})$. From $V_{\text{reg}}(\mathbb{R}) \subseteqq V_c(\mathbb{R}) \subseteqq \text{Sper}_c A \subseteqq \overline{V_{\text{reg}}(\mathbb{R})}$ we derive

$$H(\text{Sper}_c A) = H(V_c(\mathbb{R}))\,.$$

(ii) now follows from 2.1. □

Proposition 2.6. $H(A) = A$ *iff* $V(\mathbb{R})$ *is compact.*

Proof. Choose a representation $A = \mathbb{R}[x_1, \ldots, x_n]/\mathfrak{A}$. Then $V(\mathbb{R}) = \{\alpha \in \mathbb{R}^n \mid g(\alpha) = 0$ for all $g \in \mathfrak{A}\}$. Set $f = \sum_1^n \bar{x}_i^2$. If $H(A) = A$, then $f \in H(A)$ which means that f is bounded on $V(\mathbb{R})$. Hence $V(\mathbb{R})$ is closed and bounded in $\mathbb{R}^n$, thus compact.

Now suppose $V(\mathbb{R})$ is compact. Given $f \in A$, then f is bounded, since it is a continuous function on a compact set $V(\mathbb{R})$. Thus $f \in H(V(\mathbb{R}))$. Hence, by 1.8, $H(A) = A$. □

We now look at the classical notion of central points in $V(\mathbb{R})$.

Definition. Suppose $\mathscr{V}$ is a valuation ring in F with maximal ideal $m_{\mathscr{V}}$ and $A \subseteqq \mathscr{V}$. We say $\mathscr{V}$ *has center* $\wp$ *on* A if $A \subseteqq V$ and $\wp = m_{\mathscr{V}} \cap A$.

Proposition 2.7. *A prime ideal* $\wp \subseteqq A$ *is the center of a real valuation ring* $V \subseteqq F$ *iff* $\wp$ *is the support of a point in* $\text{Sper}_c A$.

Proof. Given $P \in \text{Sper}_c A$ with $\wp = \text{supp}(P)$. Then there is some $Q \in \text{Sper}\,F$ such that $P \supseteqq Q \cap A$. Let $(F, \bar{Q})$ be a real closure of Q. By [B4], 2.9, there is a place $\lambda: F \to L \cup \{\infty\}$, for some real closed L, extending $\pi: A \to A/\wp$. Let $\mathscr{V} \subseteqq F$ be the valuation ring associated to λ. Then $m_{\mathscr{V}} \cap A = \{a \in A \mid \lambda(a) = 0\} = \wp$.

Conversely, suppose $\wp = m_{\mathscr{V}} \cap A$ for some real valuation ring $\mathscr{V} \subseteqq F$ with $A \subseteqq \mathscr{V}$. Then there is some $Q \in \text{Sper}\,F$ such that $A(Q) \subseteqq \mathscr{V}$. Let $\tilde{Q}$ be the pushdown of Q along $\mathscr{V}$, i.e., the image of Q in $\mathscr{V}/m_{\mathscr{V}}$. It is well-known that $\tilde{Q}$ is an order. We have the canonical map

$$\phi: A \to A/\wp \hookrightarrow \mathscr{V}/m_{\mathscr{V}}\,.$$

Set $P = \phi^{-1}(\tilde{Q})$, then $\text{supp}(P) = \wp$ and $P \in \text{Sper}_c A$ since P specializes $Q \cap A$. □

Definition. We say A is *real complete* if every real valuation ring on F has a center on A.

Theorem 2.8. *The following are equivalent*:

(i) *A is real complete.*

(ii) $A \subseteqq H(F)$.

(iii) $V_c(\mathbb{R}) = \operatorname{Max}\operatorname{Sper}_c A$.

(iv) *$V_c(\mathbb{R})$ is compact.*

Proof. (i) $\Rightarrow$ (ii) If A is real complete, then $A \subseteqq A(P)$ for each $P \in \operatorname{Sper} F$. Hence $A \subseteqq H(F)$.

(ii) $\Rightarrow$ (iii) Assume $A \subseteqq H(F)$ and consider $P \in \operatorname{Sper} F$. We have $A \subseteqq A(P)$ by assumption which gives us a natural map $\phi: A \hookrightarrow A(P) \to \mathbb{R}$. Clearly $\phi \in V(\mathbb{R})$ and it is easy to check that ϕ maps to the maximal specialization of $(0, P) \in \operatorname{Sper} A$ under the natural embedding $V(\mathbb{R}) \hookrightarrow \operatorname{Sper} A$. Thus $\operatorname{Max}\operatorname{Sper}_c A \subseteqq V(\mathbb{R})$ and hence $V_c(\mathbb{R}) = \operatorname{Max}\operatorname{Sper}_c A$.

(iii) $\Rightarrow$ (iv) $\operatorname{Max}\operatorname{Sper}_c A$ is compact, by 1.14.

(iv) $\Rightarrow$ (i) Given $f \in A$, then f is finite in each $P \in V_c(\mathbb{R}) \subseteqq V(\mathbb{R})$. Then, using 1.1, by compactness we have $f \in H(V_c(\mathbb{R}))$. Thus $A = H(V_c(\mathbb{R}))$ and hence, by 2.5,

$$A \subseteqq H(F) \subseteqq V$$

for any real valuation ring V. □

Proposition 2.9. *If $V_{\text{sing}}(\mathbb{R})$ is compact, then $H(A) = A \cap H(F)$. In particular,*

$$V_{\text{sing}}(\mathbb{R}) = \emptyset$$

implies $H(A) = A \cap H(F)$.

Proof. By 2.6, $A = H(V_{\text{sing}}(\mathbb{R}))$. Since $V(\mathbb{R}) = V_c(\mathbb{R}) \cup V_{\text{sing}}(\mathbb{R})$, it follows that $H(A) = H(V_c(\mathbb{R}))$. Therefore $H(A) = A \cap H(F)$ by 2.5. □

§ 3. Iteration of the real holomorphy ring

Since $H(R)$ is a ring, we can compute $H(H(R))$, etc. In this section we study the rings $H^n(R) := H(H(\ldots H(R)\ldots))$, iterated n times.

Examples. (1) By [M], 3.3, [P], 1.7, [Sch], 7.6, if R is any ring with $1 + \Sigma R^2 \subseteqq R^*$, then $H(H(R)) = H(R)$ and hence $H^n(R) = H(R)$ for all n. In particular this holds for fields.

(2) In $\mathbb{R}[x_1, \ldots, x_n]$, set $u = 1 + \Sigma x_i^2$ and let $R = \mathbb{R}[x_1, \ldots, x_n]_S$, where

$$S = \{u^k \mid k \in \mathbb{N}\}.$$

By 1.9, we have $H(R) = \left\{ \frac{f}{u^k} \,\middle|\, \deg f \leqq 2k \right\}$. By 1.10 (v), $H(R)$ is an affine algebra and if $V = \operatorname{Spec} H(R)$, then $V(\mathbb{R}) = \mathbb{P}^n(\mathbb{R})$. In particular, $V(\mathbb{R})$ is compact and thus

$$H(H(R)) = H(R)$$

by 2.6.

Fix k, a totally archimedean field. For any k-algebra R, not necessarily reduced, we write $\operatorname{tr}(R)$ for the transcendence degree of R over k. Our goal is to prove the following:

Theorem 3.1. *Suppose R is a k-algebra such that $\operatorname{tr}(R/\wp) \leqq d$ for every real prime $\wp \in \operatorname{Min\,Spec} R$. Then $H^{d+1}(R) = H^d(R)$.*

Lemma 3.2. *Suppose $X \subseteqq \operatorname{Sper} R$, $\mathfrak{A}$ is an ideal in R such that $R = H(X)$ and*

$$\operatorname{Sper} R = X \cup \operatorname{Sper} R/\mathfrak{A}.$$

Then

(i) $H(R) = \{r \in R \mid \bar{r} := r + \mathfrak{A} \in H(R/\mathfrak{A})\}$,

(ii) $\mathfrak{A} \subseteqq H(R)$,

(iii) $H(R)_f = R_f$ *for all* $f \in \mathfrak{A}$,

(iv) $H(R)/\mathfrak{A} = H(R/\mathfrak{A})$.

Proof. (i) Given $r \in R$, by assumption r is bounded with respect to all $P \in X$. Hence $r \in H(R)$ iff r is bounded with respect to all $P \in \operatorname{Sper} R/\mathfrak{A}$, which clearly implies (i).

(ii) For $r \in \mathfrak{A}$, $\bar{r} = 0$, hence $\bar{r} \in H(R/\mathfrak{A})$. Thus $r \in H(R)$ by (i).

(iii) Pick any $r \in R$ and $f \in \mathfrak{A}$, then $rf \in \mathfrak{A}$, hence $rf \in H(R)$. Given any $k \in \mathbb{N}$,

$$\frac{r}{f^k} = \frac{rf}{f^{k+1}}$$

and so the natural map $H(R)_f \to R_f$ is a bijection.

(iv) follows easily from (i). □

Corollary 3.3. *Given the hypotheses of* 3.2, *then*

$$H^n(R) = \{r \in R \mid \bar{r} \in H^n(R/\mathfrak{A})\} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. For $n = 1$, this is proven in 3.2 (i). Assume $n > 1$ and the conclusion of 3.2 holds for $H^{n-1}(R)$. Then, since $\mathfrak{A} \subseteq H^{n-1}(R)$ and $H^{n-1}(R)/\mathfrak{A} = H^{n-1}(R/\mathfrak{A})$, using 1.12 we can apply 3.2 to $H^{n-1}(R)$. This yields

$$H^n(R) = H(H^{n-1}(R)) = \{r \in R \mid \bar{r} \in H(H^{n-1}(R/\mathfrak{A}))\} = \{r \in R \mid \bar{r} \in H^n(R/\mathfrak{A})\}.$$

The remaining parts of 3.2 follow easily for $H^n(R)$ and we are done by induction. □

The following is the key reduction result we need to prove Theorem 3.1:

Proposition 3.4 (Central Reduction Lemma). *Let R be a noetherian real ring of dimension d with* Sing R *closed in* Spec R. *Assume further that $R/\wp \subseteq H(R_\wp)$ for every minimal prime ideal $\wp$ of dimension d. Let*

$$M = \{\wp \in \operatorname{Spec} R \mid \wp \supseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \operatorname{Min} \operatorname{Spec} R \text{ with } \dim \mathfrak{q} < d\} \cup \operatorname{Sing} R$$

and set $\mathfrak{A} = \sqrt[\text{re}]{I}$, where I is the intersection of all primes in M. Then

(i) $R \subseteq H(\operatorname{Sper} R_f)$ *for all* $f \in \mathfrak{A}$,

(ii) $H^n(R) = \{f \in R \mid \bar{f} \in H^n(R/\mathfrak{A})\}$ *for all* $n \in \mathbb{N}$,

(iii) $\dim \mathfrak{A} < d$.

Proof. We will use results from the ideal theory of noetherian rings, see e.g. [Ku]. Since R is noetherian, R has finitely many minimal prime ideals, say $\{\wp_1, \ldots, \wp_s\}$. Let $C_i = \operatorname{Spec} R/\wp_i \subseteq \operatorname{Spec} R$. Then $\operatorname{Spec} R = \bigcup_{i=1}^{s} C_i$, $\dim C_i = \dim \wp_i$, and the C_i's are the irreducible components of Spec R. Renumber so that $C_1, \ldots, C_k$ are the components of dimension $< d$ and $C_{k+1}, \ldots, C_s$ are the components of dimension d. Then

$$M = \operatorname{Sing} R \cup \left(\bigcup_{i=1}^{k} C_i\right).$$

Let $L = \left\{(\wp, \bar{P}) \in \operatorname{Sper} R \mid \wp \in \text{Reg-Spec}\, R \cap \left(\bigcup_{j=k+1}^{s} C_i\right)\right\}$. Then clearly

$$(*) \qquad \operatorname{Sper} R = L \cup \operatorname{Sper} R/\mathfrak{A}.$$

Claim. $R = H(L)$.

Proof of claim. Given $r \in R$, fix a component $C = \operatorname{Spec} R/\wp$ of dimension d and let $\tilde{C} := \{(\wp, \bar{P}) \in \operatorname{Sper} R \mid \wp \in C\}$. By assumption, $R/\wp \subseteq H(R_\wp)$, hence by 1.15, $\bar{r}$ is bounded on $\operatorname{Sper}_c R/\wp$. Then, by 1.16, $\bar{r}$ is bounded on Reg-Sper $R/\wp$. Inside R, this means that r is bounded on Reg-Sper $R/\wp \subseteq \operatorname{Sper} R$. From

$$\tilde{C} \cap \text{Reg-Sper}\, R \subseteq \text{Reg-Sper}\, R/\wp \subseteq \operatorname{Sper} R$$

we see that r is bounded on $\tilde{C} \cap \text{Reg-Sper} R$. Since this holds for all components C of dimension d, the claim is proven.

Using the claim and (∗), (i) and (ii) of the proposition follow from 3.2 and 3.3.

It remains to prove that $\dim \mathfrak{A} < d$. Clearly it is enough to prove $\dim I < d$. Suppose $\dim I = d$, then there is some $\wp \in \operatorname{Min Spec} R$ with $\dim \wp = d$ and $I \subseteq \wp$. This means there is some component C of $\operatorname{Spec} R$ with dimension d such that $C \subseteq M$. But the C_i's are closed, as is $\operatorname{Sing} R$, and C is irreducible, which implies $C \subseteq C_i$ for some $1 \leqq i \leqq k$, or $C \subseteq \operatorname{Sing} R$. Since $\dim C_i < d$, the first is impossible and the second is impossible since

$$\operatorname{Min Spec} R \subseteq \operatorname{Reg} R .$$

Thus $\dim \mathfrak{A} < d$ and we are done. □

We begin our proof of 3.1 by studying the affine case.

Lemma 3.5. *Suppose A is a real affine algebra over a totally archimedean field k with $\dim A = d$. Set $H := H(A)$. Then:*

(i) $\operatorname{Min Spec} H$ *is finite.*

(ii) *Given* $\wp \in \operatorname{Min Spec} H$, *then* $H_\wp = \operatorname{quot}(H/\wp)$ *is a function field with* $\operatorname{tr}(H/\wp) \leqq d$.

(iii) *If* $\wp \in \operatorname{Min Spec} H$ *such that* $\operatorname{tr}(H_\wp) = d$, *then* $\wp$ *has a unique extension to* $\mathfrak{p} \in \operatorname{Min Spec} A$ *and* $H_\wp = A_\mathfrak{p}$.

Proof. By [Ku], I, 4.9, $\operatorname{Min Spec} A$ is finite and by [Bo], II, Prop. 16, every

$$\wp \in \operatorname{Min Spec} H$$

extends to $\operatorname{Min Spec} A$. (i) now follows easily.

(ii) Given $\wp \in \operatorname{Min Spec} H$, let $\mathfrak{p} \in \operatorname{Min Spec} A$ be such that $\wp = \mathfrak{p} \cap H$. Then there is a mapping

$$k \hookrightarrow H/\wp \hookrightarrow A/\mathfrak{p} \hookrightarrow A_\wp .$$

Because of $\operatorname{char} k = 0$, $H_\wp$ is a function field.

(iii) Pick $\mathfrak{p} \in \operatorname{Min Spec} A$ such that $\wp = H \cap \mathfrak{p}$. We first prove $H_\wp = A_\mathfrak{p}$. Since $H_\wp$ is a subfield of $A_\mathfrak{p}$ and they have the same transcendence degree over k, the extension $A_\mathfrak{p} | H_\wp$ is algebraic. Hence, given $x \in A$, there is some $s \in H \setminus \wp$ such that $\overline{sx}$ is integral over $H_\wp$. Thus we have an equation (depending on $\mathfrak{p}$)

$$(1) \qquad (sx)^t + a_1 (sx)^{t-1} + \cdots + a_t = b \in \mathfrak{p}$$

where $a_i \in H$. We claim we can find an s that works for all possible choices of $\mathfrak{p}$. First note that since $\operatorname{Min Spec} A$ is finite, there are only finitely many equations. If $\{s_1, s_2, \ldots, s_k\}$ are the s's corresponding to the equations, multiply the i-th equation by a suitable power of $s_1 \cdot \dots \cdot \hat{s}_i \cdot \dots \cdot s_k$ to obtain a uniform s.

Now we can multiply the equations for each $\mathfrak{p} \in \operatorname{Min Spec} A$ over $\wp$. Thus we end up with an equation

$$(2) \qquad (sx)^t + a_1(sx)^{t-1} + \cdots + a_t = b \in \cap \mathfrak{p},$$

for suitable t, where the intersection is over all $\mathfrak{p} \in \operatorname{Min Spec} A$ lying over $\wp$, $s \in H \setminus \wp$, and $a_i \in H$. Let $\{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$ be the set of elements of $\operatorname{Min Spec} A$ which do not lie over $\wp$, and set $\mathfrak{A} = \bigcap (\mathfrak{q}_i \cap H)$. Then $\mathfrak{A} \nsubseteq \wp$, else $\mathfrak{q}_i \cap H \subseteq \wp$ for some i which implies $\mathfrak{q}_i \cap H = \wp$, a contradiction. Thus we can find $u \in \mathfrak{A} \setminus \wp$. Multiplying (2) by u^t we get, since A is reduced,

$$(qx)^t + b_1(qx)^{t-1} + \cdots + b_k = c \in \bigcap_{\mathfrak{q} \in \operatorname{Min Spec} A} \mathfrak{q} = 0,$$

where $b_i \in H$ and $q \in H \setminus \wp$. Hence qx is integral over H. Thus, since H is integrally closed in A by 1.2, $qx \in H$ where $q \in H \setminus \wp$. Hence $\bar{q}\bar{x} \in \bar{H} = H/\wp$ and $\bar{q} \neq 0$. This proves that $H_\wp = A_\mathfrak{p}$.

Now we claim $\mathfrak{p} = \{x \in A \mid sx \in \wp \text{ for some } s \in H \setminus \wp\}$. Given $x \in \mathfrak{p}$, then as shown, there exists $s \in H \setminus \wp$ such that $sx \in H \cap \mathfrak{p} = \wp$. Conversely, if $x \in A$ and $xs \in \wp$ for some $s \in H \setminus \wp$, then $sx \in \mathfrak{p}$ and $s \notin \mathfrak{p}$ imply $x \in \mathfrak{p}$. Thus the claim is proven from which it follows that $\mathfrak{p}$ is the unique minimal prime lying over $\wp$.

Proposition 3.6. *If A is an affine algebra over k with* $\dim A = d$, *then Theorem* 3.1 *holds for A.*

Proof. Set $H := H(A)$. The proof is by induction on

$$d := \max\{\operatorname{tr}(A/\wp) \mid \wp \in \operatorname{Min Spec} A\}.$$

If $d = 0$ then for any $\wp \in \operatorname{Min Spec} A$, $A/\wp = A_\wp$ is an algebraic extension of k. Hence $H(A/\wp) = A/\wp$ since k is totally archimedean. Then it follows from 1.2 (ii) that $H = A$ and we are done.

Let $e := \max\{\operatorname{tr}(H/\wp) \mid \wp \in \operatorname{Min Spec} H\}$. By 3.5 (ii), we know $e \leqq d$.

Clearly $H = \varinjlim B$, where $\{B\}$ ranges over all affine subalgebras contained in H. Then by 1.11, $H^d = \varinjlim H^d(B)$. Also, note $\operatorname{tr}(B/\wp) \leqq e$ for all such B and any $\wp \in \operatorname{Min Spec} B$. If $e < d$, then by induction $H^d(H) = H^{d-1}(H)$. Hence $H^{d+1} = H^d$.

Now suppose $e = d$.

Claim. In the above limit, we can restrict to subalgebras B such that $B_\wp = H_\mathfrak{q}$ for every $\wp \in \operatorname{Min Spec} B$ and $\mathfrak{q} \in \operatorname{Min Spec} H$ with $\wp = \mathfrak{q} \cap B$.

Proof of claim. Given $B \subseteq H$ any affine algebra, then

$$\dim B = \max\{\operatorname{tr}(B/\mathfrak{q}) \mid \mathfrak{q} \in \operatorname{Min Spec} B\} \leq \max\{\operatorname{tr}(H/\wp) \mid \wp \in \operatorname{Min Spec} H\} \leqq d.$$

By 3.5(i), we have $\operatorname{Min}\operatorname{Spec} H$ finite. Given $\wp \in \operatorname{Min}\operatorname{Spec} H$ and $\mathfrak{q} \in \operatorname{Min}\operatorname{Spec} A$ with $\mathfrak{q} = \wp \cap H$, then $H_\wp$ is a subfield of the function field $A_\mathfrak{q}$. Because $\operatorname{char} k = 0$, $H_\wp$ is finitely generated over k, say of transcendence degree m. Clearly $m \leqq d$. Thus we can find $r_i \in H$ and $s_i \in H \setminus \wp$, $i = 1$ to m, such that $H_\wp = k\left(\frac{\bar{r}_1}{\bar{s}_1}, \ldots\right)$. Let E be the (finite) set of all such r_i's and s_i's corresponding to all $\wp$ and $\mathfrak{q}$'s as above. Then $B_0 := k[E]$ is an affine subalgebra of $H(R)$.

Now we restrict to all subalgebras $B \subseteqq H$ with $B_0 \subseteqq B$. For each $\mathfrak{q} \in \operatorname{Min}\operatorname{Spec} H$ such that $\wp = \mathfrak{q} \cap B \in \operatorname{Min}\operatorname{Spec} B$, consider the natural mapping

$$B \hookrightarrow H \hookrightarrow H/\mathfrak{q} \hookrightarrow H_\mathfrak{q}.$$

From this we get an injection $B/\wp \hookrightarrow H/\mathfrak{q}$ and hence the quotient field $B_\wp$ of $B/\wp$ is $H_\mathfrak{q}$. Thus the claim is proven.

Now we need only prove the proposition holds for all subalgebras B of H satisfying the conditions of the claim. If $\dim B < d$, then by induction we have $H^d(B) = H^{d-1}(B)$. Suppose $\dim B = d$, let $\wp$ be a minimal prime of dimension d. Then, by 3.5(iv), there is some minimal prime $\mathfrak{q}$ in H and a minimal prime $\mathfrak{p}$ of A such that $B_\wp = H_\mathfrak{q} = A_\mathfrak{q}$. Also, we have $B/\wp \subseteqq H/\mathfrak{q} \subseteqq H(A_\mathfrak{p}) = H(B_\wp)$. We have shown that all of the assumptions of the Central Reduction Lemma hold for B. Hence $H^d(B) = \{f \in B \mid \bar{f} \in H^d(B/\sqrt[\mathrm{re}]{\mathfrak{A}})\}$, where $\mathfrak{A}$ is defined as in the lemma. Since $\dim(B/\sqrt[\mathrm{re}]{\mathfrak{A}}) < d$, by induction we get $H^d(B) = H^{d-1}(B)$ and the proposition is proven. □

Proof of 3.1. Clearly $R = \varinjlim R_i$, where $\{R_i\}$ is the set of affine subalgebras contained in R. Then by 1.11, $H^d(R) = \varinjlim H^d(R_i)$. Also, note $\operatorname{tr}(R_i/\wp) \leqq d$ for all R_i and any $\wp \in \operatorname{Min}\operatorname{Spec} R_i$. Thus we may assume R is an affine algebra and hence we are done by 3.6. □

We end this section with an example where $H(H(R)) \neq H(R)$.

Example. Let $\phi(x, y, z) = (x^2 + y^2)(1 - x^2 - y^2 - z^2) - x^2 y^2 \in \mathbb{R}[x, y, z]$, and set $B = \mathbb{R}[x, y, z]/(\phi)$. An easy check shows ϕ is irreducible. Note that

$$V_{\mathrm{reg}}(\mathbb{R}) = \{(x, y, z) \in V(\mathbb{R}) \mid x^2 + y^2 \neq 0\} \neq \emptyset,$$

hence (ϕ) is a real prime ideal. Also note $V_{\mathrm{sing}}(\mathbb{R}) = \{(0, 0, z) \mid z \in \mathbb{R}\}$. Let $F = \operatorname{quot}(B)$.

Let $R = B_x$, then R is an affine algebra with real points

$$W(\mathbb{R}) = \{(x, y, z) \in V(\mathbb{R}) \mid x \neq 0\}.$$

Then W is an open subvariety of V, and $W(\mathbb{R}) \subseteqq V_{\mathrm{reg}}(\mathbb{R})$. Hence every real point of W is regular. Thus, by Prop. 2.9, $H(R) = R \cap H(F)$.

Consider $t = \dfrac{1-y^2-z^2}{x^2} \in R$. In F we have $(x^2+y^2)(1-x^2-y^2-z^2) = x^2y^2$ and then, since $x \neq 0$, $1-y^2-z^2 = \dfrac{x^2y^2}{x^2+y^2} + x^2$. Thus

$$t = 1 + \frac{y^2}{x^2+y^2} = 1 + \frac{1}{(x/y)^2+1} \in H(F) \cap R = H(R).$$

Claim 1. $H(R) = B[t]$.

Proof of claim. By the above, $B[t] \subseteq H(R)$. Suppose $f \in H(R)$, we can write $f = h(x,y,z) + \sum_{i=1}^{k} \dfrac{h_i(y,z)}{x^i}$. Since $h(x,y,z) \in B \subseteq B[t]$, without loss of generality we can assume $h(x,y,z) = 0$. Let $K = \{(0,y,z) \mid y^2+z^2 = 1\}$ and fix $(0, y_0, z_0) \in K$. Then there exists a sequence of points $(x_j, y_j, z_j) \in W(\mathbb{R})$ converging to $(0, y_0, z_0)$. By definition f is bounded on $W(\mathbb{R})$. Hence, for fixed i, since $h_i(y_j, z_j)$ converges to $h_i(y_0, z_0)$, we must have $h_i(y_0, z_0) = 0$ ($\frac{1}{x^i}$ has a "pole" of order i).

Fix $h = h_i$ for some i. We have $h(y,z) = 0$ for all $(0,y,z) \in K$. We claim this implies $1-y^2-z^2$ divides h. To see this, consider h as an element of $\mathbb{R}(x)[y]$ and use the division algorithm to write

$$(*) \qquad h \cdot f(x) = q(x,y)(1-x^2-y^2) + r(x) + s(x)y,$$

where $f(x), r(x), s(x) \in \mathbb{R}[x]$. Then fix x_0, $-1 < x_0 < 1$ and choose y_0 such that

$$1 - x_0^2 - y_0^2 = 0.$$

Evaluating $(*)$ at $(x_0, \pm y_0)$ we get $r(x_0) + s(x_0)y_0 = r(x_0) - s(x_0)y_0 = 0$, from which follows $r(x_0) = 0$. Since this holds for all $x_0 \in (-1,1)$, it follows that $r(x) = 0$. Similarly, we have $s(x) = 0$. Thus we have $h \cdot f(x) = q(x,y)(1-x^2-y^2)$. Assume $f \neq$ const. Since f does not divide $(1-x^2-y^2)$, it follows that f divides $q(x,y)$ and so we get $1-x^2-y^2$ divides h. This also holds if $f =$ const.

We have shown that for each i, $\dfrac{h_i(y,z)}{x^i} = t\left(\dfrac{g(y,z)}{x^{i-1}}\right)$. Repeating the above argument, by induction we obtain $f \in B[t]$, proving the claim.

Now set $D := \mathbb{R}\left[x, y, \dfrac{y^2}{x^2+y^2}\right]$.

Claim 2. D isomorphic to $\mathbb{R}[X,Y,Z]/(Y^2 - Z(X^2+Y^2))$.

Proof of claim. Set $a = Y^2$ and $b = X^2 + Y^2$. Define $\theta : \mathbb{R}[X,Y,Z] \to D$ by

$$\theta(X) = x, \quad \theta(Y) = y, \quad \text{and} \quad \phi(Z) = \frac{y^2}{x^2+y^2}.$$

Clearly, θ is onto. By [Ku], 5.10, the kernel of θ is $(a - Zb)$. (Note that, $\{x^2, x^2 + y^2\}$ is a $\mathbb{R}[x, y]$-regular sequence.) Thus θ is an isomorphism.

From the claims we see that $H(R) = B[t] = \mathbb{R}[x, y, t, z]/\mathfrak{A}$, where

$$\mathfrak{A} = \left(y^2 - t(x^2 + y^2),\ z^2 - 1 - x^2 - y^2 - x^2 t\right).$$

Thus the line $\{(0, 0, t, 1) \mid t \in \mathbb{R}\} \subseteq V(\mathbb{R})$. In particular, $V(\mathbb{R})$ is not compact and hence, by 2.6, $H(H(R)) \neq H(R)$. Note that $H^3(R) = H^2(R)$ since R has dimension 2.

§ 4. $H(R)$ and sums of higher powers

In the first section we characterized $H(R)$ using $T(R)$ and $T^+(R)$. For formally real fields K the set of quadratic sums $T(K) = \Sigma K^2$ can be replaced by the set of sums of $2n$-th powers for each $n \in \mathbb{N}$, see [B3], 2.5, [B1], 3.3. In this section we study this question for rings. We use the idea of orderings of level n as in [BG1] and [Be].

Definition. A subset $P \subseteq R$ is an *ordering of level n on R* if there exists a ring homomorphism $\varphi: R \to K$, where K is a field, and an ordering $\bar{P}$ of level n on K with

$$P = \varphi^{-1}(\bar{P}).$$

Alternatively we can define orderings of level n as pairs $P \equiv (\wp, \bar{P})$, $\wp$ a real prime in R and $\bar{P}$ an ordering of level n in $k(\wp)$. For further descriptions and results on orderings of level n, see [BG1].

Note that orderings of level 1 are precisely the orderings defined in §1, i.e., the elements of Sper R.

Definition. (i) For an ordering P of level n, we set $P^+ = P \setminus -P$.

(ii) The set of *totally positive elements of level n in R* is $T_n(R) := \bigcap P$, the intersection over all orderings of level n.

(iii) The set of *strictly totally positive elements of level n* is $T_n^+(R) := \bigcap P^+$, the intersection over all orderings of level n.

Clearly $T_1(R) = T(R)$ and $T_1^+(R) = T^+(R)$. In [Be], R. Berr generalizes Stengle's abstract Positivstellensatz to orderings of level n:

Proposition 4.1 ([Be], 1.6). *Given $a \in R$, then*

(i) $a \in T_n(R) \Leftrightarrow$ *there exists $k \in \mathbb{N}_0$ and $t, t' \in \Sigma R^{2n}$ with $at = a^{2nk} + t'$,*

(ii) $a \in T_n^+(R) \Leftrightarrow$ *there exists $t, t' \in \Sigma R^{2n}$ with $at = 1 + t'$.*

Remark. In [W], it is shown that in (ii), t can be replaced by an element of the form $1 + s$, $s \in \Sigma R^{2n}$. This can also be shown as follows: From $at = 1 + t'$ we deduce $a \cdot (at)^{2n-1} = a(1 + t')^{2n-1}$, hence $a^{2n} t^{2n-1} = a(1 + t_1)$, $t_1 \in \Sigma R^{2n}$. Finally,

$$a(1+t+t_1) = 1 + t' + a^{2n}t^{2n-1}.$$

As in §1, we have

Corollary 4.2. (i) *If $\varphi: R \to S$ in a ring homomorphism, then $\varphi(T_n(R)) \subseteq T_n(S)$ and $\varphi(T_n^+(R)) \subseteq T_n^+(S)$.*

(ii) *The mappings $R \mapsto (H(R), H(R) \cap T_n(R))$ and $R \mapsto (H(R), H(R) \cap T_n^+(R))$ are covariant functors from the category of commutative rings to the category of archimedean partially ordered rings.*

Proposition 4.3. (i) *$H(R) = \{a \in R \mid$ there exists $k \in \mathbb{N}$ with $k \pm a \in T_n(R)\}$.*

(ii) *$H(R) = \{a \in R \mid$ there exists $k \in \mathbb{N}$ with $k \pm a \in T_n^+(R)\}$.*

Proof. (i) Given $a \in H(R)$. By definition, there exists $k \in \mathbb{N}$ such that $k \pm a \in T_1(R)$. We want to show $(1+k) \pm a \in T_n(R)$. Let $\wp$ be a real prime ideal in R and set $\bar{a} := a + \wp \in k(\wp)$. Then

$$k \pm \bar{a} \in H(k(\wp)) \cap \Sigma k(\wp)^2.$$

By [B2], (1.6), for a field K and $b \in H(K) \cap \Sigma K^2$ we have

$$1 + b \in H(K)^* \cap \Sigma K^2 \subseteq \bigcap_n \Sigma K^{2n}.$$

Thus, in our case: $(1+k) \pm \bar{a} \in \Sigma k(\wp)^{2n}$. Hence, in R,

$$(1+k) \pm \bar{a} \in \cap P, \quad P \text{ an ordering of level } n, \quad \wp = P \cap -P.$$

Since every real prime ideal is of this form, we have $(1+k) \pm a \in \cap P = T_n(R)$.

(ii) follows from (i) since $1 + T_n \subseteq T_n^+ \subseteq T_n$. □

Definition. Set

$$\tilde{T}(R) := \{f \in H(R) \mid f(P) \in H(k(\wp))^* \text{ for all } P = (\wp, \bar{P}) \in \operatorname{Sper} R\}$$

and

$$S(R) = \{f \in R \mid f(P) \neq 0 \text{ for all } P \in \operatorname{Sper} R\}.$$

Note that $H(R)^* \subseteq \tilde{T}(R)$ and $\tilde{T}(R) \subseteq S(R)$.

Lemma 4.4. (i) *$\tilde{T}(R) = \{f \in H(R) \mid f$ is bounded away from 0 on $\operatorname{Sper} R$, i.e., there is some $r \in \mathbb{Q}^+$ such that $|f(P)| > r$ for all $P \in \operatorname{Sper} R\}$.*

(ii) *If $H(R) = R$, then $\tilde{T}(R) = S(R)$.*

Proof. (i) Let C denote the right-hand side of the equation, i.e., the elements of $H(R)$ that are bounded away from 0. If F is a field, it is easy to show that for $x \in H(F)$, x is a unit in $H(F)$ iff x is bounded away from 0. This implies $C \subseteq \tilde{T}(R)$.

Now suppose $f \in \tilde{T}(R)$. Then for any $P = (\wp, \bar{P}) \in \operatorname{Sper} R$, f is bounded away from 0 in $k(\wp)$. By a compactness argument, as in the proof of 1.1, we get that f is globally bounded away from 0, i.e., $f \in C$.

(ii) $\tilde{T}(R) \subseteqq S(R)$ is clear. Given $f \in S(R)$. Fix $P = (\wp, \bar{P}) \in \operatorname{Sper} R$ and suppose $f(P)$ is not bounded away from 0 with respect to $\bar{P}$ in $k(\wp)$. Then $f(P) \in I(\bar{P})$, where $I(\bar{P})$ denotes the maximal ideal of the valuation ring $A(\bar{P})$ associated to $\bar{P}$. Set

$$\mathfrak{p} := I(\bar{P}) \cap H(k(\wp)),$$

then $f(P) \in \mathfrak{p}$. As is well-known, $\mathfrak{p}$ is a real prime ideal in $H(k(\wp))$. Since $H(R) = R$, we have a map $R \to R/\wp \to H(k(\wp))/\mathfrak{p}$, which induces $Q = (\mathfrak{q}, \bar{Q}) \in \operatorname{Sper} R$. But $f(P) \in \mathfrak{p}$ implies $f(Q) = 0$, contradicting $f \in S(B)$. Hence f is bounded away from 0 with respect to all $P \in \operatorname{Sper} B$ and thus $f \in \tilde{T}(B)$ by (i). □

Theorem 4.5. (i) *$\tilde{T} \cap T(R) \subseteqq T_n(R)$ for all n.*

(ii) *If B is a subring of R such that $H(B) = B$, then $S(B) \cap T(R) \subseteqq T_n(R)$ for all n.*

Proof. (i) This is proven using an argument similar to that of 4.3.

(ii) Use (i) and 4.4 (i) and note $S(B) \cap T(R) = \tilde{T}(B) \cap T(R) \subseteqq \tilde{T}(R) \cap T(R)$. □

Remark. 4.5 generalizes Theorem 1.6 of [B2] to rings.

Example. Let $R = \mathbb{Z}[X]_{(1+x^2)(2+x^2)}$. Then $\frac{1+x^2}{2+x^2}$ is a unit in $H(R)$ and contained in $T(R)$. It follows from 4.5 that $\frac{1+x^2}{2+x^2} \in \bigcap_n T_n^+(R)$. Because of $T_n^+(R) \subseteqq \Sigma \mathbb{Q}(X)^{2n}$, this generalizes the result in [B2], 1.7. See also [R], § 5.

The above results, particularly Theorem 4.5, will allow us to analyse the multiplicative semigroup $1 + \Sigma R^{2n}$.

The following result is well-known. The equivalence of (i) and (ii) is proven for example in [L], 3.9; the remaining cases were proven by Joly [J].

Proposition 4.6. *The following are equivalent for a ring R:*

(i) $\operatorname{Sper} R \neq \emptyset$,

(ii) $-1 \notin \Sigma R^2$,

(iii) $-1 \notin \Sigma R^{2n}$ *for each* $n \in \mathbb{N}$,

(iv) $-1 \notin \Sigma R^{2n}$ *for some* $n \in \mathbb{N}$.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are clear. We thus prove (iv) ⇒ (i): Assume (iv) holds and set $S := 1 + \Sigma R^{2n}$, then $0 \notin S$. Let $\wp$ be an ideal maximal with respect to $\wp \cap S = \emptyset$. It is well-known that $\wp \in \operatorname{Spec} R$. We claim $\wp$ is real. The equivalence (ii) $\Leftrightarrow$ (iv) is known for fields, thus it is enough to show $-1 \notin \Sigma k(\wp)^{2n}$. Assume not, then there exists $s \notin \wp$ and $x_1, \ldots, x_r \in R$ with $s^{2n} + \Sigma x_i^{2n} \in \wp$. By the maximality of $\wp$ it follows that

$$(\wp + Rs) \cap S \neq \emptyset,$$

i.e., $u + as \in S$ for $u \in \wp$, $a \in R$. Taking powers, we have $(u + as)^{2n} = u' + a's^{2n} \in S$ with $u' \in \wp$, $a' \in R$. Multiplying $s^{2n} + \Sigma x_i^{2n} \in \wp$ with a' and adding u' it follows that $u' + a's^{2n} + \Sigma y_i^{2n} \in \wp$, i.e., $\wp \cap S \neq \emptyset$, a contradiction. □

Definition. Given a multiplicative semigroup S in R, then

$$\operatorname{Sat}(S) = \{s \in R \mid ss' \in S \text{ for some } s' \in S\}$$

is called the *saturated hull of S*.

Krull's Theorem on the existence of prime ideals says that

$$\operatorname{Sat}(S) = \bigcap_{\substack{\wp \in \operatorname{Spec} R \\ \wp \cap S = \emptyset}} \wp^c,$$

where $\wp^c = R \setminus \wp$.

Theorem 4.7. *For each $n \in \mathbb{N}$*

$$\operatorname{Sat}(1 + \Sigma R^2) = \operatorname{Sat}(1 + \Sigma R^{2n}) = \bigcap_{\substack{\wp \in \operatorname{Spec} R \\ \wp \text{ real}}} \wp^c.$$

Proof. If $0 \in 1 + \Sigma R^2$, then by 4.6 we have $0 \in 1 + \Sigma R^{2n}$ and $\operatorname{Sper} R = \emptyset$. Thus, in this case, we have $\operatorname{Sat}(1 + \Sigma R^{2n}) = R$ and the theorem is proven. Hence we asume $0 \notin 1 + \Sigma R^2$, thus also $0 \notin 1 + \Sigma R^{2n}$. If $\wp$ is a real prime ideal, then clearly

$$(1 + \Sigma R^{2n}) \cap \wp = \emptyset.$$

Suppose conversely that $\wp_0$ is a prime ideal with $\wp_0 \cap (1 + \Sigma R^{2n}) = \emptyset$. Then there is a prime ideal $\wp$ over $\wp_0$ which is maximal with respect to this property and hence, as in the proof of 4.6, $\wp$ is real. Thus we have

$$1 + \Sigma R^{2n} \subseteqq \wp^c \subseteqq \wp_0^c,$$

and the theorem follows. □

Corollary 4.8. *Suppose $-1 \notin \Sigma R^2$. Then for each n the natural mapping*

$$R_{1 + \Sigma R^{2n}} \rightarrow R_{1 + \Sigma R^2}$$

is an isomorphism, and every maximal ideal in $R_{1 + \Sigma R^{2n}}$ is real.

Proof. It is well-known that in general the natural mapping $R_S \to R_{\mathrm{Sat}(S)}$ is an isomorphism. The ring $R' := R_{1+\Sigma R^2}$ satisfies $1 + \Sigma R'^2 \subseteq (R')^*$ and in such a ring every maximal ideal is real cf. the following 5.1. □

By 4.8 we have for any $n, m \in \mathbb{N}$:

$$1 + \Sigma R^{2n} \subseteq \mathrm{Sat}(1 + \Sigma R^{2nm}),$$

i.e., for any $q \in \Sigma R^{2n}$ there exists $x \in R$ with

$$(1 + q)\, x \in 1 + \Sigma R^{2nm}.$$

We will now show that we can find factors x of a very special type.

Proposition 4.9. *Given $n, m \in \mathbb{N}$ and $q \in T_n(R)$.*

(i) $q^m \in T_{nm}(R)$.

(ii) *There exist $t, t' \in \Sigma R^{2nm}$ with*

$$(1 + q)^m \cdot t = 1 + t'.$$

(iii) *There exists $t, t' \in \Sigma R^{2nm}$ with*

$$[2(1 + q)^m - 1] \cdot t = 1 + t'.$$

Proof. (i) follows from the fact that if K is a field of characteristic 0, then

$$(*) \qquad (\Sigma K^{2n})^m \subseteq \Sigma K^{2nm},$$

see [B 2], 1.9. (ii) follows easily from (iii). To prove (iii) we note that if $\wp$ is any real prime ideal, then $x := 2(1 + q)^m - 1 \notin \wp$. This together with $(*)$ implies that $x \in T^+_{nm}$. (iii) now follows from 4.1 (ii). □

Remark. From 4.9 (ii) it is obvious that $\mathrm{Sat}(1 + \Sigma R^{2n}) = \mathrm{Sat}(1 + \Sigma R^2)$.

Example. (i) Let R be a ring with $\mathbb{Z} \subseteq R \subseteq \bar{\mathbb{Q}}$. Then $\wp = 0$ is the only real prime ideal and hence $\mathrm{Sat}(1 + \Sigma R^2) = R \setminus \{0\}$.

(ii) Let V be an irreducible real affine variety and let $R = \mathbb{R}[V]$. Then by [BCR], 4.4.3 we have

$$\mathrm{Sat}(1 + \Sigma R^2) = \{f \in R \mid f \text{ has no zeros on } \mathbb{R}^n\}.$$

§ 5. Rings with $1 + \Sigma R^2 \subseteq R^*$

The following proposition is well-known.

Proposition 5.1. *The following are equivalent for a ring R:*

(i) $1+\Sigma R^2 \subseteq R^*$,

(ii) $1+\Sigma R^{2n} \subseteq R^*$ *for each* $n \in \mathbb{N}$,

(iii) $1+\Sigma R^{2n} \subseteq R^*$ *for some* $n \in \mathbb{N}$,

(iv) *every maximal ideal of* R *is real.*

Proof. The implication (i) ⇒ (iv) is easy, and the implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) follow from 4.7 and the fact that $R^* = \bigcap_{\wp \in \operatorname{Max\,Spec} R} \wp^c$. □

We assume for the rest of the paper that $1+\Sigma R^2 \subseteq R^*$.

Examples 5.2. (i) Suppose A is a ring with $\operatorname{Sper} A \neq \emptyset$, then $0 \notin 1+\Sigma A^2$. Then $R := A_{1+\Sigma A^2}$ satisfies $1+\Sigma R^2 \subseteq R^*$.

(ii) Suppose V is a real affine variety with coordinate ring $\mathbb{R}[V]$, then

$$\mathcal{O}_{\mathbb{R}}(V) := \{f \in \mathbb{R}(V) \mid f \text{ is regular at every point } x \in V(\mathbb{R})\} = \mathbb{R}[V]_{1+\Sigma\mathbb{R}[V]^2}.$$

To prove this, use Stengle's Positivstellensatz, see [BCR], 4.4.5.

(iii) Let Y be a topological space, then $R := C(Y, \mathbb{R})$ satisfies $1+\Sigma R^2 \subseteq R^*$.

Proposition 5.3. *Let* V *be an irreducible real affine variety with coordinate ring* $\mathbb{R}[V]$, *and let* F *be the quotient field of* $\mathbb{R}[V]$.

(i) $V(\mathbb{R})$ *is compact iff* $H(\mathbb{R}[V]) = \mathbb{R}[V]$ *iff* $H(\mathcal{O}_{\mathbb{R}}(V)) = \mathcal{O}_{\mathbb{R}}(V)$.

(ii) $V_c(\mathbb{R})$ *is compact iff* $\mathbb{R}[V] \subseteq H(F)$ *iff* $\mathcal{O}_{\mathbb{R}}[V] \subseteq H(F)$.

Proof. (i) $V(\mathbb{R})$ compact iff $H(\mathbb{R}[V]) = \mathbb{R}[V]$ is 2.6. Suppose $V(\mathbb{R})$ is compact. Let $S = 1+\Sigma\mathbb{R}[V]^2$, then as in 5.2 (ii), $\mathcal{O}_{\mathbb{R}}[V] = \mathbb{R}[V]_S$. Hence, by 1.8,

$$H(\mathcal{O}_{\mathbb{R}}[V]) = \{f \in \mathcal{O}_{\mathbb{R}}[V] \mid f \text{ bounded on } V(\mathbb{R})\}.$$

Since $V(\mathbb{R})$ is compact, it follows that $H(\mathcal{O}_{\mathbb{R}}[V]) = \mathcal{O}_{\mathbb{R}}[V]$.

Now suppose $H(\mathcal{O}_{\mathbb{R}}[V]) = \mathcal{O}_{\mathbb{R}}[V]$, then Σx_i^2 must be bounded on $V(\mathbb{R})$, which implies $V(\mathbb{R})$ is compact.

(ii) This follows from the fact that if $y \in \Sigma F^2$, then $\frac{1}{1+y} \in H(F)$. □

Remark. For any ring R' the natural mapping $R' \to R'_{1+\Sigma R'^2}$ induces a homomorphism $\operatorname{Sper}(R'_{1+\Sigma R'^2}) \to \operatorname{Sper} R'$. From this fact we see immediately that

$$H(R') = R' \cap H(R'_{1+\Sigma R'^2})$$

(with the obvious interpretation of the right side).

Theorem 5.4. *For each* $n \in \mathbb{N}$

(i) $T_n^+(R) = R^* \cap \Sigma R^{2n}$,

(ii) $T_n^+(R) = (H(R)^* \cap T(R)) \cdot (T^+(R))^n$.

Proof. (i) The inclusion $R^* \cap \Sigma R^{2n} \subseteq T_n^+(R)$ is true for any ring R. Given $a \in T_n^+$, it follows from 4.1 (iii) that $at = 1 + t'$ for $t, t' \in \Sigma R^{2n}$. Since $1 + t'$ is a unit in R, so are a and t. Thus $a = (1 + t')t^{2n-1} \cdot (t^{-1})^{2n} \in R^* \cap \Sigma R^{2n}$.

(ii) From 4.5 we have $H(R)^* \cap T(R) \subseteq T_n^+(R)$. Then 4.9 (i) yields $(T^+)^n \subseteq T_n^+$. Hence $(H^* \cap T) \cdot (T^+)^n \subseteq T_n^+$. Now set $a = \Sigma x_i^{2n} \in T_n^+$. Then $a \in R^*$ by (i). Because every maximal ideal in R is real, this shows $\Sigma x_i^2 \in R^*$. Consider $b = \dfrac{\Sigma x_i^{2n}}{(\Sigma x_i^2)^n}$. For each real prime ideal $\wp$ we have $b + \wp,\ b^{-1} + \wp \in H(k(\wp))^*$, which implies $b \in H^* \cap T$. Clearly

$$\Sigma x_i^2 \in R^* \cap \Sigma R^2 = T^+,$$

then it follows that $b \in (H^* \cap T) \cdot (T^+)^n$. □

Example. Let $R = \mathbb{Z}[X]_{1+\Sigma\mathbb{Z}[X]^2}$. Then $\dfrac{1+X^2}{2+X^2}$ is an element of $H(R)^* \cap T(R)$, hence, using 5.4, we have $\dfrac{1+X^2}{2+X^2} \in \bigcap_n \Sigma R^{2n}$. Thus we have improved the results $\dfrac{1+X^2}{2+X^2} \in \bigcap_n \Sigma \mathbb{Q}(X)^{2n}$ from [B2].

For a ring A, let quot(A) be the total quotient ring A_N, where N is the multiplicative semigroup of non-zero-divisors of A. Clearly $A \hookrightarrow \mathrm{quot}(A)$.

Proposition 5.5. (i) $H(R) \subseteq R \subseteq \mathrm{quot}(H(R))$.

(ii) $1 + \Sigma H(R)^2 \subseteq H(R)^*$.

Proof. (i) For any $x \in R$ we have $1 + x^2 \in R^*$. Further, we see that

$$\frac{1}{1+x^2},\ \frac{x}{1+x^2} \in H(R).$$

Hence $x = \left(\dfrac{x}{1+x^2}\right)\left(\dfrac{1}{1+x^2}\right)^{-1} \in \mathrm{quot}(H(R))$.

(ii) follows from the fact that $(1+q)^{-1} \in H(R)$ for every $q \in \Sigma R^2$. □

Remark. By 5.5 (ii) and 5.1, every maximal ideal of $H(R)$ is real. In general, a prime ideal of $H(R)$ need not be real, as the next example will show. However we always have

Lemma 5.6. *Given $\wp \in \operatorname{Spec} H(R)$ and $\Sigma x_i^2 \in \wp$ such that $\Sigma x_i^2 \in R^*$. Then each $x_i \in \wp$. In this case $x_i = \left(\frac{x_i}{\Sigma x_j^2}\right) \cdot \Sigma x_j^2$ and $\frac{x_i}{\Sigma x_j^2} \in H(R)$.*

Proof. Easy. □

Example. Set $R = \mathbb{R}[X, Y]_{1+\Sigma \mathbb{R}[X,Y]^2}$. Then $\wp = \left(\frac{X^2+Y^2}{1+X^2+Y^2}\right)$ is a non-real prime ideal of $H(R) = \{f \in R \mid f \text{ bounded on } \mathbb{R}^2\}$.

Proposition 5.7. *For each $n \in \mathbb{N}$*

(i) $H(R) = \{r \in R \mid k \pm r \in \Sigma R^{2n}$ *for some* $k \in \mathbb{N}\}$,

(ii) $H(R) = \mathbb{Z}\left[\frac{1}{1+\Sigma R^{2n}}\right]$.

Proof. (i) is a consequence of 4.3 (i) and 5.4.

(ii) We first note that

$$\mathbb{Q} \subseteq \mathbb{Z}\left[\frac{1}{1+\Sigma R^{2n}}\right] \subseteq H(R).$$

Given $a \in H(R)$, by (i) we can find $k \in \mathbb{N}$ with $k + a = 1 + x$, where $x \in \Sigma R^{2n}$. Clearly $1 + x \in H$, hence $m - (1+x) = 1 + y$ for suitable $m \in \mathbb{N}$ and $y \in \Sigma R^{2n}$. We have

$$\frac{1+y}{1+x} \in R^* \cap \Sigma R^{2n},$$

thus writing $a = -k + m\left(1 + \frac{1+y}{1+x}\right)^{-1}$ we have $a \in \mathbb{Z}\left[\frac{1}{1+\Sigma R^{2n}}\right]$. □

In the first section, we considered the mapping

$$\phi : H(R) \to C(X, \mathbb{R}),$$

where $X = \operatorname{Hom}((H(R), Q^+), (\mathbb{R}, \mathbb{R}_+))$, $Q^+ = H(R) \cap T^+$. There is a mapping

$$X \xhookrightarrow{i} \operatorname{Hom}(H(R), \mathbb{R}) \xrightarrow{j} \operatorname{Max} \operatorname{Sper} H(R),$$

where i is the inclusion mapping, $\operatorname{Hom}(H(R), \mathbb{R})$ is given the subspace topology of the Tychonoff-Product space $\mathbb{R}^H$, and j is defined by:

$$\varphi \mapsto \varphi^{-1}(\mathbb{R}_+).$$

That $\varphi^{-1}(\mathbb{R}_+) \in \operatorname{Max} \operatorname{Sper} H(R)$ is shown in [KS], Kor. 5, p. 134 or [BG 2], 2.9.

Proposition 5.8. (i) *The mappings* $X \xrightarrow{i} \mathrm{Hom}(H(R), \mathbb{R}) \xrightarrow{j} \mathrm{Max\,Sper}\,H(R)$ *are homeomorphisms.*

(ii) $H(R)^* \cap \Sigma R^2 = \Phi^{-1}(C_+^*(X, \mathbb{R}))$.

(iii) $H(R)^* \cap \Sigma R^2 = \left\{ r\frac{s+q}{t+q} \,\middle|\, r, s, t \in \mathbb{Q}_*^+, q \in \Sigma R^{2n} \right\}$ *for all* $n \in \mathbb{N}$.

Proof. See [M], 3.4. In [KS], Kor. 5, p. 314, or [BG 2], 2.9, it is also shown that j is a bijection. □

For a topological space Y, let $B(Y)$ be the group of clopen subsets of Y relative to the symmetric difference. As in [B2] we set $\mathbb{R}_- = \{r \in \mathbb{R} \mid r < 0\}$ and we get:

Corollary 5.9. *The mapping* $\varepsilon \mapsto \varepsilon^{-1}(\mathbb{R}_-)$ *induces an isomorphism*

$$H(R)^* \Big/ {H(R)^* \cap \Sigma R^2} \cong B(X).$$

In the case of $R = K$, K a field, X has a natural interpretation as the space $M(K)$ of real places $\lambda: K \to \mathbb{R} \cup \infty$. This is the basis for the fact that $H(K)$ is a Prüfer ring with quotient field K. In [M], Marshall has a corresponding interpretation in the case of rings with $1 + \Sigma R^2 \subseteq R^*$. It is not proven in [M] that if R is a ring with $1 + \Sigma R^2 \subseteq R^*$, then the real holomorphy ring $H(R)$ is a Prüfer ring of R. We would like to prove this, thus we need the concept of M. Griffin [G] of Prüfer rings of A as overrings R of A in the total quotient ring quot(A). (In our situation we have $H(R) \subseteq R \subseteq \mathrm{quot}(H(R))$, see 5.5.)

Definition. Suppose A is a subring of R and $\wp \in \mathrm{Spec}\,A$.

(i) Set $A_{[\wp]} := \{r \in R \mid rs \in A$ for some $s \in A \setminus \wp\}$ and $\wp^\# := \{r \in R \mid rs \in \wp$ for some $s \in A \setminus \wp\}$.

(ii) As in [G], A is called a *Prüfer ring of* R, if for each maximal ideal $\wp$ of A the pair $(A_{[\wp]}, \wp^\#)$ is a Manis valuation ring of R, i.e., the pair satisfies: for all $x \in R \setminus A_{[\wp]}$ there is a $y \in \wp^\#$ with $xy \in A_{[\wp]} \setminus \wp^\#$.

Theorem 5.10. *Every overring of* $H(R)$ *in* R *is a Prüfer ring in* R.

Proof. Given a ring B such that $H(R) \subseteq B \subseteq R$ and $\wp \in \mathrm{Spec}\,B$. Set $H = H(R)$, $A = B_{[\wp]}$, and $\wp_0 = \wp \cap H$. For each $r \in R$ the following are in $H(R) \subseteq B \subseteq A$:

$$\frac{1}{1+r^2}, \quad \frac{r^2}{1+r^2} = 1 - \frac{1}{1+r^2}, \quad \text{and} \quad \frac{r}{1+r^2}.$$

Now given $r \in R \setminus A$, from the definition of $B_{[\wp]}$ it follows that $rs \notin H$ for each $s \in H \setminus \wp_0$. Since $r/(1+r^2)$, $1/(1+r^2) \in H$ it follows that $\frac{1}{1+r^2} \in \wp_0 \subseteq \wp^\#$. Then clearly

$$\frac{r^2}{1+r^2}\in A\backslash\wp^{\#}, \quad \text{thus} \quad \frac{r^2}{1+r^2}=r\cdot\frac{r}{1+r^2}$$

and it remains to show that $\frac{r}{1+r^2}\in\wp^{\#}$. We have $\left(\frac{r}{1+r^2}\right)^2=\frac{r^2}{1+r^2}\cdot\frac{1}{1+r^2}\in\wp^{\#}$, from which follows $\frac{r}{1+r^2}\in\wp^{\#}$. □

Remark. 5.10 can also be proven using [Po], 1.7.

Corollary 5.11. *Given $n\in\mathbb{N}$ and B a ring with $1+\Sigma B^{2n}\subseteq B^*$. If A is a subring of B with $1/(1+x)\in A$ for every $x\in\Sigma B^{2n}$, then A is a Prüfer ring of B.*

Proof. It follows from 5.1 that $1+\Sigma B^2\subseteq B^*$, and from 5.7 (ii) that $H(B)\subseteq A$. Now apply 5.10. □

Remarks. (i) For $R=C(Y,\mathbb{R})$ we saw that $H(R)=C^b(Y,\mathbb{R})$. Griffin notes that C^b is a Prüfer ring of C, see [G], p. 417.

(ii) In 5.8 we defined for the given space X a homeomorphism $X\cong\operatorname{Hom}(H(R),\mathbb{R})$. If $\varphi\in\operatorname{Hom}(H(R),\mathbb{R})$ and $\wp=\ker\varphi$, then clearly φ induces $\varphi^{\#}: H(R)_{[\wp]}\to R$ with $\ker\varphi^{\#}=\wp^{\#}$. Then, since $(H(R)_{[\wp]},\wp^{\#})$ is a Manis valuation pair, we can interpret $\varphi^{\#}$ as a place $\varphi: R\to\mathbb{R}\cup\infty$. This is the basis of Marshall's interpretation of X as a space of places.

Prüfer rings can be characterized by the property that finitely generated ideals are invertible, see [G].

Definition. Suppose H is a subring of R.

(i) An H-module $\mathfrak{a}\subseteq R$ is a *fractional ideal* (with respect to H) if $\mathfrak{a}R=R$.

(ii) $\operatorname{Inv}(H,R)=\{\mathfrak{a}\mid\mathfrak{a}$ is a fractional ideal and $\mathfrak{a}\mathfrak{b}=H$ for some fractional ideal $\mathfrak{b}\}$, the group of invertible fractional ideals.

(iii) $R^*\equiv\{aH\mid a\in R^*\}$ is the subgroup of fractional principal ideals of $\operatorname{Inv}(H,R)$. The factor group

$$\operatorname{Cl}(H,R):=\operatorname{Inv}(H,R)/_{R^*}$$

is the *class group* of (H,R) (of H for short).

Remark. Usually when defining fractional ideal, one requires $y\mathfrak{a}\subseteq H$ for some $y\in H\cap R^*$. But in our case of finitely generated H-modules this always holds, see 5.12 below. With the usual proof one shows that every invertible fractional ideal is a finitely generated H-module.

Proposition 5.12. *Set $H=H(R)$. Given $n\in\mathbb{N}$, the following are equivalent for an H-module $(a_1,\ldots,a_r):=\Sigma Ha_i\subseteq R$:*

(i) $(a_1, \dots, a_r) R = R$,

(ii) $\Sigma a_i^{2n} \in R^*$.

Proof. (i) ⇒ (ii) Assume $b := \Sigma a_i^{2n} \notin R^*$. Then there is a maximal ideal $\mathcal{M}$ of R with $b \in \mathcal{M}$. Since $\mathcal{M}$ is real, it follows that each a_i lies in $\mathcal{M}$. Thus $(a_1, \dots, a_r) R \subseteqq \mathcal{M}$ and hence (i) does not hold.

(ii) ⇒ (i) is clear since $\Sigma a_i^{2n} \in (a_1, \dots, a_r)$. □

Corollary 5.13. (i) *Every finitely generated fractional ideal is invertible.*

(ii) *For any fractional ideal* $(a_1, \dots, a_r)$ *we have* $(a_1, \dots, a_r)^{2n} = (\Sigma a_i^{2n})$.

Proof. (i) follows from (ii) using 5.12 (ii): Set $\mathfrak{a} = (a_1, \dots, a_r)$, then clearly $\Sigma a_i^{2n} \in \mathfrak{a}^{2n}$. The fractional ideal $\mathfrak{a}^{2n}$ is generated by the elements $\prod_{i=1}^{r} a_i^{t_i}$, $t_i \geqq 0$, $\Sigma t_i = 2n$, thus it remains to show that $\prod a_i^{t_i} / \Sigma a_i^{2n} \in H$. This equation holds for formally real fields, hence it holds in our situation by 1.2 (ii). □

Remark. For $\mathfrak{a} = (a_1, \dots, a_r)$, we have $\mathfrak{a}^{-1} = \left(\dots, \frac{a_i}{\Sigma a_j^2}, \dots \right)$. Hence $\mathrm{Cl}(H, R)$ is a group of exponent 2.

In the following we consider the connection between the elements of $R^* \cap \Sigma R^{2n}$ and the invertible fractional ideals.

Proposition 5.14. (i) *Suppose* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are invertible fractional ideals with* $\mathfrak{a}^t = \mathfrak{b}^t$ *for some* $t \in \mathbb{N}$. *Then* $\mathfrak{a} = \mathfrak{b}$.

(ii) *If* $\Sigma_1^r a_i^{2n}, \Sigma_1^s b_j^{2n} \in R^* \cap \Sigma R^{2n}$ *and* $\Sigma_1^r a_i^{2n} = \varepsilon \cdot \Sigma_1^s b_j^{2n}$, *where* $\varepsilon \in H^*$, *then*

$$(a_1, \dots, a_r) = (b_1, \dots, b_s).$$

Proof. (i) We have $(\mathfrak{a}\mathfrak{b}^{-1})^t = H$, thus need only show $\mathfrak{a}^t = H$ implies $\mathfrak{a} = H$. So suppose $\mathfrak{a}^t = H$ and given $a \in \mathfrak{a}$, then $a^t \in H$. Since H is a Prüfer ring in R, by [G], Prop. 6, p. 416, H is integrally closed in R. Hence $a \in H$ and thus $\mathfrak{a} \subseteqq H$. Hence $\mathfrak{a} = H$.

(ii) Set $\mathfrak{a} = (a_1, \dots, a_r)^{2n}$ and $\mathfrak{b} = (b_1, \dots, b_s)^{2n}$, then $\mathfrak{a}^{2n} = (\Sigma a_i^{2n}) = (\Sigma b_j^{2n}) = \mathfrak{b}^{2n}$. Thus $\mathfrak{a} = \mathfrak{b}$ by (i). □

Theorem 5.15. *The mapping* $\Sigma_1^r a_i^{2n} \mapsto (a_1, \dots, a_r)$, $\Sigma a_i^{2n} \in R^*$, *induces a group isomorphism*

$$R^* \cap \Sigma R^{2n} /_{(R^*)^{2n} \cdot (H^* \cap \Sigma R^2)} \cong \mathrm{Cl}(H, R).$$

Proof. We have $(a_1, \dots, a_r)(b_1, \dots, b_s) = (\dots, a_i b_j, \dots)$ and also

$$\Sigma_i a_i^{2n} \cdot \Sigma_j b_j^{2n} = \Sigma_{i,j} (a_i b_j)^{2n},$$

hence there is a group epimorphism $R^* \cap R^{2n} \to \mathrm{Cl}(H, R)$, where $\Sigma a_i^{2n} \mapsto$ the class of $(a_1, \ldots, a_r)$. Given $(a_1, \ldots, a_r) = (a)$, $a \in R^*$. Then $(\Sigma a_i^{2n}) = (a^{2n})$, where $\Sigma a_i^{2n} = \varepsilon \cdot a^{2n}$ with $\varepsilon \in H^*$. Then $\varepsilon \in H^* \cap \Sigma R^{2n} = H^* \cap \Sigma R^2$, the latter by 5.5. If $\Sigma a_i^{2n} \in (H^* \cap R^2) \cdot R^{*2n}$, it follows that $(a_1, \ldots, a_r)^{2n} = (\varepsilon \cdot a^{2n}) = (a^{2n}) = (a)^{2n}$, $a \in R^*$, i.e., $(a_1, \ldots, a_r) = (a)$. □

As in [B2] we can use the connection between $R^* \cap \Sigma R^{2n}$ and the fractional ideals to study quantitative problems on sums in R. We need to make use of the idea of Waring-constants $g(K, n)$ as in [B3], p. 887, thus we define:

Definition. Suppose A is any ring and $n \in \mathbb{N}$. Then we set

$$g^*(A, n) = \inf\{l \mid A^* \cap \Sigma A^n = A^* \cap \Sigma_1^l A^n\} \text{ or } \infty$$

the n-th Waring-constants for the units of A.

In our situation we also need

Definition. $\mu(R) = \min\{r \mid$ each invertible fractional ideal of H can be generated by r elements$\}$ or ∞.

Looking at the proof of [B2], 2.11, 2.12, we see that the facts used in the proof – for example, 1.6 (iii) with $\mathbb{Q} \subseteq H$, 5.6 (ii) – also hold in our situation. Thus we get the following results:

Proposition 5.16. (i) *$\mu(R) \leqq g^*(R, 2n)$ for all n.*

(ii) *If $\mu = \mu(R) < \infty$, then $R^* \cap \Sigma R^{2n} = (H(R)^* \cap \Sigma R^2) \cdot (R^* \cap \Sigma_1^{\mu} R^{2n})$.*

Theorem 5.17. *The following are equivalent:*

(i) $g^*(R, 2) < \infty$,

(ii) *$g^*(R, 2n) < \infty$ for some $n \in \mathbb{N}$,*

(iii) *$g^*(R, 2n) < \infty$ for all $n \in \mathbb{N}$.*

If $g^* = g^*(R, 2) < \infty$, it follows that

$$g^*(R, 2n) \leqq g^*(R, 2) \cdot \binom{2n + g^* + 2}{g^*} \cdot G(2n) \cdot \mu(R),$$

where $G(2n)$ = classical Waring-constant for the representation of sufficiently large natural numbers as sum of $2n$-th powers.

Remark. In [CDLR] many results on the Pythagoras number $P(R) = g(R, 2)$ are proven. Clearly $g(R, 2) < \infty$ implies $g^*(R, 2) < \infty$ and in the case of $\mathbb{Z}[X]$ one has $g(2, \mathbb{Z}[X]) = \infty$ and $g^*(\mathbb{Z}[X], 2) = 1$ (see [CDLR], 4.14).

Theorem 5.18. *Let V be an irreducible real affine variety of dimension d. Then*:

(i)
$$g^*(\mathcal{O}_{\mathbb{R}}(V), 2n) \leqq (2^d)^2 \binom{2n + 2^d + 2}{2^d}.$$

(ii) *If $V(\mathbb{R})$ is compact, then $\mathcal{O}_{\mathbb{R}}(V)^* \cap \Sigma \mathcal{O}_{\mathbb{R}}(V)^2 \subseteqq \Sigma \mathcal{O}_R(V)^{2n}$.*

Proof. (i) We have $g^*(\mathcal{O}_{\mathbb{R}}(V), 2) \leqq 2^d$ [Ma1]. The result now follows from 5.16 and 5.17 since in this case the factor $G(2n)$ can be dropped. □

(ii) This follows from 5.3 and 5.4. □

Remark. The statement (ii) in 5.18 is also proven by R. Berr, cf. [Be2], 4.1.

Added in proof. Rings R satisfying $H(R) = R$ are treated in [ABR], p. 164ff.

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