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Polynomials non-negative on strips and half-strips

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ABSTRACT

In 2008, Marshall (2010) [4] settled a long-standing open problem by showing that if $f(x, y) \in \mathbb{R}[x, y]$ is a polynomial that is non-negative on the strip $[0, 1] \times \mathbb{R}$, then there exist sums of squares $\sigma(x, y), \tau(x, y) \in \sum \mathbb{R}[x, y]^2$ such that $f(x, y) = \sigma(x, y) + \tau(x, y)(x - x^2)$. In this paper, we generalize Marshall's result to various strips and half-strips in the plane. Our results give many new examples of non-compact semialgebraic sets in $\mathbb{R}^2$ for which one can characterize all polynomials which are non-negative on the set. For example, we show that if U is a compact subset of the real line and $\{g_1, \ldots, g_k\}$ a specific set of generators for U as a semialgebraic set, then whenever $f(x, y)$ is non-negative on $U \times \mathbb{R}$, there are sums of squares $s_0, \ldots, s_k$ such that $f = s_0 + s_1 g_1 + \cdots + s_k g_k$.

1. Introduction

Throughout, we work in the real polynomial ring in two variables, which we denote by $\mathbb{R}[x, y]$. The set of sums of squares in $\mathbb{R}[x, y]$ is denoted by $\sum \mathbb{R}[x, y]^2$. Recently, Marshall [4] settled a long-standing open problem by proving the following:

Theorem 1. *Suppose $f(x, y) \in \mathbb{R}[x, y]$ is non-negative on the strip $[0, 1] \times \mathbb{R}$. Then there exist $\sigma(x, y), \tau(x, y) \in \sum \mathbb{R}[x, y]^2$ such that*

$$f(x, y) = \sigma(x, y) + \tau(x, y)(x - x^2).$$

An expression $f = \sigma + \tau(x - x^2)$ is an immediate witness to the positivity condition on f. In general, one wants to characterize polynomials f which are positive, or non-negative, on a semialgebraic set $K \subseteq \mathbb{R}^n$ in terms of sums of squares and the polynomials used to define K. Representation theorems of this type have a long and illustrious history, going back at least to Hilbert. There has been much interest in these questions in the last decade, in large part because of applications outside of real algebraic geometry, notably in problems of optimizing polynomial functions on semialgebraic sets. In this paper we look at some generalizations of Marshall's theorem. Our results give many new examples of non-compact semialgebraic sets in $\mathbb{R}^2$ for which one can characterize all polynomials which are non-negative on the set.

Let $\mathbb{R}[X]$ denote $\mathbb{R}[x_1, \ldots, x_n]$, the real polynomial ring in n variables, and write $\sum \mathbb{R}[X]^2$ for the sums of squares in $\mathbb{R}[X]$. Given a finite set $S = \{s_1, \ldots, s_k\} \subseteq \mathbb{R}[X]$ the *basic closed semialgebraic set in $\mathbb{R}^n$ generated by S*, denoted as K_S, is $\{a \in \mathbb{R}^n \mid s_i(a) \geq 0 \text{ for } i = 1, \ldots, k\}$. Note that the strip $[0, 1] \times \mathbb{R}$ is the basic closed semialgebraic set in $\mathbb{R}^2$ generated by $\{x - x^2\}$.

There are two algebraic objects associated with the semialgebraic set K_S: the *quadratic module generated by S*, denoted as M_S, is the set of all elements of $\mathbb{R}[X]$ which can be written as $\sigma_0 + \sigma_1 s_1 + \cdots + \sigma_k s_k$, where each $\sigma_i \in \sum \mathbb{R}[X]^2$; the *preordering* generated by S, denoted as T_S, consists of all elements of the form $\sum_{e \in \{0,1\}^k} \sigma_e s^e$, where s^e denotes $s_1^{e_1} \ldots s_s^{e_s}$ for

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$e = (e_1, \dots, e_s)$, and each $\sigma_e \in \sum \mathbb{R}[X]^2$. In general, $M_S \subsetneq T_S$, although if $|S| = 1$, then clearly $T_S = M_S$. Also, $T_S = M_S$ iff M_S is closed under multiplication iff $s_i \cdot s_j \in M_S$ for all i, j.

We recall briefly what is known about the existence of representations in T_S or M_S for polynomials positive or non-negative on K_S. Schmüdgen [11] showed that if K_S is compact, then every f which is strictly positive on K_S is in T_S, regardless of the choice of generators S. However, in general, one cannot replace $f > 0$ on K_S by $f \geq 0$ on K_S, or replace T_S by M_S. If K_S is not compact and $\dim(K_S) \geq 3$ then by Scheiderer [9, Proposition 6.1], there always exist polynomials f which are positive on K_S, but not in T_S, regardless of the choice of generators S. The same is true if $\dim(K_S) = 2$ and K_S contains an open cone, by Powers and Scheiderer [8, Proposition 3.7] (see also [2, Theorem 3.9]). By Kuhlmann and Marshall [2, Theorem 2.2], if $K_S \subseteq \mathbb{R}$ and is not compact, then T_S contains every f which is non-negative on K_S, provided one chooses the right set of generators S. If $K_S \subseteq \mathbb{R}$ and is compact, then by Kuhlmann et al. [3, Theorem 3.5], $M_S = T_S$ and T_S contains all polynomials non-negative on K_S, again provided one chooses the right set of generators.

We say that M_S (respectively, T_S) is *saturated* if for every $f \in \mathbb{R}[X]$, f non-negative on K_S implies $f \in M_S$ (respectively, in T_S). Marshall's Theorem says that the quadratic module in $\mathbb{R}^2$ generated by $x - x^2$ is saturated. This was only the second example given of a finitely generated saturated preordering in the non-compact case (the first being the preordering generated by x, $1 - x$ and $1 - xy$ given in [10, Rem. 3.14]), and settled a long-standing open problem.

Our aim in this paper is to give families of examples related to Marshall's theorem. In the next section, we generalize Marshall's result to the case $U \times \mathbb{R}$, where U is any compact set in $\mathbb{R}$; more precisely, we show that if $S \subseteq \mathbb{R}[x]$ is the "obvious" set of generators for U, then the quadratic module in $\mathbb{R}[x, y]$ generated by S is saturated. In Section 3, we look at some non-compact subsets of a strip $[a, b] \times R$ which are bounded as $y \to -\infty$; we refer to such a set as a half-strip in $\mathbb{R}^2$. We give a representation theorem for a half-strip of the form $(U \times \mathbb{R}) \cap \{y \geq q(x)\}$, where $U \subseteq \mathbb{R}$ is compact and $q(x) \geq 0$ on U. We give other examples of half-strips for which the corresponding preordering is saturated, as well as a family of negative examples.

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2. Polynomials non-negative on strips in the plane

In this section, we give representation theorems for non-compact basic closed semialgebraic sets which are contained in a subset of $\mathbb{R}^2$ of the form $[a, b] \times \mathbb{R}$ and are unbounded as $y \to \pm\infty$. We refer to such a set as a *strip* in the plane. We start with a representation theorem for strips of the form $U \times \mathbb{R}$, where $U \subseteq \mathbb{R}$ is compact. More precisely, we show that the quadratic module corresponding to $U \times \mathbb{R}$ is saturated, as long as we choose the right set of generators. We end this section with a few remarks about the more general case of $U \times W$, where $W \subseteq \mathbb{R}$ is a non-compact basic closed semialgebraic set.

For the rest of this section, fix $U \subseteq \mathbb{R}$ compact, say $U = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$, where $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. Define $S \subseteq \mathbb{R}[x]$ by

$$S = \{x - a_1, (x - a_2)(x - b_1), \dots, (x - a_k)(x - b_{k-1}), b_k - x\}.$$

Then the basic closed semialgebraic set generated by S in $\mathbb{R}$ (respectively in $\mathbb{R}^2$) is U (respectively $U \times \mathbb{R}$). Following [3], we call S the *natural set of generators* for $U \times \mathbb{R}$. The results in [3] have immediate application to our case:

Proposition 1. *Suppose U and S are as above. Let T and M be the preorder and quadratic module generated by S in $\mathbb{R}[x, y]$. Then:*

a. *$T = M$. In particular, M is closed under multiplication.*
b. *If $f \in \mathbb{R}[x]$ is non-negative on U, then $f \in M$.*

Proof. (a) By Kuhlmann et al. [3, Theorem 3.5], this is true in $\mathbb{R}[x]$, and hence it is true in $\mathbb{R}[x, y]$ as well.
(b) This follows from [3, Corollary 3.6]. □

Our goal in this section is to prove the following:

Theorem 2. *Let U and S be as above and M the quadratic module in $\mathbb{R}[x, y]$ generated by S. Then M is saturated. In other words, if $f(x, y) \in \mathbb{R}[x, y]$ is non-negative on $U \times \mathbb{R}$, then $f \in M$.*

From Proposition 1, Theorem 2 is true if f is a polynomial in x only. So suppose we have $f \in \mathbb{R}[x, y]$ such that $f \geq 0$ on $U \times \mathbb{R}$ and $\deg_y f \geq 1$. Since f is positive as $y \to \pm\infty$, it follows that f has even degree as a polynomial in y and that the leading coefficient of f as a polynomial in y is non-negative on U.

Next we show that it is enough to prove Theorem 2 for the case where the leading coefficient of f (as a polynomial in y) is positive on U. The proof is a straightforward generalization of the proof of [4, Lemma 2.1]. For ease of exposition, throughout this section let $s_0 = 1$, $s_1 = x - a_1$, $s_2 = (b_1 - x)(a_2 - x), \dots, s_k = (b_{k-1} - x)(a_k - x)$, $s_{k+1} = b_k - x$, so that $S = \{s_1, \dots, s_{k+1}\}$.

Lemma 1. *It is enough to prove Theorem 2 for $f \in \mathbb{R}[x, y]$ such that the leading coefficient of f as a polynomial in y is strictly positive on U.*

Proof. Arguing exactly as in the proof of [4, Lemma 2.1], we can reduce to showing that if $h \in \mathbb{R}[x]$ with $h \geq 0$ on U, and h is $\pm$ a product of linear factors $x - r$ with $r \in U$, then for any $f \in \mathbb{R}[x, y]$, $hf \in M$ implies $f \in M$. The proof is by induction on $\deg h$. If $\deg h = 0$, this is trivial; hence we assume $\deg h \geq 1$.

Since $hf \in M$, we have

$$hf = \sigma_0 s_0 + \sigma_1 s_1 + \cdots + \sigma_{k+1} s_{k+1}, \tag{1}$$

where each $\sigma_i \in \sum \mathbb{R}[x, y]^2$.

Given $r \in U$, suppose $x - r$ is a factor of h. There are several cases to consider.

Case 1: Suppose r is in the interior of U; then since h does not change sign at r, it follows that $(x-r)^2$ divides h. Substituting $x = r$ into both sides of (1), we have $0 = \sum_{i=0}^{k+1} \sigma_i(r, y) s_i(r)$. Since each $s_i(r)$ is positive, it follows that $\sigma_i(r, y) = 0$ for all $y \in \mathbb{R}$. Thus $\sigma_i(r, y)$ is identically zero, which implies that $x - r$ divides each coefficient of $\sigma_i(x, y)$, and consequently $x - r$ divides $\sigma_i(x, y)$. Since $\sigma_i(x, y)$ is a sum of squares, it follows that $(x-r)^2$ divides $\sigma_i(x, y)$. Dividing both sides of (1) by $(x-r)^2$, we are done by induction.

Case 2: Suppose $x - a_1$ or $x - b_k$ divides h. We give the proof for $x - a_1$; the proof for $x - b_k$ is similar. If $x - a_1$ divides h, substituting $x = a_1$ into (1), we have $0 = \sigma_0(a_1, y) + \sum_{i=2}^{k+1} \sigma_i(a_1, y) s_i(a_1)$. Since $s_i(a_1) > 0$ for $2 \leq i \leq k + 1$, arguing as in the first case, this implies that $(x - a_1)^2$ divides $\sigma_i(x, y)$ for $i = 2, \ldots, k + 1$. Let $\tau_i(x, y) = \sigma_i(x, y)/(x - a_1)^2 \in \sum \mathbb{R}[x, y]^2$. Dividing both sides of (1) by $x - a_1$, we obtain

$$\frac{h}{x - a_1} f = \tau_0(x - a_1) + \sigma_1 + \tau_2(x - a_1)s_2 + \cdots + \tau_{k+1}(x - a_1)s_{k+1}. \tag{2}$$

Since M is closed under multiplication, we have that $(x - a_1)s_i \in M$ for each i. It follows that the right-hand side of (2) is in M and we are done by induction.

Case 3: Suppose neither Case 1 nor Case 2 applies; then h contains a factor $x - a_i$ for $2 \leq i \leq k$, or $x - b_i$ for $1 \leq i \leq k - 1$. We give the proof for $x - a_i$; the proof for $x - b_i$ is the same. Since $h \geq 0$ on U and h contains no factors $x - r$ for r an interior point of U, and h does not contain a factor $x - a_1$ or $b_k - x$ either, it follows that h contains a factor $(x - a_i)^2$ or a factor $(x - a_i)(b_i - x) = s_i$. In the first case, applying the argument of Case 2 twice, we see that $(x - a_i)^2$ must divide every term on the right-hand side of (1) and we are done by induction. In the second case, we argue as in Case 2 to conclude that s_i divides every term on the right-hand side of (1) and we are again done by induction. □

Lemma 2. *We may assume that f has finitely many zeros on $U \times \mathbb{R}$.*

Proof. The proof is essentially the same as the proof of [4, Lemma 2.2]. □

Lemma 3. *Suppose $f = \sum_{i=0}^{2d} a_i(x)y^i$ is non-negative on $U \times \mathbb{R}$, f has only finitely many zeros in $U \times \mathbb{R}$, and $a_{2d} > 0$ on U. Then there exists $\epsilon(x) \in \mathbb{R}[x]$, with $\epsilon(x) \geq 0$ on U, such that $f(x, y) \geq \epsilon(x)(1 + y^2)^d$ holds on $U \times \mathbb{R}$, and for each $x \in U$, $\epsilon(x) = 0$ if and only if there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$.*

Proof. By Marshall [4, Lemma 4.2] and its proof, for $i = 1, \ldots, k$, there exists a polynomial $\epsilon_i(x) \in \mathbb{R}[x]$, with $\epsilon_i(x) \geq 0$ on $[a_i, b_i]$, such that $f(x, y) \geq \epsilon_i(x)(1 + y^2)^d$ holds on $[a_i, b_i] \times \mathbb{R}$, $\epsilon_i(x) = 0$ for $x \in [a_i, b_i]$ if and only if there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$, and $\epsilon_i(x) \neq 0$ for $x \in \mathbb{R}\backslash[a_i, b_i]$.

Dividing each ϵ_i by the maximum of $\{\epsilon_i(x) \mid x \in U\}$ and 1, we may assume that each $\epsilon_i(x) \leq 1$ on U. Let $\epsilon(x) = \left(\prod_{i=1}^{k} \epsilon_i(x)\right)^2$; then $\epsilon(x) \geq 0$ on U, and

$$f(x, y) \geq \epsilon(x)(1 + y^2)^d$$

holds on $U \times \mathbb{R}$. For each $x \in U$, the polynomial $\epsilon(x) = 0$ if and only if some $\epsilon_i(x) = 0$; hence $\epsilon(x) = 0$ if and only if there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$. □

In [4, Lemma 4.4], it is shown that if $f \in \mathbb{R}[x, y]$ such that $f \geq 0$ on $[0, 1] \times \mathbb{R}$ and the leading coefficient of f is positive on the interval $[0, 1]$, then for each $r \in [0, 1]$ there is a representation of f involving the generators of the quadratic module and functions of the form $\sum g_i^2$, where each g_i is a polynomial in y with coefficients analytic functions of x in some neighborhood of r. In our case, we need the same result with $[0, 1]$ replaced by U. This follows immediately from [4, Lemma 4.4] unless $r = a_i$ for $2 \leq i \leq k$ or $r = b_i$ for $1 \leq i \leq k - 1$; for the latter cases we need one extra step.

Lemma 4. *Suppose $f \in \mathbb{R}[x, y]$ is non-negative on $U \times \mathbb{R}$, and the leading coefficient of f as a polynomial in y is strictly positive on U. Then:*

1. *For each r in the interior of U, there exist g_1, g_2 polynomials in y with coefficients analytic functions of x in some open neighborhood $V(r)$ of r, such that $f = g_1^2 + g_2^2$ on $V(r) \times \mathbb{R}$.*
2. *There exist g_l, h_l, with $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(a_1)$ of a_1 such that $f = \sum_{l=1}^{2} g_l^2 + \sum_{l=1}^{2} h_l^2(x - a_1)$ on $V(a_1) \times \mathbb{R}$.*

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3. *For $i = 1, \ldots, k-1$, there exist g_l, h_l, with $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(b_i)$ of b_i such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (b_i - x)(a_{i+1} - x)$ on $V(b_i) \times \mathbb{R}$.*
4. *For $i = 1, \ldots, k-1$, there exist g_l, h_l, $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(a_{i+1})$ of a_{i+1} such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (b_i - x)(a_{i+1} - x)$ on $V(a_{i+1}) \times \mathbb{R}$.*
5. *There exist g_l, h_l, with $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(b_k)$ of b_k, such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (b_k - x)$ on $V(b_k) \times \mathbb{R}$.*

Proof. (1), (2) and (5) follow from [4, Lemma 4.4], using a change of variables, if necessary.

For (3), if x is sufficiently close to b_i, by Marshall [4, Lemma 4.4], there exist $\varphi_l(x, y)$, $\psi_l(x, y)$, $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(b_i)$ of b_i, such that

$$f = \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \psi_l^2 \,(b_i - x).$$

We have

$$\begin{aligned} f &= \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \frac{\psi_l^2}{(a_{i+1} - x)} \,(b_i - x)(a_{i+1} - x) \\ &= \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \left(\frac{\psi_l}{\sqrt{a_{i+1} - x}} \right)^2 \,(b_i - x)(a_{i+1} - x). \end{aligned}$$

As $\dfrac{1}{\sqrt{a_{i+1} - x}}$ is analytic for x close to b_i, by taking $g_l = \varphi_l$ and $h_l = \dfrac{\psi_l}{\sqrt{a_{i+1} - x}}$, we get the desired result.

A similar proof shows that (4) holds. □

We need the following version of the Weierstrass Approximation Theorem, which is an immediate generalization of [4, Proposition 4.5]

Proposition 2. *Suppose $\phi, \psi : U \to \mathbb{R}$ are continuous functions, where $U \subseteq \mathbb{R}$ is compact, $\phi(x) \le \psi(x)$ for all $x \in U$, and $\phi(x) < \psi(x)$ for all but finitely many $x \in U$. If ϕ and ψ are analytic at each point $a \in U$ where $\phi(a) = \psi(a)$ then there exists a polynomial $p(x) \in \mathbb{R}[x]$ such that $\phi(x) \le p(x) \le \psi(x)$ holds for all $x \in U$.*

We are now ready to prove Theorem 2. As above, we denote the elements of S by $\{s_1, \ldots, s_{k+1}\}$. Let $f(x, y) = \sum_{j=0}^{2d} a_j(x) y^j$, where $d \ge 1$, $a_{2d}(x) > 0$ on U, and $f(x, y)$ has only finitely many zeros in $U \times \mathbb{R}$. By Lemma 3, we have $\epsilon(x) \in \mathbb{R}[x]$ such that $\epsilon(x) \ge 0$ on U, $f(x, y) \ge \epsilon(x)(1 + y^2)^d$, and $\epsilon(x) = 0$ iff $y \in U$ such that $f(x, y) = 0$. Let $f_1(x, y) := f(x, y) - \epsilon(x)(1 + y^2)^d$; then $f_1 \ge 0$ on $U \times \mathbb{R}$. Replacing $\epsilon(x)$ by $\frac{\epsilon(x)}{N}$, $N > 1$, if necessary, we can assume f_1 has degree $2d$ as a polynomial in y, and the leading coefficient of f_1 is positive on U.

By Lemma 4, for each $r \in U$, there exists an open neighborhood $V(r)$ of r such that

$$f_1 = \sum_{j=1}^{2} g_{0,j,r}(x, y)^2 + \sum_{j=1}^{2} g_{1,j,r}(x, y)^2 \, s_1 + \cdots + \sum_{j=1}^{2} g_{k+1,j,r}(x, y)^2 \, s_{k+1} \tag{3}$$

on $V(r) \times \mathbb{R}$, where $g_{i,j,r}(x, y)$ are polynomials in y of degree $\le d$ with coefficients analytic functions of x in $V(r)$, for $i = 0, \ldots, k+1$ and $j = 1, 2$. If r is in the interior of U, note that $g_{i,j,r} = 0$ for $i \neq 0$. If $r = a_1$, then $g_{i,j,r} = 0$ for $i \neq 1$, etc.

Since U is compact, there are finitely many $V(r_1), \ldots, V(r_p)$ which cover U and, since $\epsilon(x)$ has only finitely many roots in U, we choose the open cover such that no $V(r_l)$ contains more than one root of $\epsilon(x)$, and no root is in more than one $V(r_l)$. Let $1 = \nu_1 + \cdots + \nu_p$ be a partition of unity corresponding to the open cover of U, and note that by construction, if a root u of $\epsilon(x)$ is in $V(r_l)$, then $\nu_l(x) = 1$ for x close to u. Since U is compact, there are finitely many $V(r_1), \ldots, V(r_p)$ which cover U.

Define $\varphi_{i,j,l}$, polynomials in y with coefficients functions of x as follows: the coefficient of y^q in $\varphi_{i,j,l}$ is $\sqrt{\nu_l(x)}$ times the coefficient of y^q in g_{i,j,r_l}. Then we have

$$f_1 = \sum_{l=1}^{p} \nu_l f_1 = \sum_{l=1}^{p} \left(\sum_{j=1}^{2} \varphi_{0,j,l}^2 + \sum_{j=1}^{2} \varphi_{1,j,l}^2 \, s_1 + \cdots + \sum_{j=1}^{2} \varphi_{k+1,j,l}^2 \, s_{k+1} \right) \tag{4}$$

on $U \times \mathbb{R}$.

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We approximate the coefficients of the $\varphi_{i,j,l}$'s by polynomials, using Proposition 2. Fix $\varphi_{i,j,l}$ and a coefficient $u(x)$. Define $\phi, \psi : U \to \mathbb{R}$ by $\phi(x) = u(x) - \frac{2}{5}\epsilon(x)$, and $\psi(x) = u(x) + \frac{2}{5}\epsilon(x)$. Then by our construction, $\phi(x)$ and $\psi(x)$ satisfy all of the conditions of Proposition 2, and so there exists $w \in \mathbb{R}[x]$ such that

$$u(x) - \frac{2}{5}\epsilon(x) \le w(x) \le u(x) + \frac{2}{5}\epsilon(x), \quad \text{for each } x \in U. \tag{5}$$

Now we use these $w(x)$'s to define, for each triple i, j, l, a polynomial $h_{i,j,l}$, where $\deg_y h_{i,j,l} = \deg_y \varphi_{i,j,l}$, and if $u(x)$ is the coefficient of y in φ, and $w(x)$ is the coefficient of y in h, then (5) holds. Finally, let

$$h_l(x, y) := \sum_{j=1}^{2} h_{0,j,l}(x, y)^2 + \sum_{j=1}^{2} h_{1,j,l}(x, y)^2 s_1 + \cdots + \sum_{j=1}^{2} h_{k+1,j,l}(x, y)^2 s_{k+1}.$$

We have polynomials h_l and $\delta \in \mathbb{R}[x, y]$ such that

$$f_1 = \left(\sum_{l=1}^{p} h_l(x, y)\right) + \delta(x, y),$$

where $\delta(x, y) = \sum_{i=0}^{2d} c_i(x)y^i$ and $|c_i(x)| \le \frac{2}{5}\epsilon(x)$ on U, for all i.

This yields $f(x, y) = f_1(x, y) + \epsilon(x)(1 + y^2)^d = \sum_{l=1}^{p} h_l(x, y) + t_1(x, y) + t_2(x, y)$, where

$$t_1(x, y) := \frac{2}{5}\epsilon(x)(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} c_i(x)y^i,$$

$$t_2(x, y) := \epsilon(x)\left[(1 + y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d})\right].$$

We have $\sum_{l=1}^{p} h_l(x, y) \in T$ and we can prove that $t_1, t_2 \in T$ exactly as in [4]. Therefore $f(x, y) \in T$. This completes the proof of Theorem 2.

Suppose $\tilde{U} \subseteq \mathbb{R}$ is a non-compact basic closed semialgebraic set. An obvious question to ask is what happens if we replace $U \times \mathbb{R}$ by $U \times \tilde{U}$? First we note that by Powers and Reznick [6, Theorem 2], if $S \subseteq \mathbb{R}[x, y]$ such that $K_S = U \times \mathbb{R}^+$, then M_S cannot be saturated, regardless of the choice of generators S. Furthermore, if $S \subseteq \mathbb{R}[x]$ generates $\tilde{U}$ as a semialgebraic set in $\mathbb{R}$, then T_S is saturated iff S contains the natural set of generators [2, Theorem 2.2]. This means that the best theorem that we could hope for is the following: let $S_1 \subseteq \mathbb{R}[x]$ be the natural set of generators for U and $S_2 \subseteq \mathbb{R}[y]$ the natural set of generators for $\tilde{U}$; then the preordering in $\mathbb{R}[x, y]$ generated by $S_1 \cup S_2$ is saturated. We have the following partial result, which is [5, Corollary 11]:

Theorem 3. *Let U, $\tilde{U}$, S_1, and S_2 be as above and T the preordering in $\mathbb{R}[x, y]$ generated by $S_1 \cup S_2$. If $f = \sum_{i=0}^{d} a_i(x)y^i \subseteq \mathbb{R}[x, y]$ such that $f > 0$ on $U \times \tilde{U}$ and $a_d > 0$ on U, then $f \in T$.*

We conjecture that the above theorem is true without one or both of the assumptions on f.

3. Half-strips and further examples

In this section we look at non-compact basic closed semialgebraic subsets of a strip $[a, b] \times \mathbb{R}$ which are bounded as $y \to -\infty$. We refer to such a set as a *half-strip* in $\mathbb{R}^2$. We give a representation theorem for the half-strip $\{(x, y) \in \mathbb{R}^2 \mid x \in U, y \ge q(x)\}$, where $U \subseteq \mathbb{R}$ and $q(x) \in \mathbb{R}[x]$ with $q(x) \ge 0$ on U. This follows from Theorem 2 by an elementary argument. We give a few other examples of saturated preorderings in the half-strip case as well as a family of negative examples. (See Figs. 1–3.) Finally, we give an example of a non-compact surface in $\mathbb{R}^3$ for which the corresponding preordering is saturated.

Remark 1. Suppose $U \subseteq \mathbb{R}$ is compact and S the natural choice of generators for U. We saw in the previous section that in $\mathbb{R}[x, y]$, the preordering generated by S and the quadratic module generated by S are the same. However, in [7, Theorem 2], it is shown that if S any set of generators in $\mathbb{R}[x]$ for $[0, 1]$, then the quadratic module generated by S and y is not saturated. Hence in the half-strip case, our representation theorems will hold only for preorderings and not quadratic modules as in the strip case.

Theorem 4. *Given compact $U \subseteq \mathbb{R}$ with the natural choice of generators $\{s_1, \ldots, s_k\}$ and $q(x) \in \mathbb{R}[x]$ with $q(x) \ge 0$ on U, let $S = \{s_1, \ldots, s_k, y - q(x)\}$ and K be the half-strip K_S. If T is the preordering in $\mathbb{R}[x, y]$ generated by S, then T is saturated.*

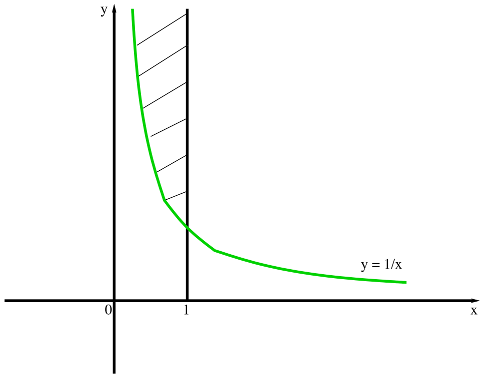

Fig. 1. Half-strip cut by $xy = 1$.

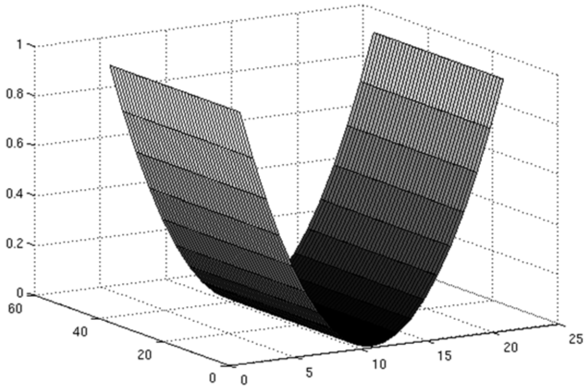

Fig. 2. Half-strip in $\mathbb{R}^3$.

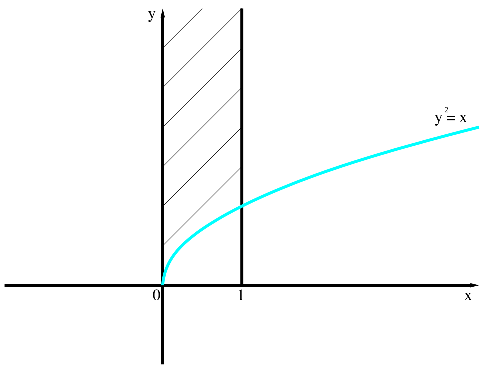

Fig. 3. Half-strip cut by $y^2 = x$.

Proof. We first claim that it is enough to prove the theorem for $q(x) = 0$, i.e., the case where $S = \{s_1, \ldots, s_k, y\}$ and the corresponding semialgebraic set is $U \times \mathbb{R}^+$. Suppose that the preordering $W \subseteq \mathbb{R}[u, v]$ generated by $\{s_1(u), \ldots, s_k(u), v\}$ is saturated and we have that $f(x, y) = \sum_{i=0}^{k} a_i(x)y^i$ is non-negative on K. Define g in $\mathbb{R}[u, v]$ by $g(u, v) := \sum a_i(u)(q(u)+v)^j$. Then $f(x, y) \geq 0$ on K implies $g(u, v) \geq 0$ on $U \times \mathbb{R}^+$. Hence $g \in W$. Substituting $u = x$, $v = y - q(x)$ in a representation of g in W, we obtain a representation of f in T.

We are reduced to proving the theorem for $S = \{s_1, \ldots, s_k, y\}$. If $f(x, y) \geq 0$ on $U \times \mathbb{R}^+$, then $f(x, y^2) \geq 0$ on $U \times \mathbb{R}$. Then, by Theorem 2, we can write $f(x, y^2)$ as a sum of terms of the form $(\sum_{i=0}^{m} h_i(x, y)^2)s_j$ for $j = 0, 1, \ldots, k$ (where we set $s_0 = 1$).

We have

$$f(x, y^2) = \frac{1}{2}f(x, y^2) + \frac{1}{2}f(x, (-y)^2) = \frac{1}{2}\sum_j \left(\sum h_i(x, y)^2\right) s_j + \frac{1}{2}\sum_j \left(\sum h_i(x, -y)^2\right) s_j.$$

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Using the standard identity

$$\frac{1}{2}\left(\sum_i a_i y^i\right)^2 + \frac{1}{2}\left(\sum_i a_i(-y)^i\right)^2 = \left(\sum_j a_{2j}y^{2j}\right)^2 + \left(\sum_j a_{2j+1}y^{2j}\right)^2 \cdot y^2$$

we have that $f(x, y^2)$ can be written as a sum of polynomials of the form

$$\left(\sum_i \sigma_i(x, y^2)^2 + \tau_i(x, y^2)^2 \cdot y^2\right) s_j.$$

Replacing y^2 by y yields a representation of $f(x, y)$ in T. □

Combining Theorem 4 with a substitution technique from work of Scheiderer [10], we can obtain more examples of half-strips for which the corresponding preordering is saturated.

Example 1. Let $S = \{x - x^2, xy - 1\}$ so that K_S is the upper half $[0, 1] \times \mathbb{R}^+$ cut by $xy = 1$. We claim that T_S is saturated.

Suppose $f(x, y) \geq 0$ on K_S. Pick an integer $n \geq 0$ large enough that $x^{2n}f \in \mathbb{R}[x, xy]$. Define g in $\mathbb{R}[u, v]$ by $g(u, v) := u^{2n}f(u, \frac{v}{u})$, i.e., $g(x, xy) = x^{2n}f(x, y)$. As $f(x, y) \geq 0$ on K_S, $g(u, v) \geq 0$ on $[0, 1] \times \mathbb{R}^+$; hence, by Theorem 4, there exist $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \sum \mathbb{R}[u, v]^2$ such that

$$g(u, v) = \sigma_0 + \sigma_1(u - u^2) + \sigma_2(v - 1) + \sigma_3(u - u^2)(v - 1).$$

Then

$$x^{2n}f(x, y) = \sigma_0(x, xy) + \sigma_1(x, xy)(x - x^2) + \sigma_2(x, xy)(xy - 1) + \sigma_3(x, y)(x - x^2)(xy - 1). \tag{6}$$

Define $s_m := \dfrac{\sigma_m}{x^{2n}}$, $m = 0, \ldots, 3$. As x^{2n} divides each of the sums on the right-hand side of (6), the s_m are sos in $\mathbb{R}[x, y]$. Thus f can be written as

$$f(x, y) = s_0(x, y) + s_1(x, y)(xy - 1) + (s_2(x, y) + s_3(x, y)(xy - 1))(x - x^2).$$

Hence $f \in T_S$.

Next we give an example of $S \subseteq \mathbb{R}[x, y, z]$ such that K_S is non-compact of dimension 2, and T_S is saturated.

Example 2. Suppose $S = \{1 - x^2, z - x^2, x^2 - z\}$ so that K_S $\{(x, y, z) \mid -1 \leq x \leq 1, z = x^2\}$. We claim that T_S is saturated.

Given $f(x, y, z) \geq 0$ on K_S, write $f = \sum g_i(x, y)z^i = \sum g_i(x, y)(z^i - x^{2i}) + \sum g_i(x, y)x^{2i}$, where $g_i(x, y) \in \mathbb{R}[x, y]$. Then $\sum g_i(x, y)(z^i - x^{2i})$ is in the ideal generated by $z - x^2$ and hence in T_S. Let $g(x, y) = \sum g_i(x, y)x^{2i} = f(x, y, x^2)$. Since $f(x, y, z) \geq 0$ on K, this implies that $g(x, y) \geq 0$ on $[-1, 1] \times \mathbb{R}$. By Theorem 1, we have $g(x, y) = \sigma(x, y) + \tau(x, y)(1 - x^2)$, where $\sigma, \tau \in \sum \mathbb{R}[x, y]^2$. Thus $f \in T_S$.

We end with a family of examples of half-strips for which no corresponding finitely generated preordering is saturated. This is a generalization of an example due to Netzer, see [1, Lemma 7.4].

Proposition 3. *Suppose $m \in \mathbb{N}$ is even and $q(x) \in \mathbb{R}[x]$ with $\deg q$ odd and $q(x) \geq 0$ on $[0, 1]$. Let $K = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y^m \geq q(x), y \geq 0\}$. Then there is no finite set of generators $S \subseteq \mathbb{R}[x, y]$ with $K_S = K$ such that T_S is saturated.*

Proof. Suppose $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[x, y]$ is such that $K_S = K$ and T_S is saturated. For $c \in [0, 1]$, let T_c be the preordering in $\mathbb{R}[x]$ generated by $\{g_1(c, y), \ldots, g_s(c, y)\}$; then T saturated implies that T_c is saturated. Since $\{g_1(c, y) \geq 0, \ldots, g_s(c, y) \geq 0\} = [q(c)^{\frac{1}{m}}, \infty)$, by Theorems 2.1 and 2.2 in [2], $y - q(c)^{\frac{1}{m}}$ must be among the $g_i(c, y)$ up to a constant factor. Without loss of generality, we can assume

$$g_1(c, y) = r(c)\left(y - q(c)^{\frac{1}{m}}\right),$$

for infinitely many $c \in [0, 1]$ and some positive function r. Let d be the degree of $g_1(x, y)$ in y, and write $g_1(x, y) = \sum_{i=0}^{d} a_i(x)y^i$ with $a_i(x) \in \mathbb{R}[x]$. Then

$$g_1(c, y) = r(c)\left(y - q(c)^{\frac{1}{m}}\right) = a_0(c) + a_1(c)y + \cdots + a_d(c)y^d$$

for infinitely many $c \in [0, 1]$. Comparing coefficients, this implies $a_0(c) = -r(c)q(c)^{\frac{1}{m}}$ and $a_1(c) = r(c)$ for infinitely many $c \in [0, 1]$. Hence, since a_0, a_1 are polynomials, $a_0(x)^m = a_1(x)^m q(x) \in \mathbb{R}[x]$. But this is a contradiction, since the degree of the left-hand side is $m \cdot \deg a_0(x)$ while the degree of the right-hand side is $m \cdot \deg a_1(x) + \deg q(x)$, which implies that one is even and one is odd. □

Example 3. Suppose $S = \{x - x^2, y^2 - x, y\}$, so that K_S is the half-strip $[0, 1] \times \mathbb{R}^+$ cut by the parabola $y^2 = x$. Then, by the previous proposition, no finitely generated preordering corresponding to K_S is saturated.

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