Positive Polynomials and Sums of Squares: Theory and Practice

Victoria Powers *

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Abstract

If a real polynomial f can be written as a sum of squares of real polynomials, then clearly f is nonnegative on $\mathbb{R}^n$, and an explicit expression of f as a sum of squares is a certificate of positivity for f. This idea, and generalizations of it, underlie a large body of theoretical and computational results concerning positive polynomials and sums of squares. In this survey article, we review the history of the subject and give an overview of recent results, both theoretical results concerning the existence of certificates of positivity and work on computational and algorithmic issues.

Keywords: sums of squares, positive polynomials, certificates of positivity, Hilbert's 17th Problem, positivstellensätze.

> In theory, theory and practice are the same. In practice, they are different. - *A. Einstein*

If a real polynomial f in n variables can be written as a sum of squares of real polynomials, then clearly f must take only nonnegative values in $\mathbb{R}^n$. This simple, but powerful, fact and generalizations of it underlie a large body of theoretical and computational results concerning positive polynomials and sums of squares.

An explicit expression of f as a sum of squares is a *certificate of positivity for f*, i.e., a polynomial identity which gives an immediate proof of the positivity of of f on $\mathbb{R}^n$. In recent years, much work has been devoted to the study of certificates of positivity for polynomials. In this paper we will give an overview of some recent results in the theory and practice of positivity and sums of squares, with detailed references to the literature. By "theory", we mean theoretical results concerning the existence of certificates of positivity. By "practice", we mean work on computational and algorithmic issues, such as finding certificates of positivity for a given polynomial.

For the most part, we restrict results to those in a real polynomial ring. This is somewhat misleading, since it is impossible to prove most

*Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322. Email: vicki@mathcs.emory.edu.

of the results for polynomials without using a more abstract approach. For example, in order to obtain a solution to Hilbert's 17th problem, it was necessary for Artin (along with Schreier) to first develop the theory of ordered fields! The reader should keep in mind that underneath the theorems in this paper lie the elegant and beautiful subjects of Real Algebra and Real Algebraic Geometry, among others.

The subject of positivity and sums of squares has been well-served by its expositors. There are a number of books and survey articles devoted to various aspects of the subject. Here we mention a few of these that the interested reader could consult for more details and background on the topics covered in this paper, as well as related topics that are not included: There are the books by Prestel and Delzell [69] and Marshall [44] on positive polynomials, a survey article by Reznick [75] about psd and sos polynomials with a wealth of historical information, and a recent survey article by Scheiderer [84] on positivity and sums of squares which discusses results up to about 2007. Finally, there is a survey article by Laurent [42] which discusses positivity and sums of squares in the context of applications to polynomial optimization.

1 Preliminaries and background

In this section, we introduce the basic concepts and review some of the fundamental results in the subject, starting with results in the late 19th century. For a fuller account of the historical background, see the survey [75]. For a more detailed survey of the subject up to about 2007, readers should consult the survey article [84].

1.1 Notation

Throughout, we fix $n \in \mathbb{N}$ and let $\mathbb{R}[X]$ denote the real polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$. We denote by $\mathbb{R}[X]^+$ the set of polynomials in $\mathbb{R}[X]$ with nonnegative coefficients. The following monomial notation is convenient: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let X^α denote $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For a commutative ring A, we denote the set of sums of squares of elements of A by $\sum A^2$.

We define the basic objects studied in real algebraic geometry. Given a set G of polynomials in $\mathbb{R}[X]$, the closed semialgebraic set defined by G is

$$\mathcal{S}(G) := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \text{ for all } g \in G\}.$$

If G is finite, $\mathcal{S}(G)$ is the basic closed semialgebraic set generated by G.

The basic algebraic objects of interest are defined as follows. For a finite subset $G = \{g_1, \ldots, g_r\}$ of $\mathbb{R}[X]$, the preordering generated by G is

$$PO(G) := \{ \sum_{e=(e_1,\ldots,e_r)\in\{0,1\}^r} s_e g_1^{e_1} \ldots g_r^{e_r} \mid \text{ each } s_e \in \sum \mathbb{R}[X]^2 \}.$$

The quadratic module generated by G is

$$M(G) := \{ s_0 + s_1 g_1 + \cdots + s_r g_r \mid \text{ each } s_i \in \sum \mathbb{R}[X]^2 \}.$$

Notice that if $f \in PO(G)$ or $f \in M(G)$, then f is clearly positive on $\mathcal{S}(G)$ and an identity $f = \sum_{e\in\{0,1\}^r} s_e g_1^{e_1} \ldots g_r^{e_r}$ or $f = s_0 + s_1 g_1 + \cdots + s_r g_r$ is a certificate of positivity for f on $\mathcal{S}(G)$.

Traditionally, a result implying the existence of certificates of positivity for polynomials on semialgebraic sets is called a *Positivstellensatz* or a *Nichtnegativstellensatz*, depending on whether the polynomial is required to be strictly positive or non-strictly positive on the set. We will use the term "representation theorem" for any theorem of this type and refer to a "representation of f" (as a sum of squares, in the preordering, etc.), meaning an explicit identity for f.

1.2 Classic results

A polynomial $f \in \mathbb{R}[X]$ is positive semidefinite, psd for short, if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. We say f is sos if $f \in \sum \mathbb{R}[X]^2$. Of course, f sos implies that f is psd, and for $n = 1$, the converse follows from the Fundamental Theorem of Algebra.

We begin our story in 1888, when the 26-year-old Hilbert published his seminal paper on sums of squares [28] in which he showed that for $n \geq 3$, there exist psd forms (homogenous polynomials) in n variables which are not sums of squares. In the same paper, he proved that every psd ternary quartic – homogenous polynomial of degree 4 in 3 variables – is a sum of squares. [1] Hilbert was able to prove that for $n = 3$, every psd form is a sum of squares of rational functions, but he was not able to prove this for $n > 2$. This became the seventeenth on his famous list of twenty-three mathematical problems that he announced at the 1900 International Congress of Mathematicians in Berlin. In 1927, E. Artin [2] settled the question:

Theorem 1 (Artin's Theorem)**.** *Suppose $f \in \mathbb{R}[X]$ is psd, then there exists nonzero $g \in \mathbb{R}[X]$ such that $g^2 f$ is sos.*

[1]Hilbert worked with forms, however for the purposes of this paper we prefer to work in a non-homogenous setting. A form can be dehomogenized into a polynomial in one less variable and the properties of being psd and sos are inherited under dehomogenization. When discussing work related to Hilbert's work, we will use the language of forms, otherwise, we state results in terms of polynomials.

The following Positivstellensatz has until recently been attributed to Stengle [93], who proved it in 1974. It is now known that the main ideas were in a paper of Krivine's from the 1960's.

Theorem 2 (Classical Positivstellensatz)**.** *Suppose $S = \mathcal{S}(G)$ for finite $G \subseteq \mathbb{R}[X]$ and $f \in \mathbb{R}[X]$ with $f > 0$ on S. Then there exist $p, q \in PO(G)$ such that $pf = 1 + q$.*

1.3 Bernstein's and Pólya's theorems

Certificates of positivity for a univariate $p \in \mathbb{R}[x]$ such that $p \geq 0$ or $p > 0$ on an interval $[a, b]$ have been studied since the late 19th century. Questions about polynomials positive on an interval come in part from the relationship with the classic Moment Problem, in particular, Hausdorf's solution to the Moment Problem on $[0, 1]$ [27].

In 1915, Bernstein [6] proved that if $p \in \mathbb{R}[x]$ and $p > 0$ on $(-1, 1)$, then p can be written as a positive linear combination of polynomials $(1 - x)^i(1 + x)^j$ for suitable integers i and j; however, it might be necessary for $i + j$ to exceed the degree of p. Notice that writing p as such a positive linear combination is a certificate of positivity for p on $[-1, 1]$.

Pólya's Theorem, which he proved in 1928 [58], concerned forms positive on the standard $n-1$-simplex $\Delta_{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1\}$.

Theorem 3 (Pólya's Theorem)**.** *Suppose $f \in \mathbb{R}[X]$ is homogeneous and is strictly positive on Δ_{n-1}, then for sufficiently large N, all of the coefficients of $(X_1 + \cdots + X_n)^N f$ are positive.*

Here "all coefficients are positive" means that every monomial of degree $\deg f + N$ appears with a strictly positive coefficient.

Bernstein's result is equivalent to the one-variable dehomogenized version of Pólya's Theorem: If $p \in \mathbb{R}[x]$ is positive on $(0, \infty)$, then there exists $N \in \mathbb{N}$ such that $(1 + x)^N p$ has only positive coefficients. The equivalence is immediate by applying the "Goursat transform" which sends p to

$$(x + 1)^d p\left(\frac{1 - x}{1 + x}\right),$$

where $d = \deg p$.

1.4 Schmüdgen's Theorem and beyond

In 1991, Schmüdgen [88] proved his celebrated theorem on representations of polynomials strictly positive on compact basic closed semialgebraic sets. This result began a period of much activity in Real Algebraic Geometry, which continues today, and stimulated new directions of research.

Theorem 4 (Schmüdgen's Positivstellensatz)**.** *Suppose G is a finite subset of $\mathbb{R}[X]$ and $\mathcal{S}(G)$ is compact. If $f \in \mathbb{R}[X]$ is such that $f > 0$ on $\mathcal{S}(G)$, then $f \in PO(G)$.*

Schmüdgen's theorem yields "denominator-free" certificates of positivity, in contrast to Artin's theorem and the Classic Positivstellensatz. The underlying reason that such certificates exist is that the preordering $PO(G)$ in this case is *archimedean*: Given any $h \in \mathbb{R}[X]$, there exists $N \in \mathbb{N}$ such that $N \pm h \in PO(G)$. Equivalently, there is some $N \in \mathbb{N}$ such that $N - \sum X_i^2 \in PO(G)$. It is a fact that if $\mathcal{S}(G)$ is compact, then $PO(G)$ is archimedean. This follows from Schmüdgen's proof of his theorem; there is a direct proof due to Wörmann [95].

The definition of archimedean for a quadratic module M is the same as for a preordering. If $M(G)$ if archimedean, then it is immediate that $\mathcal{S}(G)$ is compact; the converse is not true in general. In 1993, Putinar [70] gave a denominator-free representation theorem for archimedean quadratic modules.

Theorem 5 (Putinar's Positivstellensatz)**.** *Suppose G is a finite subset of $\mathbb{R}[X]$ and $M(G)$ is archimedean. If $f \in \mathbb{R}[X]$ is such that $f > 0$ on $\mathcal{S}(G)$, then $f \in M(G)$.*

In 1999, Scheiderer began a systematic study of questions concerning the existence of certificates of positivity in a broader setting. Let A be a commutative ring, then $a \in A$ is called psd if its image is nonnegative in every element of the real spectrum of A. One then asks when does psd = sos in A? In a series of fundamental papers, Scheiderer settles this question in many cases for coordinate rings of real affine varieties, and more general rings [80], [81], [83], [86], [87]. This work led to many new representation theorems for polynomial rings. See [84] for a detailed account.

2 Theory: Certificates of Positivity

In this section we look at very recent theoretical results concerning sums of squares, psd polynomials, and certificates of positivity. We start with some modern riffs on Hilbert's 1888 paper. We then look at the sums of squares on algebraic curves. We discuss stability in quadratic modules, a topic which is important in computational questions and applications. Finally, we look at recent work concerning sums of squares in cases where the polynomials have some special structure.

2.1 Psd ternary quartics

Hilbert's 1888 proof that a psd ternary quartic is a sum of three squares of quadratic forms is short, but difficult; arguably a high point of 19th century algebraic geometry. Even today the proof is not easy to understand and Hilbert's exposition lacks details in a number of key points. Several authors have given modern expositions of Hilbert's proof, with details filled in.

There is an approach due to Cassels, published in Rajwade's book Squares [72, Chapter 7], and articles by Rudin [77] and Swan [94]. In 1977, Choi and Lam [15] gave a short elementary proof that a psd ternary quartic must be a sum of five squares of quadratic forms. In 2004, Pfister [54] gave an elementary proof that a psd ternary quartic is a sum of four squares of quadratic forms and he gave an elementary and constructive argument in the case that the ternary quartic has a non-trivial real zero. Very recently, Pfister and Scheiderer [55] gave a complete proof of Hilbert's Theorem, different from Hilbert's proof. Although the proof is not easy, it uses only elementary techniques such as the theorems on implicit functions and symmetric functions.

In the "Practice" section of this paper, we will discuss computational issues around Hilbert's theorem on ternary quartics.

2.2 Hilbert's construction of psd, not sos, polynomials

In Hilbert's 1888 paper, he described how to find psd forms which are not sums of squares. However, his construction did not yield an explicit example of a psd, not sos, polynomial. It took nearly 80 years for an explicit example of a psd, not sos, polynomial to appear in the literature; the first published example was due to Motzkin. Since then, other examples and families of examples have been produced (see the survey [75] for a detailed account), however only recently has there been attempts to exploit the constructive side of Hilbert's proof.

Reznick [73] has isolated the underlying mechanism of Hilbert's construction and shown that it applies to more general situations than those considered by Hilbert. He is then able to produce many new examples of psd, not sos, polynomials.

Hilbert's proof, and Reznick's modern exposition and generalization, use the fact that forms of degree d satisfy certain linear relations, known as the Cayley-Bacharach relations, which are not satisfied by forms of full degree $2d$. Very recently, Blekherman [9] shows that the Cayley-Bacharach relations are, in fact, the fundamental reason that there are psd polynomials that are not sos. In small cases, he is able to give a complete characterization of the difference between psd and sos forms. For example, the result for forms of degree 6 in 3 variables is the following:

Theorem 6 ([9],Theorem 1.1)**.** *Let $H_{3,6}$ be the vector space of degree* 6 *forms in* 3 *variables. Suppose $p \in H_{3,6}$ is psd and not sos. Then there exist two real cubics q_1, q_2 intersecting in 9 (possible complex) projective points $\gamma_1, \ldots, \gamma_9$ such that the values of p on γ_i certify that p is not a sum of squares in the following sense: There is a linear functional l on $H_{3,6}$, defined in terms of the γ_i's, such that $l(q) \geq 0$ for all sos q and $l(p) < 0$.*

2.3 The gap between psd and sos polynomials

It is often useful to view the sets of psd and sos polynomials as cones in the vector space of all polynomials. One might ask how big is the "gap" between the sos and psd polynomials, i.e., what is the quantitative relationship between the cones of psd and sos polynomials. Blekherman [8] showed that for fixed degree, there are significantly more psd polynomials than sos polynomials, in a precise quantitative sense. He gives asymptotic bounds for the sizes of these sets as the number of variables grows.

On the other hand, there are results which show that if the degree is variable, then in some sense sos polynomials are plentiful among psd polynomials. Berg, Christensen, and Ressel [4] showed that sos polynomials are dense among polynomials which are non-negative on the unit cube $[-1, 1]^n$ with respect to the l_1-norm of coefficients. An explicit version of this result is given by Lasserre and Netzer [41]. Lasserre [39], [40] showed that psd polynomials can be approximated coefficient-wise by sos polynomials. Of course, the degrees of the approximating polynomials go to infinity in these results.

Finally, we mention recent work of Chesi [14], who gives a matrix characterization of psd, not sos, polynomials. This characterization is based on eigenvector and eigenvalue decompositions.

2.4 Denominators in Artin's Theorem

Artin's Theorem says that if $f \in \mathbb{R}[X]$ is psd, then there exists nonzero $p \in \mathbb{R}[X]$ such that $p^2 f$ is a sos. We can think of p^2 as a denominator in a representation of f as a sum of squares of rational functions. Artin's proof was not constructive, which leads to a natural question: Given psd f, what type and degree of denominators can occur? In this section, we discuss both classical and more recent work related to denominators in Artin's Theorem.

In 1893, Hilbert showed that if $f \in \mathbb{R}[x, y]$ is psd of degree m, then there exists psd $p \neq 0$ of degree $m - 4$ so that pf is a sum of three squares. This implies that there is a representation of p as a quotient of sums of squares with denominator of degree $\leq (m-4)(m-8)\ldots$. In [21], de Klerk and

Pasechnik use Hilbert's result to give an an algorithm for finding p and writing pf as a sum of squares.

Pólya's Theorem implies that if f is both positive definite and even, then for sufficiently large r, f is a sum of squares of rational functions with common denominator $(1 + \sum X_i)^r$. Habicht [26] used Pólya's Theorem to show that a positive definite form is a quotient of two sums of squares of monomials. It follows that if f is positive definite, then f can be written as a sum of squares of rational functions with positive denominators. In 1995, in [74], Reznick showed that for a positive definite form f, there is N so that $(\sum X_i)^n f$ is sos. He gave a bound on N, in terms of the degree, number of variables, and a measure of how close f is to having a zero.

In the above examples for positive definite forms f, a "uniform" denominator is obtained in the sense that for every such f, $(\sum X_i)^r$ will serve as a denominator. The restriction to positive definite forms in necessary, due to the fact that there exist psd forms f in $n \geq 4$ variables so that if $p^2 f$ is sos, the p must have a specified zero. The existence of these so-called "bad points" means that $(\sum X_i)^r f$ can never be sos. Bad points were first noticed by Straus and were extensively studied by Delzell in his PhD thesis [22].

Scheiderer [83] has shown that in the case of forms of three variables (dehomogenizing, polynomials of two variables), there is a uniform denominator and in fact, any positive de

nite quadratic form will serve. This shows that ternary forms do not have bad points. On the other hand, Reznick [76] showed that for any given n, m, there does not exist a single form p which serves as a denominator for every psd form f in n variables of degree m.

Very recently, Guo, Kaltofen, and Zhi [25] developed an algorithmic method for proving lower bounds for the degree of the denominator in any representation in $\sum \mathbb{R}[X]^2$ of a specified psd polynomial. As an example, they look at some symmetric forms of degree 6 in four and

five variables and prove that any representation as a quotient of sums of squares must have denominator degree at least 4 and 6, respecitively. This will be discussed further in §3.3.

2.5 Polynomials positive on noncompact semialgebraic sets

We now turn to representation theorems for polynomials positive on noncompact basic closed semialgebraic sets. Given finite $G \subseteq \mathbb{R}[X]$, let $\mathcal{S} = \mathcal{S}(G)$ and suppose that $\mathcal{S}$ is not compact. Let $\mathcal{P} = PO(G)$ and $M = M(G)$. We would like to know if Schmüdgen's Theorem or Putinar's Theorem extends to this case: Given $f > 0$ on $\mathcal{S}$, is $f \in \mathcal{P}$ or $f \in M$? More generally, we can ask whether this holds for $f \geq 0$ on $\mathcal{S}$, in which case we say that $\mathcal{P}$

or M is *saturated*. We have the following negative results due to Scheiderer:

Theorem 7 ([80]). *1. Suppose* $\dim \mathcal{S} \geq 3$. *Then there exists* $p \in \mathbb{R}[X]$ *such that* $p \geq 0$ *on* $\mathbb{R}^n$ *and* $p \notin \mathcal{P}$.

2. If $n = 2$ *and* S *contains an open* 2*-dimensional cone, then there is* $p \in \mathbb{R}[X]$ *with* $p \geq 0$ *on* $\mathbb{R}^2$ *and* $p \notin \mathcal{P}$.

In contrast to these, the $n = 1$ case has been completely settled, by Kuhlmann and Marshall [35], extending work of Berg and Maserick [5]. In this case, the preordering $\mathcal{P}$ is saturated, provided one chooses the right set of generators.

Definition 1 ([35], 2.3). Suppose $\mathcal{S}$ is a closed semialgebraic set in $\mathbb{R}$, then $\mathcal{S}$ is a union of finitely many closed intervals and points. Define a set of polynomials F in $\mathbb{R}[x]$ as follows:

- If $a \in \mathcal{S}$ and $(-\infty, a] \cap \mathcal{S} = \emptyset$, then $x - a \in F$.
- If $a \in \mathcal{S}$ and $(a, \infty) \cap \mathcal{S} = \emptyset$, then $ax \in F$.
- If $a, b \in \mathcal{S}$ and $(a, b) \cap \mathcal{S} = \emptyset$, then $(x - a)(x - b) \in F$.

It is easy to see that $\mathcal{S}(F) = \mathcal{S}$; F is called the **natural choice of generators for** $\mathcal{S}$.

Theorem 8 ([35],Thm. 2.2, Thm. 2.5). *Let* $\mathcal{S}$ *be as above and suppose* G *is any finite subset in* $\mathbb{R}[X]$ *such that* $\mathcal{S}(G) = S$. *Let* $\mathcal{P} = PO(G)$ *and let* F *be the natural choice of generators.*

1. *Every* $p \in \mathbb{R}[x]$ *such that* $p \geq 0$ *on* $\mathcal{S}$ *is in* $\mathcal{P}$ *iff the set of generators* G *of* $\mathcal{S}$ *contains* F.
2. *Let* $M = M(F)$, *then every* $p \in \mathbb{R}[x]$ *such that* $p \geq 0$ *on* $\mathcal{S}$ *is in* M *iff* $|F| \leq 1$, *or* $|F| = 2$ *and* $\mathcal{S}$ *has an isolated point.*

One case not covered by the above results is that of noncompact semialgebraic subsets of $\mathbb{R}^2$ which do not contain a 2-dimensional cone. We write $\mathbb{R}[x]$ for the polynomial ring in one variable and $\mathbb{R}[x, y]$ for the polynomial ring in two variables. The first example given of a noncompact basic closed semialgebraic set in $\mathbb{R}^2$ for which the corresponding preordering is saturated is due to Scheiderer [83]. His example is the preordering in $\mathbb{R}[x, y]$ generated by $\{x, 1 - x, y, 1 - xy\}$. Powers and Reznick [63] studied polynomials positive on noncompact rectangles in $\mathbb{R}^2$ and obtained some partial results. They showed that if $F = \{f_1, \ldots, f_r, y\}$ with $f_1, \ldots, f_r \in \mathbb{R}[x]$ and $\mathcal{S}(F)$ is the half-strip $[0, 1] \times \mathbb{R}^+$, then there always exists $g > 0$ on $[0, 1] \times \mathbb{R}^+$ with $g \notin M(F)$. On the other hand, it is shown that under a certain condition,

$g \geq 0$ on $[0,1] \times \mathbb{R}$ implies $g = s + t(x - x^2)$ with $s, t \in \sum \mathbb{R}[x,y]^2$. Recently, Marshall proved this without the condition on g, settling a long-standing open problem.

Theorem 9 ([45])**.** *Suppose $p \in \mathbb{R}[x,y]$ is non-negative on the strip $[0,1] \times \mathbb{R}$. Then there exist $s, t \in \sum \mathbb{R}[x,y]^2$ such that $p = s + t(x - x^2)$.*

In other words, any p which is nonnegative on the strip $[0,1] \times \mathbb{R}$ is in the quadratic module $M(x - x^2)$. This result has been extended by H. Nguyen in her PhD thesis [48] and by Nguyen and Powers.

Theorem 10 ([49], Thm. 2)**.** *Suppose $U \subseteq \mathbb{R}$ is compact and F is the natural choice of generators for U. Let $\mathcal{S} = U \times \mathbb{R} \subseteq \mathbb{R}^2$ and let M be the quadratic module in $\mathbb{R}[x,y]$ generated by F. Then every $p \in \mathbb{R}[x,y]$ with $p \geq 0$ on $\mathcal{S}$ is in M.*

By the result from [63], we know that this does not generalize to the half-strip case, however we do obtain a representation theorem if the quadratic module is replaced by a preordering and we use the natural choice of generators.

Theorem 11 ([49], Thm. 3)**.** *Given compact $U \subseteq \mathbb{R}$ with natural choice of generators $\{s_1, \ldots, s_k\}$ and $q(x) \in \mathbb{R}[x]$ with $q(x) \geq 0$ on U, let $F = \{s_1, \ldots, s_k, y - q(x)\}$, so that $\mathcal{S}(F)$ is the upper half of the strip $U \times \mathbb{R}$ cut by $\{q(x) = 0\}$. If $\mathcal{P}$ is the preordering in $\mathbb{R}[x,y]$ generated by F, then $\mathcal{P}$ is saturated.*

There are also examples for which no corresponding finitely generated preorder is saturated. The following from [49] is a generalization of an example from [17] due to Netzer.

Example 1. Suppose $F = \{x - x^2, y^2 - x, y\}$, so that $\mathcal{S} = \mathcal{S}(F)$ is the half-strip $[0,1] \times \mathbb{R}^+$ cut by the parabola $y^2 = x$. Then for any $\tilde{F} \subseteq \mathbb{R}[x,y]$ such that $\mathcal{S}(\tilde{F}) = \mathcal{S}$, there is some $p \in \mathbb{R}[x,y]$ such that $p \geq 0$ on $\mathcal{S}$ and $p \notin PO(\tilde{F})$.

For all of the positive examples above, the fibers $\mathcal{S} \cap \{y = a\}$ are connected. It is not known if there are positive examples for which this doesn't hold, e.g., we have the following open problem:

Question: Let $\mathcal{S} = \mathcal{S}(\{x - x^2, y^2 - 1\})$ in $\mathbb{R}^2$, so that $\mathcal{S} = [0,1] \times ((-\infty, -1] \cup [1, \infty))$. Given $g \in \mathbb{R}[x,y]$ such that $g \geq 0$ on $\mathcal{S}$, is $g \in PO(\{x - x^2, y^2 - 1\})$?

2.6 Sums of squares on real algebraic varieties

We now look at a more general setting than polynomial rings. Let V be an affine variety defined over $\mathbb{R}$, $\mathbb{R}[V]$ the coordinate ring of V, and $V(\mathbb{R})$

the set of real points of V. Then $f \in \mathbb{R}[V]$ is psd if $f(x) \geq 0$ for all $x \in V(\mathbb{R})$, and f is sos if f is a finite sum of squares of elements of $\mathbb{R}[V]$. It is interesting to ask whether psd = sos in this more general setting.

If $\dim(V) \geq 3$, then Hilbert's result that psd $\neq$ sos has been extended extended to $\mathbb{R}[V]$ by Scheiderer [80] . In the dimension 2 case, Scheiderer proves the surprising theorem that if V is a nonsingular affine surface and $V(\mathbb{R})$ is compact, then psd = sos holds on V, see [83]. There is a nice application of this to Hilbert's 17th problem: If $f \in \mathbb{R}[x, y, z]$ is a psd ternary forms and g is any positive definite ternary form, then there exists $N \in \mathbb{N}$ such that $g^N f$ is sos.

The case where $\dim(V) = 1$ (real algebraic curves) is completely understood in the case where V is irreducible, again due to Scheiderer [81]. In 2010, Plaumann [56] showed that in the reducible case, the answer depends on the irreducible components of the curve, and also on how these irreducible components are configured with respect to each other. He gives necessary and sufficient conditions for psd = sos in this case. He shows, for example, that for the family of curves $C_a = \{(y - x^2)(y - a) = 0\}$ for $a \in \mathbb{R}$ (the union of a parabola and a line), psd $\neq$ sos always.

2.7 Stability

A quadratic module $M = M(g_1, \ldots, g_k)$ in $\mathbb{R}[X]$ is **stable** if there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that the following holds: For every $d \in \mathbb{N}$ and every $f \in M$ with $\deg f \leq d$, there is a representation of f in M, $f = s_0 + s_1 g_1 + \cdots + s_k g_k$ such that for all i, $\deg s_i \leq \phi(d)$. A similar definition can be made for preorders, although stability has been studied mostly in the quadratic module case. The notion of stability was introduced in [66], where it was used to study the multivariable Moment Problem for noncompact semialgebraic sets.

The easiest example of a stable quadratic module in $\mathbb{R}[X]$ is $\sum \mathbb{R}[X]^2$: If f is sos and $f = h_1^2 + \cdots + h_r^2$, then for all i, $\deg h_i^2 \leq \deg f$, since the leading forms of the h_i^2's cannot cancel. A generalization of this simple argument yields families of stable preorderings in [66]. (The arguments apply immediately to quadratic modules as well.) On the other hand, if $\mathcal{S}(G)$ has dimension ≥ 2 and $M(G)$ is archimedean, then $M(G)$ is never stable; this follows from [82, Thm. 5.4].

The notion of stability is important for computational problems as well as applications to the Moment Problem. It is this key property of stability that allows for effective algorithms for the problem of deciding whether $f \in \mathbb{R}[X]$ is sos, and finding an explicit representation if so. See §3.1 for further discussion of these algorithims. In the case of compact semialgebraic sets, the non-stability of the underlying preordering or quadratic module means the problem of finding representations of polynomials positive on the set must be difficult.

Netzer [47] generalizes the idea of stability of a quadratic module to the notion of stable with respect to a given grading on a polynomial ring. The usual notion of stability is then stability with respect to the standard grading. Considering stability with respect to other gradings allows the development of tools to prove stability with respect to the standard grading by proving it first for finitely many non-standard ones. The paper [47] contains interesting new examples of stable quadratic modules.

2.8 Certificates of positivity for polynomials with special structure

If a polynomial f for which there is a certificate of positivity has some special structure, it can happen that there exists a certificate of positivity with nice properties related to the structure. This can have implications for applications, since it can imply the existence of smaller certificates for f than the general theory implies.

2.8.1 Invariant sums of squares

In practical applications of sums of squares, there is often some inherent symmetry in the problem. This symmetry can be exploited to yield finer representation theorems which in turn can lead to a reduction in problem size for applications.

Consider the following general situation: Suppose K is a closed subset of $\mathbb{R}^n$ which is invariant under some subgroup G of the general linear group. Can we characterize G-invariant polynomials which are positive on K? For example, can they be described in terms of invariant sums of squares, or even sums of squares of invariant polynomials?

Gatermann and Parrilo [23] considered these questions in the context of finding effective sum of squares decompositions of invariant polynomials. They look at finding a decomposition of an sos polynomial f which is invariant under the action of a finite group. Cimpric, Kuhlmann, and Scheiderer [17] consider a more general set-up: G is a reductive group over $\mathbb{R}$ acting on an affine $\mathbb{R}$-variety V with an induced dual action on the coordinate ring $\mathbb{R}[V]$ and on the linear dual space of $\mathbb{R}[V]$. In this setting, given an invariant closed semialgebraic set K in $\mathbb{R}^n$, they study the problem of representations of invariant polynomials that are positive on K using invariant sums of squares. Most of their results apply in the case where the group $G(\mathbb{R})$ is compact. They obtain a generalization of the main theorem of [23] and apply their results to an investigation of the equivariant version of the K-moment problem.

2.8.2 Polynomials with structured sparsity

We discuss a "sparse" version of Putinar's theorem, where the variables consist of finitely many blocks that are allowed to overlap in certain ways, and we seek a certificate of positivity for a polynomial f that is sparse in the sense that each monomial in f involves only variables in one block. Then there is a representation of f in the quadratic module in which the sums of squares respect the block structure.

For $I \subseteq \{1, \ldots, n\}$, let X_I denote the set of variables $\{X_i \mid i \in I\}$ and $\mathbb{R}[X_I]$ the polynomial ring in the variables X_I. Suppose that $I_1, \ldots I_r$ are subsets of $\{1, \ldots, n\}$ satisfying the *running intersection property*: For all $i = 2, \ldots, r$, there is some $k < i$ such that $I_i \cap \bigcup_{j<i} I_j \subseteq I_k$. Suppose that for each i, $i = 1, \ldots, r$, we are given a finite set of polynomials $G_i = \{g_1^{(j)}, \ldots, g_{l_j}^{(j)}\}$ in $\mathbb{R}[X_{I_j}]$. Then let $S_j = \mathcal{S}(G_j)$ and let M_j be the quadratic module in $\mathbb{R}[X_{I_j}]$ generated by G_j. Also, let $S = \cap_j S_j$. The following theorem was proven by Lasserre [38] in the case where S has non-empty interior and in the general case by Kojima and Muramatsu [34]:

Theorem 12. *Suppose all of the quadratic modules M_j are archimedean, and $f \in \mathbb{R}[X_{I_1}] + \cdots + \mathbb{R}[X_{I_k}]$ is strictly positive on S. Then $f \in M_1 + \cdots + M_k$.*

Notice that the case $r = 1$ is Putinar's Theorem. Grimm, Netzer, and Schweighofer [24] gave a new simple proof of the theorem.

2.9 Pure states and sums of squares

Recently, a new approach to certificates of positivity for polynomials nonnegative on compact basic closed semialgebraic has been introduced by Burgdorf, Scheiderer, and Schweighofer [11]. Their techniques allow simple, uniform proofs of already known representation theorems, as well as several new results.

This new approach is based on pure states of convex cones in $\mathbb{R}[X]$. The techniques come from the Eidelheit-Kakutani separation theorem for convex sets in a real vector space V and when combined with the Kerin-Milman theorem yield a sufficient condition for membership in a convex cone $C \subseteq V$ provided that C has an order unit (an algebraic interior point). This condition can then be applied to preorderings and quadratic modules in $\mathbb{R}[X]$. Here is a concrete example of the type of results that are proven. Recall that a *semiring* in commutative ring is a subset containing $\{0, 1\}$ and closed under addition and multiplication.

Theorem 13 ([11],Theorem 7.8)**.** *Let $K \subseteq \mathbb{R}^n$ be a nonempty compact convex polyhedron defined by linear inequalities $g_1 \geq 0, \ldots g_s \geq 0$. Let S be* *the semiring in $\mathbb{R}[X]$ generated by $\mathbb{R}^+$ and the polynomials $g_1, \ldots, g_s$. Let F be a face of K and suppose $f \in \mathbb{R}[X]$ satisfies $f\mid_F = 0$ and $f \mid K \setminus F > 0$. For every $z \in F$ and every $y \in K \setminus F$, assume $D_{y-z}f(z) > 0$. Then $f \in S$.*

Here $D_v f(z)$ denotes the directional derivative of f at z in the direction of v. Roughly speaking, the last assumption in the theorem says that every directional derivative of f at a point of F pointing into K and not tangential to F should be strictly positive.

Previous to this work, examples of Nichtnegativstellensätze required that the nonnegative polynomial f on a compact basic closed semialgebraic set $\mathcal{S}$ have discrete zeros in $\mathcal{S}$. Results in [11] are the first that allow f to have arbitrary zeros in $\mathcal{S}$.

Example 2 ([11], Example 7.13)**.** Suppose M is an archimedean quadratic module in $\mathbb{R}[x, y, z]$, $K = \{x \in \mathbb{R}^3 \mid g(x) = 0 \text{ for all } g \in M\}$ and let $Z = \{(0, 0, t) \mid t \in \mathbb{R}\}$, the z-axis in $\mathbb{R}^3$. Assume $p, q, r \in \mathbb{R}[x, y, z]$ are such that

$$f = x^2p + y^2q + 2xyr,$$

$f > 0$ on $\mathcal{S} \setminus Z$, and $f = 0$ on Z. Then if p and $pq - r^2$ are strictly positive on $Z \cap \mathcal{S}$, $f \in M$.

3 Practice: Computational and algorithmic issues

Recently, there has been much interest in developing algorithms for deciding positivity of a polynomial and finding certificates of positivity, in part because of the many applications of these algorithms. In this section, we discuss computational problems and issues related to postivity and sums of squares. We will discuss algorithms for finding explicit certificates of positivity for $f \in \mathbb{R}[X]$, both in the global case (sums of squares) and for f positive on a compact basic closed semialgebraic set (algorithmic Schmüdgen and Putinar theorems). We also discuss computational issues around Bernstein's Theorem and Pólya's Theorem as well as quantitative questions on psd ternary quartics (Hilbert's Theorem).

3.1 Finding sum of squares representations

For $f \in \mathbb{R}[X]$, suppose we would like to decide if f is sos and if so, find an explicit representation of f as a sum of squares. The method we describe, sometimes called the *Gram matrix method* reduces the problem to linear algebra. For more details and examples, see e.g. [16], [67], [42, §3.3].

Suppose $f \in \mathbb{R}[X]$ has degree $2d$, let $N = \binom{n-1+d}{d}$ and let V be the $N \times 1$ vector of all monomials in $\mathbb{R}[X]$ of degree at most d. Then f is sos iff there exists an $N \times N$ symmetric psd matrix $\mathcal{A}$ such that

$$f(X) = V \cdot \mathcal{A} \cdot V^T, \tag{1}$$

The set of matrices $\mathcal{A}$ such that (1) holds is an affine subset $\mathcal{L}$ of the space of $N \times N$ symmetric matrices; a matrix in $\mathcal{L}$ is often called a Gram matrix for f. Then f is sos iff $\mathcal{L} \cap P_N \neq \emptyset$, where P_N is the convex cone of psd symmetric $N \times N$ matrices over $\mathbb{R}$. Finding this intersection is a *semidefinite program* (SDP). There are good numerical algorithms – and software – for solving semidefinite programs. For details on using SDPs to find sum of squares representations , see e.g. [52], [68].

Since there is an *a priori* bound on the size of the SDP corresponding to writing a particular f as a sum of squares, this gives an exact algorithm. However, since we are using numerical software, there are issues of exact versus numerical answers.

Consider the following example, due to C. Hillar: Suppose

$$f = 3 - 12y - 6x^3 + 18y^2 + 3x^6 + 12x^3y - 6xy^3 + 6x^2y^4,$$

is f sos? If we try to decide this with software we might get the answer "yes" and a decomposition similar to this:

$$f = (x^3 + 3.53y + .347xy^2 - 1)^2 + (x^3 + .12y + 1.53xy^2 - 1)^2 + \\ (x^3 + 2.35y - 1.88xy^2 - 1)^2. \tag{2}$$

The coefficients of the right-hand side of (2) are not exactly the same as the coefficients of f, so we might wonder if f is really sos. It turns out that f is sos, and (2) is an approximation of a decomposition for f of the form

$$(x^3 + a^2y + bxy^2 - 1)^2 + (x^3 + b^2y + cxy^2 - 1)^2 + (x^3 + c^2y + axy^2 - 1)^2,$$

where a, b, c are real roots of $x^3 - 3x + 1$.

In theory, a SDP problem can be solved purely algebraically, for example, using quantifier elimination. In practice, this is impossible for all but trivial problems. Work by Nie, Ranestand, and Sturmfels [50] shows that optimal solutions of relatively small SDP's can have minimum defining polynomials of huge degree, and hence we could encounter sos polynomials of relatively small size which have decompositions using algebraic numbers of large degree.

Since solving the underlying SDP exactly is impossible in most cases, we are led to the following question: Suppose $f \in \sum \mathbb{Q}[X]^2$ and we find a numerical (approximate) certificate $f = \sum g_i^2$ (via SDP software, say), can we find an exact decomposition of f in $\sum \mathbb{Q}[X]^2$? Recent approaches using hybrid symbolic-numeric approaches are very promising.

Peyrl and Parrilo [53] give an algorithm for converting a numerical sos decomposition into an exact certificate, in some cases. The idea: Given $f \in \sum \mathbb{Q}[X]^2$, we want to find a symmetric psd matrix $\mathcal{A}$ with rational entries so that

$$f = V \cdot \mathcal{A} \cdot V^T \tag{3}$$

The SDP software will produce a psd matrix A which only *approximately* satisfies (3). The idea is to project A onto the affine space of solutions to (3) in such a way that the projection remains in the cone of psd symmetric matrices. The Peyrl-Parrilo method is (theoretically!) guaranteed to work if there exists a rational solution and the underlying SDP is strictly feasible, i.e., there is a solution with full rank. Kaltofen, Li, Yang, and Zhi [31] have generalized the technique of Peyrl and Parrilo and used these ideas to find sos certificates certifying rational lower bounds for several well-known problems.

3.2 Certificates of positivity via Artin's Theorem

Recall Artin's solution to Hilbert's 17th Problem which says that if $f \in \mathbb{R}[X]$ is psd, then f is a sum of squares in the rational function field $\mathbb{R}(X)$, i.e., f can be written as a quotient of sos polynomials. Recent work of Kaltofen, Li, Yang, and Zhi [32] turns Artin's theorem into a symbolic-numeric algorithm for finding certificates of positivity for any psd $f \in \mathbb{Q}[X]$. They extend the hybrid symbolic-numeric approaches to finding an exact sos representation of a polynomial discussed above. The algorithm finds a numerical representation of f as a quotient g/h, where g and h are sos, and then converts this to an exact rational identity using techniques described above. The algorithm has been implemented as software called ArtinProver. Kaltofen, Yang, and Zhi have used this technique and the software to settle the dimension 4 case of the Monotone Column Permanent Conjecture, see [33].

3.3 Cerificates of impossibility of sos representability

The proof that the Motzkin form $M(x, y)$ is not sos involves a term-by-term inspection of the equation $M(x, y) = \sum h_i(x, y)^2$. Proofs for examples of Choi and Lam were done using a similar term-inspection method. This term-inspection method was generalized by Choi, Lam, and Reznick [16] using the Newton polytope of a polynomial. Proofs for other known examples, e.g. the Robinson example, involve the zeros of the polynomial.

Recently, another method for proving that a given polynomial is not sos, and producing a certificate of impossibility, was given by Ahmadi and Parrilo [1], using a generalization of Farkas Lemma to semidefinite programming. As discussed in §3.1, determining if a given polynomial is sos is equivalent to deciding if a certain semidefinite program has a solution. This method produces a certificate of infeasibility for the semidefinite program via a separating hyperplane.

This method has been generalized by Guo, Kaltofen, and Zhi [25], who developed an algorithm, using semidefinite programming and Farkas Lemma, for certifying a lower bound on the degree of the denominator in any representation in $\sum \mathbb{R}(X)^2$ of a specified psd polynomial.

3.4 Schmüdgen's and Putinar's theorems

Let $G \subseteq \mathbb{R}[X]$ be a finite and suppose $\mathcal{S} := \mathcal{S}(G)$ is compact. Set $\mathcal{P} = PO(G)$. Recall Schmüdgen's Theorem says that every polynomial that is strictly positive on $\mathcal{S}$ is in $\mathcal{P}$, regardless of the choice of generating polynomials G. Schmüdgen's proof uses functional analytic methods and is not constructive in the sense that no information is given concerning how to find an explicit certificate of positivity in $\mathcal{P}$ for a given f which is strictly positive on $\mathcal{S}$.

3.4.1 Algorithmic Schümdgen Theorem

In 2002, Schweighofer [90] gave a proof of Schmüdgen's Theorem which is algorithmic, apart from an application of the Classical Positivstellensatz. The idea of the proof is to reduce to Pólya's Theorem (in a larger number of variables). The Classical Positivstellensatz is used to imply the existence of a "certificate of compactness" for $\mathcal{S}$, i.e., the existence of $s, t \in \mathcal{P}$ and $r \in \mathbb{R}$ such that

$$s(r^2 - \sum X_i^2) = 1 + t \tag{4}$$

3.4.2 Degree bounds for Schmüdgen Theorem

Unlike the global (sum of squares) case, in general, there is no bound on the degree of the sums of squares in a representation of f in $\mathcal{P}$ in terms of the degree of f only. This has obvious implications for applications of Schmüdgen's Theorem, for example in recent work on the approximation of polynomial optimization problems via semidefinite programming. Using model and valuation theoretic methods, Prestel [69, Theorem 8.3.4] showed that there exists a bound on the degree of the sums of squares which depends on three parameters, namely, the polynomials G used to define $\mathcal{S}$, the degree of f, and a measure of how close f is to having a zero on $\mathcal{S}$. Schweighofer [91] used his algorithmic proof of the result to give a bound on the degree of the sums of squares in a representation of f in $\mathcal{P}$. Roughly speaking, the bound makes explicit the dependence on the second and third parameter in Prestel's theorem. The first parameter appears in the bound as a constant, which depends only on the polynomials G, and which comes from the compactness certificate (4). The exact result is as follows:

Theorem 14 ([91],Theorem 3)**.** *Let $G = \{g_1, \dots, g_k\}$, $\mathcal{S}$, and $\mathcal{P}$ be as above and suppose $\mathcal{S} \subseteq (-1,1)^n$. Then there exists $c \in \mathbb{N}$ so that for every $f \in \mathbb{R}[X]$ of degree d with $f > 0$ on $\mathcal{S}$ and $f^* = \min\{g(x) \mid x \in S\}$,*

$$f = \sum_{e \in \{0,1\}^k} s_e g_1^{e_1} \dots g_k^{e_k},$$

where $s_e \in \sum \mathbb{R}[X]^2$ and $s_e = 0$ or

$$\deg(s_e g_1^{e_1} \dots g_k^{e_k}) \leq c d^2 \left(1 + \left(d^2 n^d \frac{||f||}{f^*}\right)^c\right).$$

Here $||f||$ is a measure of the size of the coefficients of f. The constant c depends on the polynomials G in an unspecified way, however in concrete cases one could (in theory!) obtain an explicit c from the proof of the theorem.

3.4.3 Putinar's Theorem

Let G and $\mathcal{S}$ be as above and set $M = M(G)$. Recall Putinar's Theorem says that if M is archimedean, then every $f > 0$ on $\mathcal{S}$ is in M. Again, Putinar's proof is functional analytic and does not show how to find an explicit certificate of positivity for f in M. In [92], Schweighofer extends the algorithmic proof of Schmüdgen's Theorem to give an algorithmic proof of Putinar's Theorem. Nie and Schweighofer [51] then use this proof to give a bound for the degree of the sums of squares in a representation, similar to Theorem 14. Recently, Putinar's Theorem has been used by Lasserre to give an algorithm for approximating the minimum of a polynomial on a compact basic closed semialgebraic set, see [37]. The results in [51] yields information about the convergence rate of the Lasserre method.

3.5 Rational certificates of positivity

In §3.1, an algorithm for finding sum of squares certificates of positivity for sos polynomials f is described, using semidefinite programming. This technique can also be used to find certificates of positivity for a polynomial f which is positive on a compact semialgebraic set. However, there is another question which arises when we are using numerical software: All polynomials found in a certificate of positivity, for example in the sums of squares, will have rational coefficients. But do we know that such a certificate exists, even if we start with $f \in \mathbb{Q}[X]$?

3.5.1 Sums of squares of rational polynomials

Sturmfels asked the following question: Suppose $f \in \mathbb{Q}[X]$ is in $\sum \mathbb{R}[X]^2$, is $f \in \sum \mathbb{Q}[X]^2$? Here is a trivial, but illustrative example: The rational polynomial $2x^2$ is a square, since $2x^2 = (\sqrt{2}x)^2$. But $2x^2$ is also in $\sum \mathbb{Q}[x]^2$ since $2x^2 = x^2 + x^2$. Less trivially, recall the Hillar example:

$$f = 3 - 12y - 6x^3 + 18y^2 + 3x^6 + 12x^3y - 6xy^3 + 6x^2y^4,$$

as noted above, f is a sum of three squares in $\mathbb{R}[x, y]$. It turns out that f is a sum of five squares in $\mathbb{Q}[x, y]$:

$$f = (x^3+xy^2+\frac{3}{2}y-1)^2+(x^3+2y-1)^2+(x^3-xy^2+\frac{5}{2}y-1)^2+(2y-xy^2)^2+\frac{3}{2}y^2+3x^2y^4.$$

Partial results on Sturmfels question have been given: In the univariate case, the answer is “yes”; proofs have been given by Landau [36] and Schweighofer [89]. Pourchet [59] showed that at most five squares are needed. Hillar [29] showed that the answer to Sturmfel’s question is “yes” if $f \in \sum K^2$, where K is a totally real extension of $\mathbb{Q}$, and he gave bounds for the number of squares needed. There is a simple proof of a slightly more general result with a better bound given (independently) by Scheiderer [78] and Quarez [71].

Recently, Scheiderer [79] answered Sturmfels question in the negative. He constructed families of polynomials with rational coefficients that are sums of squares over $\mathbb{R}$ but not over $\mathbb{Q}$. He showed that these counterexamples are the only ones in the case of ternary quartics.

Remark 1. The proof of Artin’s Theorem shows immediately that if $f \in \mathbb{Q}[X]$ is psd, then there always exist $g, h \in \sum \mathbb{Q}[X]^2$ such that $f = g/h$. The rationality question is not an issue in this case.

3.5.2 Rational certificates of positivity on compact sets

There is an obvious analog of Sturmfels’ question for the case of polynomials positive on compact semialgebraic sets. Let $\mathcal{P} = PO(G)$ for finite $G \subseteq \mathbb{Q}[X]$. If $f \in \mathbb{Q}[X]$ is in $\mathcal{P}$, does there exist a representation of f in $\mathcal{P}$ such that the sums of squares that occur are in $\sum \mathbb{Q}[X]^2$? We can ask a similar question for the quadratic module $M(G)$. In [60], it is shown that the answer is “yes” for $\mathcal{P}$ in the compact case and “yes” for M with an additional assumption.

Theorem 15. *Let $G = \{g_1, \ldots, g_r\} \subseteq \mathbb{Q}[X]$ and suppose $\mathcal{S} = \mathcal{S}(G)$ is compact. Let $\mathcal{P} = PO(F)$ and $M = M(F)$. Given $f \in \mathbb{Q}[X]$ such that $f > 0$ on $\mathcal{S}$, then*

1. *There is a representation of f in the preordering $\mathcal{P}$,*

$$f = \sum_{e \in \{0,1\}^r} \sigma_e g_1^{e_1} \dots g_r^{e_r},$$

with all $\sigma_e \in \sum \mathbb{Q}[X]^2$.

2. *There is a rational representation of f in M provided one of the generators is $N - \sum X_i^2$. More precisely, there exist $\sigma_0 \dots \sigma_s, \sigma \in \sum \mathbb{Q}[X]^2$ and $N \in \mathbb{N}$ so that*

$$f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s + \sigma(N - \sum X_i^2).$$

The proof of the first part follows from an algebraic proof of Schmüdgen's Theorem, due to T. Wörmann, which uses the Abstract Positivstellensatz. Wörmann's proof can be found in [7] or [69, Thm. 5.1.17]. The second part follows from Schweighofer's algorithmic proof of Putinar's Theorem.

3.6 Certificates of positivity using Bernstein's and Pólya's theorems

Using Bernstein's Theorem and Pólya's Theorem, certificates of positivity for polynomials positive on simplices can be obtained. Furthermore, this approach yields degree bounds for the certificates and, in some cases, practical algorithms for finding certificates.

3.6.1 The univariate case

For $k \in \mathbb{N}$, define in $\mathbb{R}[x]$:

$$\mathcal{B}_k := \left\{ \sum_{i+j \leq k} c_{ij}(1-x)^i(1+x)^j \mid c_{ij} \geq 0 \right\}.$$

Suppose a univariate $p \in \mathbb{R}[x]$ is strictly positive on $[-1, 1]$, then Bernstein's Theorem says that there is some $r = r(p)$ such that $p \in \mathcal{B}_r$. Suppose $p \in \mathbb{R}[x]$ has degree d, then let $\tilde{p}$ denote the Goursat transform applied to p, i.e.,

$$\tilde{p}(x) = (1+x)^d p\left(\frac{1-x}{1+x}\right).$$

Powers and Reznick gave a bound on $r(p)$ in terms of the minimum of p on $[-1, 1]$ and size of the coefficients of $\tilde{p}$, which in turn yields a bound for the size of a certificate of positivity for p.

More recently, F. Boudaoud, F. Caruso, and M.-F. Roy [10] obtain a local version of Bernstein's Theorem which yields a better bound. They show that if $\deg p = d$ and $p > 0$ on $[-1,1]$, then there exists a subdivision $-1 = y_1 < \cdots < y_t = 1$ of $[-1,1]$ such that Bernstein-like certificates of positivity for p can be obtained on each interval $[y_i, y_{i+1}]$. This yields a certificate of positivity for p on $[-1,1]$ of bit-size $O((d^4(\tau + \log_2 d))$, where $d = \deg p$ and the coefficients of p have bit-size $\leq \tau$. Moreover, their result holds with $\mathbb{R}$ replaced by any real-closed field, which is not true for Bernstein's Theorem.

3.6.2 Polynomials positive on a simplex

Recall that Pólya's Theorem says that if a form (homogeneous polynomial) f is strictly positive on the standard simplex $\Delta_{n-1} := \{x \in \mathbb{R}^n \mid x_i \geq 0$ for all i and $\sum x_i = 1\}$, then for sufficiently large $N \in \mathbb{N}$, all coefficients of $(\sum X_i)^N f$ are strictly positive. Powers and Reznick [62] gave a bound on N, in terms of the degree of f, the minimum of f on Δ_{n-1}, and the size of the coefficients. This result has been used in several applications, for example the algorithmic proof of Schmüdgen's theorem given by Schweighofer discussed in §3.4.1. Also, de Klerk and Pasechnik [20] used it to give results on approximating the stability number of a graph.

In theory, the bound for Pólya's Theorem could be used to obtain certificates of positivity on the simplex, however in practice the bounds require finding minimums of forms on closed subsets of the simplex and so are not of much practical use. Another, more feasible, approach to certificates of positivity for polynomials positive on a simplex, due to R. Leroy [43], uses the multivariable Bernstein polynomials and a generalization of the ideas in [10]. The Bernstein polynomials are more suitable that the standard monomial basis in this case since this approach gives results for an arbitrary non-degenerate simplex and yields an algorithm for deciding positivity of a polynomial on a simplex. The idea is to subdivide the simplex and obtain local certificates so that the sizes of the local certificates are smaller than those of a global certificate.

Let V be a non-degenerate simplex in $\mathbb{R}^n$, i.e., the convex hull of $n+1$ affinely independent points $v_0, v_1, \ldots, v_n$ in $\mathbb{R}^n$. The barycentric coordinates of V, $\lambda_1, \ldots, \lambda_k$, are linear polynomials in $\mathbb{R}[X]$ such that

$$\sum_{i=0}^{n} \lambda_i = 1, \qquad (X_1, \ldots, X_n) = \sum_{i=1}^{n} \lambda_i(X) v_i.$$

Then for $d \in \mathbb{N}$, the Bernstein polynomials of degree d with respect to V are $\{B_\alpha^d \mid \alpha \in \mathbb{N}^{n+1}, |\alpha| = d\}$, where

$$B_\alpha^d = \frac{d!}{\alpha_0! \alpha_1! \cdots \alpha_n!} \prod_{i=0}^{n} \lambda_i^{\alpha_i}.$$

They form a basis for the vector space of polynomials in $\mathbb{R}[X]$ of degree $\leq d$, hence any $f \in \mathbb{R}[X]$ of degree $\leq d$ can be written uniquely as a linear combination of the B_α^d's. The coefficients are called the Bernstein coefficients of f. If $f > 0$ on V, then for sufficiently large D, the Bernstein coefficients using the B_α^D's are nonnegative, which yields a certificate of positivity for f on V.

This can be made computationally feasible, as well as lead to an algorithm for deciding if f is positive on V. The idea is to triangulate V into smaller simplices and look for certificates of positivity on the sub-simplices. A stopping criterion is obtained using a lower bound on the minimum of a positive polynomial on V, in terms of the degree, the number of variables, and the bitsize of the coefficients. This was proven by S. Basu, Leroy, and Roy [3] and later improved by G. Jeronimo and D. Perrucci [30].

3.6.3 Pólya's Theorem with zeros

What can we say if the condition "strictly positive on Δ_{n-1}" in Pólya's Theorem is replaced by "nonnegative on Δ_{n-1}"? It is easy to see that in this case we must use a slightly relaxed version of Pólya's Theorem, replacing the condition of "strictly positive coefficients" by "nonnegative coefficients". Let $Po(n, d)$ be the set of forms of degree d in n variables for which there exists an $N \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^N p \in \mathbb{R}^+[X]$. In other words, $Po(n, d)$ are the forms which satisfy the conclusion of Pólya's Theorem, with "positive coefficients" replaced by "nonnegative coefficients."

It is easy to see that $p \in Po(n, d)$ implies $p \geq 0$ on Δ_{n-1} and that $p > 0$ on the interior of Δ_{n-1}. Further, $Z(p)$, the zero set of p, must be a union of faces of Δ_{n-1}. Pólya's Theorem and the bound are generalized to forms that are positive on the simplex apart from zeros on the corners (zero dimensional faces) of Δ_{n-1}, in papers by Powers and Reznick [64] and M. Castle, Powers, and Reznick [12]. See also work by H.-N. Mok and W.-K. To [46], who give a sufficient condition for a form to satisfy the relaxed version of Pólya's Theorem, along with a bound in this case.

Very recently, Castle, Powers, and Reznick [13] give a complete characterization of forms that are in $Po(n, d)$ along with a a recursive bound for the N needed. Before stating the main theorem of [13], we need a few definitions.

Definition 2. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{N}^n$.

1. We write $\alpha \preceq \beta$ if $\alpha_i \leq \beta_i$ for all i, and $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.
2. Suppose F is a face of Δ_{n-1}, say $F = \{(x_1, \ldots, x_n) \in \Delta_{n-1} \mid x_i = 0 \text{ for } i \in I\}$ for some $I \subseteq \{1, 2, \ldots, n\}$. Then we denote by α_F the vector $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \in \mathbb{N}^n$, where $\tilde{\alpha}_i = \alpha_i$ for $i \in I$ and $\tilde{\alpha}_j = 0$ for $j \notin I$.

3. For a form $p \in \mathbb{R}[X]$, let $\Lambda^+(p)$ denote the exponents of p with positive coefficients and $\Lambda^-(p)$ the exponents of p with negative coefficients.

4. For a face F of Δ_{n-1} and a subset $S \subseteq \mathbb{N}$, we say that $\alpha \in \mathbb{N}$ is *minimal in S with respect to F* if there is no $\gamma \in S$ such that $\gamma_F \prec \alpha_F$.

Theorem 16. *Given $p = \sum a_\beta X^\beta$, a nonzero form of degree d, such that $p \geq 0$ on Δ_{n-1} and $Z(p) \cap \Delta_{n-1}$ is a union of faces. Let $\Lambda^+(p)$ denote the exponents of p with positive coefficients and $\Lambda^-(p)$ the exponents of p with negative coefficients. Then $p \in Po(n,d)$ if and only if for every face $F \subseteq Z(p)$ the following two conditions hold:*

1. *For every $\beta \in \Lambda^-(p)$, there is $\alpha \in \Lambda^+(p)$ so that $\alpha_F \preceq \beta_F$.*

2. *For every $\alpha \in \Lambda^+(p)$ which is minimal on $\Lambda^+(p)$ with respect to F, the form $\sum$ is strictly positive on the relative interior of F.*

3.6.4 Certificates of positivity on the hypercube

Finally, we mention briefly some recent work by de Klerk and Laurent [19] concerning polynomials positive on a hypercube $Q = [0,1]^n$. Using Bernstein approximations, they obtain bounds for certificates of positivity for a polynomial f which is strictly positive on Q, in terms of the degree of f, the size of the coefficients, and the minimum of f on Q. They also give lower bounds, and sharper bounds in the case where f is quadratic.

3.7 Psd ternary quartics

Recall Hilbert's 1888 theorem that says every psd ternary quartic (homogeneous polynomial of degree 4 in 3 variables) is a sum of three squares of quadratic forms. Hilbert's proof in non-constructive in the sense that it gives no information about the following questions: Given a psd ternary quartic, how can one find three such quadratic forms? How many "fundamentally different" ways can this be done?

Several recent works have addressed these issues. In [61], Powers and Reznick describe methods for finding and counting representations of a psd ternary quartic and answer these questions completely for some special cases. In several examples, it was found that there are exactly 63 inequivalent representations as a sum of three squares of complex quadratic forms and, of these, 8 correspond to representations as a sum of squares of real quadratic forms. By "inequivalent representations" we mean up to orthogonal equivalence; two representations are equivalent iff they have the same Gram matrix (see §3.1).

The fact that a psd ternary quartic f has 63 inequivalent representations as a sum of squares of complex quadratic forms is a result due to Coble [18].

In 2004, Powers, Reznick, Scheiderer, and Sottile [65] showed that for every real psd ternary quartic f such that the complex plane curve Q defined by $f = 0$ is smooth, exactly 8 of the 63 inequivalent representations correspond to a sum of three squares of real quadratic forms. More recently, in [85], Scheiderer extends this analysis to the singular case and computes the number of representations, depending on the configuration of the singular points. For example, if f is a psd singular ternary quartic and Q has a real double point, then there are exactly four inequivalent representations of f as a sum of three squares of quadratic forms.

Information about the number of representations also follows from the elementary proof of Hilbert's Theorem on ternary quartics given by Pfister and Scheiderer in [55]. The quantitative analysis in [65] and [85] uses tools of modern algebraic geometry and is not in any sense elementary. The work in [55] yields a new, elementary proof of the fact that for a generically chosen psd ternary quartic f, there are exactly 8 inequivalent representations and when f is generically chosen with a real zero, there are 4 inequivalent representations.

Finally, we mention very recent work on quartic curves due to Plaumann, Sturmfels, and Vinzant [57]. They give a new proof of the Coble result which yields an algorithm for computing all representations of a smooth ternary quartic as a sum of squares of three complex quadratic forms.

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