Power in the US Legislature

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Abstract

Using a standard model of the US legislative system as a monotonic simple game, we look at rankings of the four types of players – the president, the vice president, senators, and representatives – induced by power indices. We show that regardless of the power index used, the president is always ranked above the other players, and a senator is always ranked above the vice president and a representative. For most power index rankings, including the Banzhaf and Shapley-Shubik power indices, the vice president is ranked above a representative, however, there exist power indices ranking a representative above the vice president. Our results apply to more general yes-no voting systems.

1 Introduction

This is the introduction.

2 Simple games and power indices

Fix a set of players $N = \{1, 2, \ldots, n\}$ and let $\mathcal{P}(N)$ denote the power set of N, i.e., the set of all subsets of N. Elements of $\mathcal{P}(N)$ are **coalitions**. A monotonic **simple game** v (on N) is a function $v \colon \mathcal{P}(N) \to \{0, 1\}$ such that $v(N) = 1$ and for all $S, T \in \mathcal{P}(N)$, if $v(S) = 1$ and $S \subseteq T$, then $v(T) = 1$. If $v(S) = 1$, we say S is a **winning coalition**; S is **losing** if $v(S) = 0$. Let $\mathcal{W}$ denote the set of winning coalitions. The **minimal winning coalitions** are the winning coalitions for which no proper subset is winning. The set of winning coalitions is determined by the minimal ones, thus specification of the minimal winning coalitions determines v.

A monotonic simple game can be viewed as a model of a yes-no voting system in which the players are deciding on a single alternative such as a bill or amendment. The winning coalitions are precisely the sets of players that can force a bill to pass if they all support it.

Given a simple game v and a coalition S containing player i, we say i is **critical** in S if S is winning and $S \setminus \{i\}$ is losing. For $i \in N$ and $1 \leq k \leq n$, let

$$\mathcal{C}_i = \{S \in \mathcal{W} \mid i \text{ is critical in } S\}, \quad \mathcal{C}_i(k) = \{S \in \mathcal{C}_i \mid |S| = k\}.$$

Finally, let $c_i = |\mathcal{C}_i|$ and $c_i(k) = |\mathcal{C}_i(k)|$.

We define a binary relation on N by $i \succeq j$ iff $\mathcal{C}_i(k) \geq \mathcal{C}_j(k)$ for $1 \leq k \leq n$. In [2] this is called the **weak desirability** relation.

2.1 Power indices

Power indices are a way to measure the relative power of the players in a simple game. The most famous of these are the Shapley-Shubik index [5] and the Banzhaf index [1]. Semivalues were introduced in 1979 by Weber [6] as a generalization of the notion of a power index. Dubey et al. [3] show that semivalues can be characterized in terms of a **weighting vector** $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_k \geq 0$ for all k and $\sum_{k=1}^{n} \lambda_k \binom{n-1}{k-1} = 1$.

Given a semivalue Φ with weighting vector $(\lambda_1, \ldots, \lambda_n)$, the Φ-power of a player i is defined by

$$\Phi(i) := \sum_{k=1}^{n} \lambda_k \, c_i(k).$$

Thus the λ_i's give a weighting of a player's contribution to coalitions of size i. Let $p = \sum_{i=k}^{n} \Phi(k)$, the **total power** in the game, then the power index determined by Φ is

$$(p_1^{\Phi}, \ldots, p_n^{\Phi}) := \left(\frac{\Phi(1)}{p}, \ldots, \frac{\Phi(n)}{p} \right).$$

We interpret p_i^{Φ} as the fraction of the power held by player i.

The Shapley-Shubik power index is defined by weighting coefficients $\lambda_k = 1/n\binom{n-1}{k-1}$ and the Banzhaf power index is defined by weighting coefficients $\lambda_k = 1/2^{n-1}$.

Any semivalue Φ defines a ranking on the set of players in a simple game by $i \succeq_{\Phi} j$ iff $\Phi(i) \geq \Phi(j)$. Clearly, different semivalues can lead to different rankings for the same game. In [4], Saari and Sieberg look at rankings of players coming from semivalues in cooperative games. They show that different indices can generate radically different rankings and that there can be many different rankings even for games with a relatively small number of players.

The following results follow easily from the definitions, see also [2, Theorem 3.4].

Proposition 1. *Let i and j be two players in a simple game.*

(a) If $i \succeq j$, then for any semivalue Φ, $i \succeq_{\phi} j$. Hence, if the weak desirability relation is complete, i.e., for all players i, j either $i \succeq j$ or $j \succeq i$, then every semivalue gives the same ranking of the players.

(b) Suppose there exist k, m such that $c_i(k) > c_j(k)$ and $c_i(m) < c_j(m)$. Then there exist power indices Φ and Ψ such that $i \succeq_{\Phi} j$ and $j \succeq_{\Psi} i$.

Our main theorem is that there weak desirability relation is almost complete in the sense that we have an ordering of all pairs of players apart from the vice president and a representative. It follows that there are only two possible power index rankings. The president is always ranked above the other players, and a senator is always ranked above the vice president and a representative. For most power index rankings, the vice president is ranked above a representative, however, there exist power indices ranking a representative above the vice president.

3 The US legislative system

The US legislative system. The US legislative system consists of the president, vice president, 100 senators in the Senate, and 435 representatives in the House of Representatives. A bill passes if a majority of the senators and representatives vote yes and the president signs the bill. If the president does not sign the bill, it can be passed with a supermajority of at least 67 senators and 290 representatives. The role of the vice president is to break ties in the Senate.

We use a well-known model of the US legislative system as a monotonic simple game with these 537 players. Our goal is to understand the weak desirability relation in this case and to determine possible semivalue rankings of the players.

There are three types of minimal winning coalitions:

I. 51 senators, 218 representatives, and the president;

II. 50 senators, 218 representatives, the president, and the vice president;

III. 67 senators and 290 representatives.

We look at critical instances for the four types of players in order to compare the numbers $c_i(k)$. Note that if a winning coalition contains exactly 51 senators, then every senator is critical and adding the vice president yields a coalition in which no senator is critical. Apart from this case, if a player who is not the vice president is critical in a coalition that does not contain the vice president, then this player is still critical if the vice president is added to the coalition.

For ease of exposition, let p denote the president and v the vice president. Throughout this section, s denotes a fixed senator and r denotes a fixed representative. We write $c_p(k)$ (resp. $c_v(k)$, $c_s(k)$, $c_r(k)$) for number of coalitions of size k in which p (resp. v, s, r) is critical.

It is easy to see that the coalitions in which v is critical are those consisting of v, 50 senators, $218 - 435$ representatives, and the president. These range in size from 270 to 487. The following table lists the different types of coalitions, along with their sizes, in which the president (P1 - P5), a senator s (S1 - S4), or a representative r (R1 - R4) are critical.

Type	Members	Size
P1	p, v, 50 senators, 218-435 representatives	$270 - 487$
P2	p, $51 - 66$ senators, $218 - 435$ representatives	$270 - 502$
P3	p, v, $51 - 66$ senators, $218 - 435$ representatives	$271 - 503$
P4	p, $67 - 100$ senators, $218 - 289$ representatives	$286 - 390$
P5	p, v, $67 - 100$ senators, $218 - 289$ representatives	$287 - 391$
S1	p, v, s, 49 other senators, 218-435 representatives	$270 - 487$
S2	p, s, 50 other senators, $218 - 435$ representatives	$270 - 487$
S3	s, 66 other senators, $290 - 435$ representatives	$357 - 502$
S4	s, 66 other senators, $290 - 435$ representatives, v	$358 - 503$
R1	p, r, 217 other representatives, $51 - 100$ senators	$270 - 319$
R2	p, v, r, 217 other representatives, $50 - 100$ senators	$270 - 320$
R3	r, 289 other representatives, $67 - 100$ senators	$357 - 390$
R4	r, 289 other representatives, $67 - 100$ senators, v	$358 - 391$

Proposition 2. *For all $k \in \mathbb{N}$ such that $270 \leq k \leq 503$,*
(a) $c_p(k) > c_v(k)$
(b) $c_p(k) > c_s(k)$.

Proof. (a): Every coalition in which v is critical contains p, and p is also critical, thus $c_v(k) \leq c_p(k)$ for all k. In addition, given $S \in \mathcal{C}_v(k)$, the coalition formed by removing v and adding a senator not already in S is in $\mathcal{C}_p(k)$. Hence $c_p(k) > c_v(k)$ for all k.

(b): Fix $k \in \mathbb{N}$ such that $270 \leq k \leq 503$ and define $\phi : \mathcal{C}_s(k) \to \mathcal{C}_p(k)$ as follows: Given S in $\mathcal{C}_s(k)$, if S is type S1 or S2, then p is critcal in S and we define $\phi(S) = S$. If S is type S3 or S4, then the coalition $S' = (S \setminus \{s\}) \cup \{p\}$ is in $\mathcal{C}_p(k)$ since it contains only 66 senators, and we define $\phi(S) = S'$. Then ϕ is clearly injective, hence $c_s(k) \leq c_p(k)$. To show that the inequality is strict we need only show that ϕ is not surjective.

If $270 \leq k \leq 356$, then there are no type S3 or S4 coalitions in $\mathcal{C}_s(k)$. Thus any coalition in $\mathcal{C}_p(k)$ that does not contain s is not in Im ϕ, and there are clearly many of these. Now suppose $357 \leq k \leq 502$ and let $\tilde{S}$ be a coalition in Im ϕ of the form $(S \setminus \{s\}) \cup \{p\}$ with $S \in \mathcal{C}_s(k)$. Then we can construct a new coalition in $\mathcal{C}_p(k)$ by replacing any senator in $S \setminus \{s\}$ by a representative not already in $\tilde{S}$. This clearly yields a coalition that is not in Im ϕ.

For $k = 503$, coalitions in $\mathcal{C}_s(k)$ consist of s plus 66 other senators, 435 representative, and v; while those in $\mathcal{C}_p(k)$ consist of p plus 66 senators, 435 representatives, and v. Then $c_p(503) = \binom{100}{66} > \binom{99}{66} = c_s(k)$. Thus in all cases we have $c_p(k) > c_s(k)$.

□

Proposition 3. *Let $k \in \mathbb{N}$ such that $270 \leq k \leq 503$.*
(i) If $270 \leq k \leq 356$, then $c_s(k) = c_v(k)$.
(ii) If $357 \leq k \leq 503$, then $c_s(k) > c_v(k)$.

Proof. Fix k and let $j = k - 52$. Coalitions in $\mathcal{C}_v(k)$ consist of v plus 50 senators, j representatives, and the president, hence for $270 \leq k \leq 487$,

$$c_v(k) = \binom{100}{50} \cdot \binom{435}{j}.$$

(i) Since $k \leq 356$, coalitions in $\mathcal{C}_s(k)$ consist of type S1 and S2 only, thus they consist of s plus 49 other senators, j representatives, v and p; or s plus 50 other senators, j representatives, and p. Hence

$$c_s(k) = \binom{99}{50} \cdot \binom{435}{j} + \binom{99}{49} \cdot \binom{435}{j} = 2\binom{99}{50} \cdot \binom{435}{j} = \binom{100}{50} \cdot \binom{435}{j} = c_v(k).$$

(ii) For $488 \leq k \leq 503$, $c_s(k) > 0$ and $c_v(k) = 0$, so this is clear. For $357 \leq k \leq 487$, we note that in addition to the coalitions in $\mathcal{C}_s(k)$ of type (d) and (e) above, there are coalitions of type (f) and thus $c_k(s) > c_v(k)$.

$\square$

Remark. For coalitions which require the president, the vice president plays the same role as a senator and thus the result in (i) above is what we would expect.

Comparing the numbers $c_s(k)$ and $c_r(k)$ is more complicated. The following facts about binomial coefficients will be useful.

Lemma 1. *Let $m, n, k \in \mathbb{N}$.*
(a) If $k < m$, then $\binom{m}{k} + \binom{m}{k+1} = \binom{m+1}{k+1}$.
(b) If $m > n$, then for all integers i, $0 \leq i < n$, $\binom{2m+1}{m+i} \cdot \binom{2n}{n} > \binom{2m}{m} \cdot \binom{2n+1}{n+i}$.
(c) If $m > n$, then $\binom{3m+2}{2m-1+i} \cdot \binom{3n}{2n} > \binom{3m-1}{2m-1} \cdot \binom{3n+2}{2n+i}$,

Proof. (a) This is well-known.
(b) For $p \in \mathbb{N}$ and an integer i with $0 \leq i < p$, define

$$f(p,i) := \binom{2p+1}{p+i} / \binom{2p}{p}.$$

We will prove by induction on i that if $m > n$, $f(m,i) > f(n,i)$ for $0 \leq i < n$.

We have $f(m,0) > f(n,0)$ iff $\dfrac{2m+1}{m} > \dfrac{2n}{n+1}$, which follows easily from $m > n$.

Suppose $0 \leq i < n$ and $f(m,i) > f(n,i)$. We have

$$f(m,i+1)/f(m,i) = \binom{2m+1}{m+i+1} / \binom{2m+1}{m+i} = \frac{m+1-i}{m+1+i},$$

hence $f(m,i+1) = f(m,i) \cdot \frac{m+1-i}{m+1-i}$. Similarly, we have $f(n,i+1) = f(n,i) \cdot \frac{n+1+i}{n+1-i}$.

Since $f(m,i) > f(n,i)$ by the inductive hypothesis, we need only show that $\frac{m+1-i}{m+1-i} > \frac{n+1+i}{n+1-i}$ for $0 \leq i < n$. This follows easily from $m > n$. Hence, by induction, $f(m,i) > f(n,i)$ for all $1 \leq i \leq n$, and the inequality follows.

The proof of (c) is similar to the proof of (b).

$\square$

Proposition 4. *For all $k \in \mathbb{N}$ such that $c_s(k) \neq 0$, $c_s(k) > c_r(k)$.*

Proof. If $321 \leq k \leq 356$ or $392 \leq k \leq 503$, then $c_r(k) = 0$ and $c_s(k) \neq 0$, so there is nothing to prove. We break the remaining values of k into three cases: $270 \leq k \leq 320$, $k = 357$, and $358 \leq k \leq 391$.

Case 1. For $270 \leq k \leq 320$, let $j = k - 270$, so that $0 \leq j \leq 49$. The coalitions in $\mathcal{C}_s(k)$ are of type S1 or S2 and hence consist of s plus 50 other senators, $218+j$ representatives and the president, or s plus 49 other senators, $218+j$ representatives, the president and the vice president. Coalitions in $\mathcal{C}_r(k)$ are of type R1 or R2 and hence consist of r plus 217 other representatives, the president, and either $50+j$ senators or $50+j-1$ senators and the vice president. Thus, by Lemma 1,

$$
\begin{aligned}
c_s(k) &= \cdot \binom{435}{217+j}\left(\binom{99}{50} + \binom{99}{49}\right) = \binom{100}{50} \cdot \binom{435}{217+j}\\
c_r(k) &= \cdot \binom{434}{217}\left(\binom{100}{50+j} + \binom{100}{49+j}\right) = \binom{101}{50+j} \cdot \binom{434}{217}
\end{aligned}
$$

By Lemma 1 (b) with $n = 50$ and $m = 217$, we have $c_s(k) > c_r(k)$.

Case 2. $k = 357$. Coalitions in $\mathcal{C}_s(357)$ consist of s, 66 other senators, and 290 representatives, while coalitions in $\mathcal{C}_r(357)$ consist of r, 289 other representatives, and 67 senators. Thus, using Lemma 1,

$$c_s(357) = \binom{99}{66} \cdot \binom{435}{290} > \binom{100}{67} \cdot \binom{434}{289} = c_r(357).$$

Case 3. For $358 \leq k \leq 391$, let $j = k - 357$ so that $1 \leq j \leq 134$. The coalitions in $\mathcal{C}_s(k)$ consist of s, 66 other senators and either $290 + j$ representatives, or v and $289 + j$ representatives. The coalitions in $\mathcal{C}_r(k)$ consist of r, 289 other representatives, and either $67 + j$ senators or v and $66 + j$ senators. Thus

$$
\begin{aligned}
c_s(k) &= \binom{99}{66}\left(\binom{435}{289+j} + \binom{435}{288+j}\right) = \binom{99}{66} \cdot \binom{436}{289+j},\\
c_r(k) &= \binom{434}{289}\left(\binom{100}{66+j} + \binom{100}{65+j}\right) = \binom{434}{289} \cdot \binom{101}{66+j}.
\end{aligned}
$$

Using Lemma 1 (c), we see that $c_s(k) > c_r(k)$.

□

Remark. The proof of the proposition does not depend on the specific numbers for representatives and senators and thus can be easily generalized. For example, removing the president and vice president from the proof of Case 2 shows that in a bicameral legislature where bills are passed by a simple majority in each house, the players in the smaller of the two houses are weakly dominant and thus rank higher in any semivalue ranking.

Finally, we compare the numbers $c_v(k)$ and $c_r(k)$. Apart from a narrow range of k's, $c_v(k)$ is the larger of the two.

Proposition 5. *Suppose* $k \in \mathbb{N}$, $270 \leq k \leq 503$. *If* $270 \leq k \leq 356$ *or* $380 \leq k \leq 487$, $c_v(k) > c_r(k)$. *For the remaining* k, *i.e.,* $357 \leq k \leq 379$, $c_r(k) > c_v(k)$.

Proof. Recall $c_v(k) \neq 0$ for $270 \leq k \leq 391$ and that coalitions in $C_v(k)$ consist of v, 50 senators, the president, and $k-52$ representatives. Thus, for all such values of k, we have $c_v(k) = \binom{100}{50}\binom{435}{k-52}$.

For $270 \leq k \leq 356$, $c_v(k) = c_s(k) > c_r(k)$. For $k = 357$, the only coalitions in which r is critical are of type R3 and consist of r plus 289 other representatives and 67 senators. Hence we have

$$c_v(357) = \binom{100}{50}\binom{435}{305} > \binom{434}{289}\binom{100}{67} = c_r(357).$$

For $358 \leq k \leq 390$, there are coalitions of type R3 and R4, so that

$$A(j) := c_r(k) = \binom{434}{289}\left(\binom{100}{67+j} + \binom{100}{66+j}\right) = \binom{434}{289}\binom{101}{66+i}, \tag{1}$$

for $1 \leq j \leq 33$. We rewrite $c_v(k)$ using the parameter j:

$$B(j) := c_v(k) = \binom{100}{50}\binom{435}{305+j}. \tag{2}$$

Using computer algebra software such as Mathematica, we find that for $1 \leq j \leq 22$, $B > A$ and for $23 \leq i \leq 33$, $A > B$. This yields the claimed result.

Finally, for $k = 391$ we have

$$c_v(391) = \binom{100}{50}\binom{435}{239} > \binom{434}{289} = c_r(391).$$

□

Putting this all together we have $p \succeq s \succ r$ and $s \succeq v$.

Theorem 1. *(a) For any semivalue* Φ, *we have* $p \succeq_\Phi s \succeq r$ *and* $s \succeq v$.
(b) If Φ *is the Banzhaf or Shapley-Shubik index, we have* $p \succeq_\Phi s \succeq_\Phi v \succeq_\Phi r$.

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