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CHORDED K-PANCYCLIC AND WEAKLY K-PANCYCLIC GRAPHS

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Abstract

As natural relaxations of pancyclic graphs, we say a graph G is k-pancyclic if G contains cycles of each length from k to $|V(G)|$ and G is *weakly pancyclic* if it contains cycles of all lengths from the girth to the circumference of G, while G is *weakly k-pancyclic* if it contains cycles of all lengths from k to the circumference of G. A cycle C is chorded if there is an edge between two vertices of the cycle that is not an edge of the cycle. Combining these ideas, a graph is *chorded pancyclic* if it contains chorded cycles of each length from 4 to the circumference of the graph, while G is *chorded k-pancyclic* if there is a chorded cycle of each length from k to $|V(G)|$. Further, G is *chorded weakly k-pancyclic* if there is a chorded cycle of each length from k to the circumference of the graph. We consider conditions for graphs to be chorded weakly k-pancyclic and chorded k-pancyclic.

Keywords: cycle, chord, pancyclic, weakly pancyclic.

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1. Introduction

The study of cycles has a long and diverse history. Many different peoperties have been developed concerning cycles. For example, early on Bondy [2] studied one of the most important of these; pancyclicity. We say a graph G is *pancyclic* if G contains a cycle of each length from three to the order of G and G is *k-pancyclic* if it contains cycles of all lengths from k to the order of the graph. Natural relaxations of pancyclic graphs have also been developed. In his thesis, Brandt [3] introduced one such variation of pancyclic graphs. A graph is *weakly pancyclic* if it contains cycles of all lengths from the girth to the circumference of the graph. Further, a graph is *weakly k-pancyclic* if it contains cycles of all lengths from k to the circumference (see for example, [5]).

Another, more recent cycle variation is that of chorded cycles. We say an edge between two vertices of a cycle is a *chord* if it is not an edge of the cycle. We say cycle C is a *chorded cycle* if the vertices of C induce at least one chord. Pósa [13] asked what conditions imply a graph contains a chorded cycle. This question has seen considerable interest lately (see for example [7], [8], [9]).

In this paper we consider a merging of the ideas we have discussed. We say a graph is *chorded k-pancyclic* if it contains chorded cycles of all lengths from k to $|V(G)|$ (see for example [10]). Further, G is *chorded weakly k-pancyclic* if G contains chorded cycles of each length from k to the circumference of the graph. Note that we did not say chorded cycles existed from the girth on up, since the smallest chorded cycle contains a smaller cycle.

We consider only simple graphs in this paper. We use the standard notation of $V(G)$, $E(G)$, and $\delta(G)$ for the vertex set, edge set, and minimum degree of the graph G. Let $K_{a,b}$ denote the complete biartite graph with parts of order a and b. Let C_k deonte the cycle of order k and P_k denote the path of order k. Let $N_H(x)$ denote the set of neighbors of the vertex x in the graph (or subgraph) H and let $< S >$ denote the graph induced by the vertex set S. Given an orientation of some path or cycle, we denote by x^+ and x^- the successor and predessor of the vertex x following the given orientation. Further, let $x^{+2} = (x^+)^+$ and similarly, let $x^{-2} = (x^-)^-$, etc. Similarly, $N_C^+(x)$ denotes the set of successors of the neighbors of x on the cycle C following the given orientation. Let $d(u, v)$ denote the distance in the graph between vertices u and v. Given a subgraph or vertex subset S let $G - S$ be the graph obtained by removing S from G. The girth is the length of the shortest cycle and the circumference is the length of a longest cycle. For terms not defined here see [11].

In his thesis, Brandt [3] showed the following result.

Theorem 1. *Let $G \neq C_5$ be a nonbipartite triangle-free graph of order n. If $\delta(G) > n/3$, then G is weakly pancyclic with girth 4 and circumference min$\{2(n-\alpha(G)), n\}$.*

In [4] it is shown that Theorem 1 is best possible.

Brandt, Faudree, and Goddard [5] provided another result on weak pancyclic graphs, removing the triangle free condition of the previous result.

Theorem 2. *Every nonbipartite graph G of order n with minimum degree $\delta(G) \geq (n+2)/3$ is weakly pancyclic with girth 3 or 4.*

This result is almost best possible. The graph formed from K_{m+1} and $K_{m,m}$ ($m \geq 3$) by identifying a vertex from each has order $n = 3m$ and minimum degree $m = \frac{n}{3}$, but contains no odd cycle of length more than $m+1$, while having all even cycles up to $2m$.

We extend each of these last two results as follows.

Theorem 3. *Let G be a nonbipartite triangle-free graph of order $n \geq 13$. If $\delta(G) \geq \frac{n+1}{3}$, then G is chorded weakly 6-pancyclic with circumference min$\{2(n - \alpha(G)), n\}$.*

Theorem 4. *Every nonbipartite graph G of order $n \geq 13$ with minimum degree $\delta(G) \geq (n+2)/3$ is chorded weakly 6-pancyclic.*

Theorem 3 is best possible in the sense that as G is triangle-free, it contains no chorded 4 or 5-cycles. We will prove Theorems 3 and 4 in Section 2.

Our second goal concerns the following. A well-known result of Chvátal and Erdős relates connectivity ($\kappa(G)$) and independence number ($\alpha(G)$) to cycle length.

Theorem 5 Chvátal-Erdős [6]**.** *If G is a graph of order $n \geq 3$ such that $\alpha(G) \leq \kappa(G)$, then G is hamiltonian, that is, it contains a spanning cycle.*

Amar et al. [1] conjectured that if $\alpha(G) \leq \kappa(G)$ and G is not bipartite, then G has cycles of every length from 4 to $|V(G)|$. Lou [12] considered this conjecture and proved the following.

Theorem 6. *Let G be a triangle-free graph of order $n \geq 4$ with $\alpha(G) \leq \kappa(G)$. Then G is 4-pancyclic or $G = K_{\frac{n}{2},\frac{n}{2}}$, or $G = C_5$.*

Our goal is to extend Lou's Theorem as follows.

Theorem 7. *Let G be a triangle-free graph of order $n \geq 13$ with $\alpha(G) \leq \kappa(G)$. Then G is chorded weakly 8-pancyclic, or $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Note that since G is triangle-free, there cannot be a chorded C_4 or C_5 in G. In Section 3 we will prove Theorem 7 and provide examples to show there may not be chorded 6 and 7-cycles in such graphs. Thus, in general, this result is best possible.

2. PROOFS OF THEOREMS 3 AND 4

In this section we prove Theorems 3 and 4. In order to do so, we begin with several general lemmas that will apply in both proofs.

Lemma 8. *Let G be a graph of order $n \geq 12$ with $\delta(G) \geq \frac{n+1}{3}$. If H is a subgraph of G of order $6+t$ $(0 \leq t \leq 5)$ and x, y, z are vertices of H such that $d = \deg_H(x) + deg_H(y) + deg_H(z) \leq 6+t$, and*

$$N_{G-H}(x) \cap N_{G-H}(y) = \emptyset = N_{G-H}(x) \cap N_{G-H}(z),$$

then $|N_{G-H}(y) \cap N_{G-H}(z)| \geq 1$.

Proof. Since $\delta(G) \geq \frac{n+1}{3}$, we see that $3\delta(G) - d \geq n - 5 - t$. But from the neighborhood intersection conditions, since $|V(G-H)| = n - 6 - t$, it then follows that $|N_{G-H}(y) \cap N_{G-H}(z)| \geq 1$. ■

Lemma 9. *If G has order $n \geq 12$ and $\delta(G) \geq \frac{n+1}{3}$, then G contains a chorded 6-cycle.*

Proof. By Theorem 1 we know G contains 6-cycles. Suppose that G satisfies the conditions of the Theorem and further, suppose the result fails to hold. Let $C : v_1, v_2, v_3, \ldots, v_6, v_1$ be a chordless 6-cycle in G and let $H = C$.

Case 1. Assume that no two consecutive vertices of C have a common neighbor in $G - C$.

Consider the vertices v_1, v_2, v_3. By our assumption and Lemma 8, we see that there exists a vertex x with $x \in N_{G-H}(v_1) \cap N_{G-H}(v_3)$. Let $H_1 =< V(C) \cup \{x\} >$ and now consider v_2, v_3, v_4. Again by Lemma 8, we can select a vertex y with $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_4)$. But then, the cycle $v_1, x, v_3, v_4, y, v_2, v_1$ is a 6-cycle with chord v_2v_3.

Case 2. Assume that there are two consecutive vertices of C with at least one neighbor in $G - H$.

Without loss of generality, we may assume that $x \in N_{G-H}(v_1) \cap N_{G-H}(v_2)$. Let $H_1 =< V(C) \cup \{x\} >$ and consider x, v_2, v_5. If there exists a vertex y with $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$ then $v_1, x, v_2, y, v_5, v_6, v_1$ is a 6-cycle with chord v_1v_2. Similarily, if $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_5)$ then $v_1, v_2, x, y, v_5, v_6, v_1$ is a 6-cycle with chord xv_1. If both these fail to hold, then by Lemma 8, we conclude instead that $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_2)$ and let $H_2 =< V(H_1) \cup \{y\} >$.

Now consider v_6, x, y. If there exists a vertex $z \in N_{G-H_2}(v_6) \cap N_{G-H_2}(x)$ then $v_1, v_2, y, x, z, v_6, v_1$ is a 6-cycle with chord xv_2. If instead $z \in N_{G-H_2}(y) \cap N_{G-H_2}(v_6)$ then $v_1, v_2, x, y, z, v_6, v_1$ is a 6-cycle with chord xv_1. If both of these fail to hold, we conclude from Lemma 8 that $z \in N_{G-H_2}(x) \cap N_{G-H_2}(y)$ and we let $H_3 =< V(H_2) \cup \{z\} >$.

Now consider v_1, v_3, z. If there exists a vertex $w \in N_{G-H_3}(v_1) \cap N_{G-H_3}(v_3)$ we then have a 6-cycle $v_1, w, v_3, v_2, y, x, v_1$ with chord xv_2. But, if instead $w \in N_{G-H_3}(z) \cap N_{G-H_3}(v_3)$ then $v_2, y, x, z, w, v_3, v_2$ is a 6-cycle with chord xv_2. Finally, if both of these fail to hold, then by Lemma 8, $w \in N_{G-H_3}(v_1) \cap N_{G-H_3}(z)$, then $v_1, w, z, x, y, v_2, v_1$ is a 6-cycle with chord xv_2, completing the proof. ■

Lemma 10. *If G has order $n \geq 12$ and $\delta(G) \geq \frac{n+1}{3}$, then G contains a chorded 7-cycle.*

Proof. By Theorem 1 we know G contains a 7-cycle. Let G be as stated, and suppose the result fails to hold. Let $C : v_1, v_2, v_3, \ldots, v_7, v_1$ be a chordless 7-cycle in G and let $R = G - C$ and $H = C$. We now consider the following cases.

Case 1. Suppose that no two consecutive vertices of C have a common neighbor in R.

Consider v_1, v_2, v_3. By our assumption and Lemma 8 we see that there exists a vertex $x \in N_R(v_1) \cap N_R(v_3)$. Let $H_1 =< H \cup \{x\} >$. Now consider v_2, v_5, v_6. If there exists a vertex $w \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$, then $v_1, v_2, w, v_5, v_4, v_3, x, v_1$ is a 7-cycle with chord v_2v_3. If instead $w \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6)$, then $v_1, v_7, v_6, w, v_2, v_3, x, v_1$ is a 7-cycle with chord v_1v_2. However, by our assumption and Lemma 8, one of these two facts must hold.

Case 2. Suppose there are two consecutive vertices on C with a common neighbor in R.

Without loss of generality let $x \in N_R(v_1) \cap N_R(v_2)$, set $H_1 =< C \cup \{x\} >$ and consider v_2, v_5, x. If there exists $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$, then

$$v_1, x, v_2, y, v_5, v_6, v_7, v_1$$

is a 7-cycle with chord v_1v_2. If instead $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_5)$, then $v_1, v_2, x, y, v_5, v_6, v_7, v_1$ is a 7-cycle with chord xv_1. If both of these fail to happen, then by Lemma 8 there exists $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(x)$. Let $H_2 =< H_1 \cup \{y\} >$.

Now consider x, y, v_6. If there exists a vertex $z \in N_{G-H_2}(v_6) \cap N_{G-H_2}(x)$, then $v_1, v_2, y, x, z, v_6, v_7, v_1$ is a 7-cycle with chord xv_1. If instead, $z \in N_{G-H_2}(y) \cap N_{G-H_2}(v_6)$, then $v_1, v_2, x, y, z, v_6, v_7, v_1$ is a 7-cycle with chord xv_1. Otherwise, by Lemma 8, there is a vertex $z \in N_{G-H_2}(x) \cap N_{G-H_2}(y)$. We now consider v_3, v_7, z, with $H_3 =< H_2 \cup \{z\} >$.

If there exists a vertex w such that $w \in N_{G-H_3}(z) \cap N_{G-H_3}(v_7)$, then $v_1, v_2, y, x, z, w, v_7, v_1$ is a 7-cycle with chord xv_1. If instead $w \in N_{G-H_3}(z) \cap N_{G-H_3}(v_3)$, then $v_2, v_1, x, y, z, w, v_3, v_2$ is a 7-cycle with chord yv_2. Otherwise, by Lemma 8, there us a vertex $w \in N_{G-H_3}(v_3) \cap N_{G-H_3}(v_7)$ and then

$$v_1, x, y, v_2, v_3, w, v_7, v_1$$

is a 7-cycle with chord v_1v_2. This completes the proof of the Lemma. ■

Lemma 11. *Let G have order $n \geq 12$ and $\delta(G) \geq \frac{n+1}{3}$, then G contains a chorded 8-cycle.*

Proof. By Theorem 1 we know that G contains an 8-cycle. Suppose all 8-cycles are chordless and consider the 8-cycle $C : v_1, v_2, v_3, \ldots, v_8, v_1$ and let $H = C$. We now consider two cases.

Case 1. Suppose no two consecutive vertices on C have a common neighbor in $G - H$.

Consider v_1, v_2, v_3. Then, by our assumption and by Lemma 8 there exists a vertex $x \in N_{G-H}(v_1) \cap N_{G-H}(v_3)$. Let $H_1 =< H \cup \{x\} >$. Similarly, there exists a vertex y with $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_4)$. Let $H_2 =< H_1 \cup \{y\} >$.

Next consider v_8, v_1, v_2. Again, by our assumption and Lemma 8, there exists a vertex z such that $z \in N_{G-H_2}(v_2) \cap N_{G-H_2}(v_8)$. Then, $v_1, x, v_3, v_4, y, v_2, z, v_8, v_1$ is an 8-cycle with chord v_1v_2.

Case 2. Suppose there is a pair of consecutive vertices on C with a common neighbor in $G - H$.

Without loss of generality, let $x \in N_{G-H}(v_1) \cap N_{G-H}(v_2)$ and $H_1 =< H \cup \{x\} >$. Now consider v_2, x, v_5. If there exists y with $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_5)$ then, $v_1, v_2, x, y, v_5, v_6, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If instead $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$ then, $v_1, x, v_2, y, v_5, v_6, v_7, v_8, v_1$ is an 8-cycle with chord v_1v_2. If both these cases fail to hold, then by Lemma 8 there exists y with $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_2)$.

Let $H_2 =< H_1 \cup \{y\} >$ and now consider x, y, v_6. If there exists $w \in N_{G-H_2}(x) \cap N_{G-H_2}(v_6)$, then $v_1, v_2, y, x, w, v_6, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If instead $w \in N_{G-H_2}(y) \cap N_{G-H_2}(v_6)$, then $v_1, v_2, x, y, w, v_6, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If both of these cases fail to hold, then again by Lemma 8, there eists $w \in N_{G-H_2}(x) \cap N_{G-H_2}(y)$.

Now let $H_3 =< H_2 \cup \{w\} >$ and consider v_7, w, v_4. If there exists $z \in N_{G-H_3}(v_7) \cap N_{G-H_3}(v_4)$, then $v_1, x, v_2, v_3, v_4, z, v_7, v_8, v_1$ is an 8-cycle with chord v_1v_2. If instead $z \in N_{G-H_3}(w) \cap N_{G-H_3}(v_7)$, then $v_1, v_2, y, x, w, z, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If both the previous cases fail to hold, then by Lemma 8 there exists $z \in N_{G-H_3}(v_4) \cap N_{G-H_3}(w)$, in which case $v_2, v_1, x, y, w, z, v_4, v_3, v_2$ is an 8-cycle with chord xv_2. This completes the proof of the Lemma. ■

Lemma 12. *Let G be a graph of order $n \geq 13$ with $\delta(G) \geq \frac{n+1}{3}$. Then G contains chorded cycles of each length from 9 to the circumference of the graph.*

Proof. By Theorem 1, G contains cycles of each length from 9 to the circumference of G. Let G be as stated and suppose G has no chorded k-cycle for some $k \geq 9$. Let $C = C_k : v_1, v_2, \ldots, v_k$ be such a cycle in G. Further, let $H = G - C_k$. We consider the following cases.

Case 1: Suppose, no two consecutive vertices of C_k have a common neighbor off C.

By our assumption and Lemma 8, for any three consecutive vertices on C_k, v_i, v_{i+1}, v_{i+2}, $N_{G-H}(v_i) \cap N_{G-H}(v_{i+2}) \neq \emptyset$. Let $w \in N_{G-H}(v_1) \cap N_{G-H}(v_3)$ and let $H_1 =< H \cup \{w\} >$. If $N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6) \neq \emptyset$, then take $w_2 \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6)$ and note that

$$v_1, w, v_3, v_2, w_2, v_6, v_7, \ldots, v_k, v_1$$

is a k-cycle with chord v_1v_2. Thus, we may assume $N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6) = \emptyset$ and by symmetry $N_{G-H_1}(v_2) \cap N_{G-H_1}(v_{k-2}) = \emptyset$.

If $N_{G-H_1}(v_3) \cap N_{G-H_1}(v_7) \neq \emptyset$, then let $w_3 \in N_{G-H_1}(v_3) \cap N_{G-H_1}(v_7)$. Now, by our assumptions there exists $w_k \in N_{G-H_1}(v_k) \cap N_{G-H_1}(v_2)$. Now by Lemma 8, we have that

$$v_2, v_1, w, v_3, w_3, v_7, v_8, \ldots, v_k, w_k, v_2$$

is a k-cycle with chord v_2v_3. Note that if any pair of vertices v_i, v_{i+4} for $i = 2, 3, \ldots, k-3$ share a common neighbor off C, then we can always find a chorded k-cycle in a similar fashion. So we may assume this never happens. Then, in particular, considering v_2, v_5, v_6, we know by our assumptions there exists a vertex $x \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$, and let $H_2 =< H_1 \cup \{x\} >$, Similarly, considering v_5, v_8, v_9, we know there exists a vertex $y \in N_{G-H_2}(x_5) \cap N_{G-H_2}(v_8)$. Now $v_1, w, v_3, v_2, x, v_5, y, v_8, v_9, \ldots, v_k, v_1$ is a k-cycle with chord v_1v_2, completing this case.

Case 2: Suppose two consecutive vertices of $C = H$ do have a common neighbor in $G - H$.
Without loss of generality, say $w \in N_{G-H}(v_1) \cap N_{G-H}(v_2)$. and let $H_1 =< H \cup \{w\} >$. Then if any pair v_i, v_{i+3} for $i = 2, 3, \ldots, k-3$ satisfies $N_{G-H_1}(v_i) \cap N_{G-H_1}(v_{i+3}) \neq \emptyset$ with a vertex $x \in N_{G-H_1}(v_i) \cap N_{G-H_1}(v_{i+3})$, there exists a k-cycle $v_1, w, v_2, v_3, \ldots, v_i, x, v_{i+3}, \ldots, v_k, v_1$ with chord v_1v_2. Thus, assume no such pair exists. Then, in particular, considering v_2, v_5, v_8 we see that there exists a vertex, say w_2, such that $w_2 \in N_{G-H}(v_2) \cap N_{G-H}(v_8)$, and considering the triple v_3, v_6, v_9, there must exists a vertex $w_3 \in N_{G-H}(v_3) \cap N_{G-H}(v_9)$ and considering v_4, v_7, v_{10} (here v_{10} may be v_1) we have a vertex $w_4 \in N_{G-H}(v_4) \cap N_{G-H}(v_{10})$. Then the cycle $v_1, v_2, w_2, v_8, v_9, w_3, v_3, v_4, w_4, v_{10}, v_{11}, \ldots, v_1$ (note again that it is possible that $v_1 = v_{10}$) is a k-cycle with chord v_2v_3. This completes the proof. ∎

Note that Case 2 may require at least 13 vertices, hence the condition that $n \geq 13$. As this Lemma is used in Theorems 3 and 4, the condition that $n \geq 13$ must be assumed in each result.

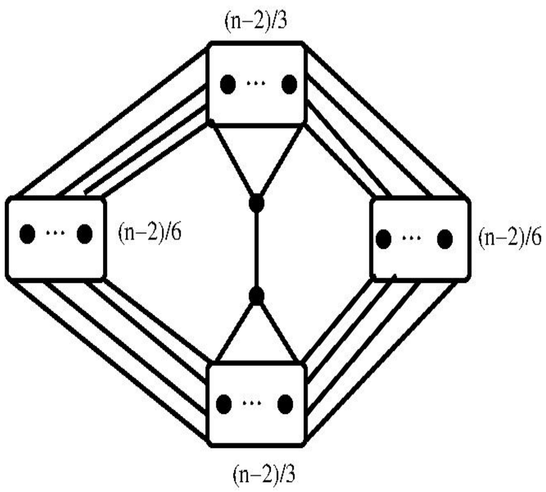

Figure 1. Sharpness example for Theorem 6.

We are now ready to prove Theorem 3.

Proof of Theorem 3.

Proof. By Theorem 1, G is weakly pancyclic with girth 3 or 4. Let G be a graph of order $n \geq 13$ with $\delta(G) \geq \frac{n+1}{3}$. Then by Lemmas 9, 10, 11, and 12 we see that G contains chorded cycles of length 6 up to the circumference of G. ■

As G is triangle-free, there can be no chorded 4 or 5-cycles, thus the result is best possible.

Example for Theorem 3 We construct a graph G as follows. Begin with a copy of $C_4 = v_1, v_2, v_3, v_4, v_1$. Blowup each of the vertices v_1 and v_3 into sets of $\frac{n-2}{3}$ independent vertices and blowup the vertices v_2 and v_4 into sets of $\frac{n-2}{6}$ independent vertices. For any edge of C_4 insert all edges between the corresponding sets. Finally, insert two new vertices x and y that are themselves adjacent and join x to all vertices in the blowup of v_1 and y to all vertices in the blowup of v_3 (see Figure 1). Note that $\delta(G) = \frac{n+1}{3}$. Further, it is easy to see that G is chorded weakly 6-pancyclic.

Proof of Theorem 4.

Proof. By Theorem 2, G is weakly pancyclic with girth 3 or 4. Again by Lemmas 9, 10, 11, and 12, we see that G has chorded cycles of each length from 6 to the circumference of G. ■

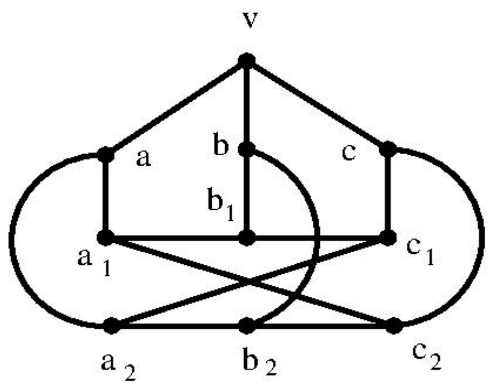

Figure 2. Here $\alpha(G) = 3$.

3. Proof of Theorem 7

The following from [12] will be useful.

Lemma 13. *If G is a triangle-free graph of order $n \geq 4$ and C is a cycle in G, then for every vertex $v \in G - C$, the set $N_C^+(v)$ is non-empty and $N_C^+(v)$ is not an independent set, hence $|N_C^+(v)| \geq 2$.*

The next Lemma has appeared in numerous papers, thus we attribute it to folklore.

Lemma 14. *Let C be a cycle in a graph G and $v \in V(G - C)$. If there is an edge in $N_C^+(v)$, then G contains a cycle D with $V(D) = V(C) \cup \{v\}$.*

We now state, in more detail, what Lou [12] proved.

Theorem 15. *If G ($G \neq K_{m,m}$ or C_5) is a triangle-free graph with $\alpha(G) \leq \kappa(G)$, then*

1. *G is k-regular and*
2. *$k = \alpha(G) = \kappa(G) = \kappa$ and G is κ-regular*
3. *G has diameter 2, and*
4. *G contains cycles of length 4 up to $|V(G)|$.*

What Lou proved actually puts some real restrictions on graphs G that satisfy the conditons of being triangle-free with $\alpha(G) = \kappa(G)$. The most severe is a bound on the order of G.

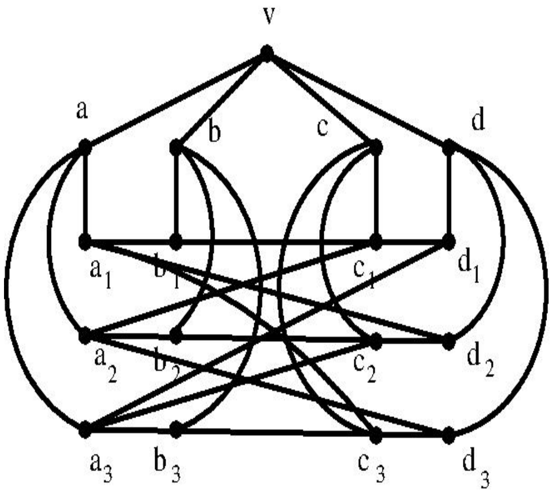

Figure 3. Here $\alpha(G) = 4$.

Lemma 16. *If G is a triangle-free graph of order n with $\alpha(G) = \kappa(G) = k$, then $n \leq k^2 + 1$.*

Proof. Let G be as stated above. Select any vertex v. Then v has exactly k mutually nonadjacent neighbors and each of these vertices may have at most $k-1$ distinct new neighbors. If there are any other vertices, say x, then $d(v, x) > 2$ and there is no way to create a path to v that would be of length at most 2. Thus, no such x exists and so $n \leq 1 + k + k(k-1) = k^2 + 1$. ■

This Lemma provides another simple observation that if $\alpha(G) = 2$, then G is either C_5 or C_4, and if $\alpha(G) = 3$, then $n \leq 10$. Thus, from now on we need only consider $\alpha(G) \geq 4$.

The graphs in Figures 2 and 3 show that the conditions of being triangle-free with $\alpha(G) = \kappa(G)$ are not enough to guarentee that 6 and 7-cycles are chorded.

We now present our proof of Theorem 7, the extension of Lou's Theorem, which utilizes an expansion of the ideas in his approach.

Proof. Suppose the result fails to hold. Then by Theorem 6, there must exist an integer k with $4 \leq k \leq |V(G)| - 2$, such that G contains a C_k but no chorded C_{k+2}. We next show that each of the following structures (see Figure 4) on a C_k, actually provides a chorded C_{k+2}.

To see this for structure (I), consider the $(k+2)$-cycle $a, u, v, a^+, a^{+2}, \ldots, a$ with chord aa^+.

For structure (III), consider the $(k+2)$-cycle $a, u, b, b^-, \ldots, a^+, v, b^+, b^{+2}, \ldots, a$ with chord aa^+.

For structure (IV), consider the $(k+2)$-cycle $a, u, v, b, b^-, \ldots a^+, b^+, b^{+2}, \ldots, a$ with chord aa^+.

In order to handle structure II we first need to develop several facts.

Claim 1: Every vertex off the k-cycle $C_k : x_1, x_2, \ldots, x_k, x_1$ $(k \geq 6)$ has at least one adjacency on C_k.

Proof. Suppose there is a vertex $v \notin V(C_k)$ such that v has no adjacencies on C_k. Then since G is triangle-free, $E(N(v)) = \emptyset$. However, for any $x_i \in V(C_k)$, $N(v) \cup \{x_i\}$ is a set of cardinality $\kappa(G) + 1$. If this set is independent, a contradiction arises to Theorem 15. Thus, every vertex on C_k is adjacent to at least one vertex in $N(v)$. Without loss of generality, say $x_1v_1 \in E(G)$ for some $v_1 \in N(v)$. Now $d(v, x_3) > 2$. Either there exists $v_3 \in N(v)$ such that $v_3 \neq v_1$ with $v_3x_3 \in E(G)$ or $x_3v_1 \in E(G)$.

First suppose latter happens, then $d(v, x_4) > 2$. Since G is triangle-free, $v_1x_4 \notin E(G)$, which implies there exists $v_4 \in N(v)$ such that $v_4x_4 \in E(G)$. Next note that $d(v, x_2) > 2$. If $x_2v_4 \in E(G)$ we obtain structure III, and hence a chorded $(k+2)$-cycle exists in G, a contradiction. Further, to avoid a triangle, $x_2v_1 \notin E(G)$, so there exists $v_2 \in N(v)$ $(v_2 \neq v_1, v_4)$ such that $v_2x_2 \in E(G)$. Note that we can extend C_k to a $(k+2)$-cycle

$$C^* = x_2, v_2, v, v_4, x_4, x_5, \ldots, x_1, x_2.$$

Now $d(x_2, x_5) > 2$. Further, $x_2x_5 \notin E(G)$ as that would provide a chord for C^*. Also $x_1x_5 \notin E(G)$ for the same reason, and $x_3x_5 \notin E(G)$ since G is triangle-free. Thus, there exists some $w \notin V(C_k)$ such that $x_2w, x_5w \in E(G)$. Now $x_1, v_1, v, v_2, x_2, w, x_5, x_6, \ldots, x_k, x_1$ is a $(k+2)$-cycle with chord x_1x_2, a contradiction.

Now consider the former case, that is, that there exists $v_3 \notin V(C_k)$ such that $v_3 \neq v_1$ and $v_3v, v_3x_3 \in E(G)$. Since $d(v, x_2) > 2$ and $v_1x_2, v_3, x_2 \notin E(G)$, there exists a $v_2 \in N(v)$ with $v_2x_2 \in E(G)$. Again there is a $(k+2)$-cycle

$$C' : x_1, v_1, v, v_3, x_3, x_4, \ldots, x_k, x_1.$$

If there are any chords in C_k not involving x_2, then C' is chorded, a contradiction. Now $d(x_2, x_5) > 2$. If there exists $w \in N_{G-C_k}(x_2) - \{x_1\}$ such that $wx_5 \in E(G)$, then

$$x_1, v_1, v, v_2, x_2, w, x_5, x_6, \ldots, x_k, x_1$$

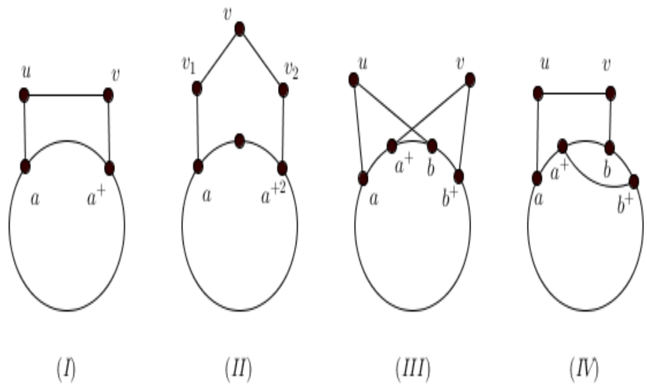

Figure 4. Four structures producing chorded $(k+2)$-cycles.

is a $(k+2)$-cycle with chord x_1x_2. So we may assume $x_2x_5 \in E(G)$. Next note that $d(x_3, x_k) > 2$. If $v_3x_k \in E(G)$, then again C' is chorded with chord v_3x_k. So we may assume there exists a vertex $w \in N_{G-C_k}(x_3)$ with $wx_k \in E(G)$. Now we see that

$$x_3, v_3, v, v_2, x_2, x_5, x_6, \ldots x_k, w, x_3$$

is a $(k+2)$-cycle with chord x_2x_3, a contradiction completing this case and the proof of the claim. $\square$

Claim 2: If structure II exists in G, then a chorded $(k+2)$-cycle exists in G.

Proof. Suppose structure II arises and let $C_k = x_1, x_2, , \ldots, x_k$. Without loss of generality suppose that $v_1, v_3 \in N_{G-C}(v)$ and v_1x_1 and v_3x_3 are edges of G. Now, by Claim 1, vertex v must have an adjacency on the cycle C_k. If $vx_2 \in E(G)$, then structure I is formed and we have a chorded $(k+2)$-cycle, a contradiction to our assumption. Thus, assume $vx_2 \notin E(G)$. Further, $vx_1, vx_3 \notin E(G)$, since either edge would create a triangle in G. Thus, $vx_i \in E(G)$ for some i, $4 \leq i \leq k$. Now this edge is a chord of the $(k+2)$-cycle $x_1, v_1, v, v_3, x_3, x_4, \ldots, x_1$, a contradiction which completes the proof of the claim. $\square$

Next choose a cycle C of length m such that $r = \max_{v \in G-C} |N_C(v)|$ (that is, over all vertices off C, v has the maximum number of adjacencies, and the maximum is taken over all choices of cycles of length m). Take a vertex v from $G - C$ with

$|N_C(v)| = r$ and another vertex $u \in V(G-C)$ which is, if possible, adjacent to v. By Lemma 13, the vertex u must have two neighbors y_1 and y_2 on C such that $y_1^+ y_2^+ \in E(G)$. Thus, by Lemma 14, there is an $(m+1)$-cycle D with $V(D) = V(C) \cup \{u\}$. If $r \geq \kappa - 1$, then v has all of its neighbors on D, so again by Lemma 14 there is a cycle on $|V(D) \cup \{v\}| = m+2$ vertices with chord $y_1 y_1^+$, a contradiction.

Now we may assume that $r \leq \kappa - 2$. Then $uv \in E(G)$ and v has another neighbor $w \in G - C$. By Lemma 14, w also has two neighbors z_1, z_2 on C such that $z_1^+ z_2^+ \in E(G)$. Since G is triangle-free, in any direction on C, there are at least two vertices between z_1 and z_2, otherwise $< z_1^+, z_2, z_2^+ >= K_3$. Thus, we may assume $z_1 \notin \{y_1^-, y_1, y_1^+\}$. Fix an orientation on D with the path $y_2^+, \ldots, y_1^-, y_1$ and let $S \subset D$ be the set of vertices y of C satisfying $y^- \in N_C(v)$. We wish to show that $N_{G-C}(v) \cup S \cup \ \{z_1^{+2}\}$ (with respect to the orientation on D) is an independent set with cardinality $\kappa + 1$, the final contradiction. In order to do this note that on C, the vertex $z_1 \neq {x_1}^{+2}, x_1^{-2}$ or structure II results, a contradiction. Thus, the edges $z_1 z_1^+$ and $z_1^+ z_2^+$ on D are also on C. Similarly, avoiding structure I, we see that there is no edge between any vertex in $N_{G-C}(v)$ and any vertex of S. For the same reason, on D, $z_1^{+2} \notin S$ and, in particular, v is not adjacent to z_1^+ on D. Also, on D, $vz_1^{+2} \notin E(G)$ otherwise,

$$z_1^{+2}, z_1^{+3}, \ldots, z_1, w, v, z_1^{+2}$$

is an $(m+2)$-cycle with chord wz_2. Moreover, z_1^{+2} is not adjacent to a vertex $y \in S$ since otherwise $z_1, w, v, y^-, y^{-2}, \ldots z_1^{+2}, y, y^+, \ldots z_1^-, z_1$ is an $(m+2)$-cycle with chord wz_2, again a contradiction to our assumption. Since an edge in S also creates an $(m+2)$-cycle with chord yy^-, all that remains is to show that z_1^{+2} is not adjacent to any vertex of $N_{G-C}(v)$. For any vertex of $N_{G-C}(v)$ other than w, this follows, as structure II would be formed and hence a chorded $(m+2)$-cycle would exist. But if z_1^{+2} is adjacent to w, we can form a new m cycle C' with w replacing z_1^+. As v is adjacent to w but not to z_1^+, v now has $r+1$ neighbors on C', contradicting our choice of cycle C. Thus, the set $N_{G-C}(v) \cup \{z_1^{+2}\} \cup S$ is independent and has cardinality at least $r+1$, a contradiction completing the proof.

■

4. CONCLUSION

It is clear from recent work that many conditions implying various cycle properties in a graph can be used to show stronger results concerning cycles with chords. This paper is just one such situation. Broading our meta-conjecture from [9]:

Almost any condition that implies some cycle property in a graph also implies a chorded cycle property, possibly with some families of exceptional graphs, and small order exceptions.

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