On Spanning Trees with few Branch Vertices, all with degree 3

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Abstract Let G be a connected claw-free graph. A conjecture of Matsuda, Ozeki, and Yamashita [9] posits the existence of either a spanning tree with few branch vertices or a large independent set of small degree sum. The possible appearances of this minimal spanning tree may be ruled out piecemeal. This paper treats one particular class, which may be the easiest to rule out but also fits, to our knowledge, all proposed sharpness examples.

Keywords: Spanning trees, Branch vertices, Claw-free graphs

Introduction

In a tree, vertices of degree one and vertices of degree at least three are called *leaves* and *branch vertices*, respectively. A hamiltonian path can be regarded as a spanning tree with maximum degree at most two, a spanning tree with at most two leaves, or a spanning tree with no branch vertex. A natural extension of the hamiltonian path problem is, therefore, to look for conditions that guarantee the existence of a spanning tree with low maximum degree, few leaves, or few branch vertices. Many researchers have investigated independence number conditions and degree sum conditions for the existence of such spanning trees; low maximum degree [3, 8, 11, 14], few leaves [1, 13, 15], and few branch vertices [2, 4, 5, 6, 9]. A paper of Matsuda, Ozeki, and Yamashita [9] conjectures a particular condition on connected claw-free graphs which ensures the existence of a spanning tree with at most k branch vertices.

Conjecture 1. *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{2k+3} \geq n-2$, then G has a spanning tree with at most k branch vertices.*

With an eye toward proving this conjecture, we will pose a slightly stronger one, but must first lay out a few definitions:

Definition 1. *Let $B(T)$ denote the set of branch vertices of a tree T, and let $L(T)$ denote the set of leaves.*

Definition 2. *For any rooted spanning tree T with root $r \in B(T)$, each branch vertex $x \in B(T) \setminus \{r\}$ has a* distance pair $(d(x,r), \deg_T(x))$*. We order the pairs lexicographically (shortest distance first, and smallest degree first given equal distance). The tree T is* minimal *if:*

(T1) $B(T)$ is as small as possible.

(T2) The distance pairs of $B(T) \setminus \{r\}$ are lexicographically as small as possible, subject to (T1).

Given the above definition of *minimal*, the general truth of the following conjecture will imply that of Conjecture 1 (though not for each individual value of k):

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Conjecture 2. *If G is connected, claw-free, and has some minimal rooted spanning tree (T, r) with $k+1$ branch vertices, then there is some independent set X of size $2k+3$, and $\sum_{x \in X} \deg_G(x) \leq n-3$.*

If the conditions in Definition 2 are kept consistent, Conjecture 2 may be provable by considering classes of spanning trees at a time. In this paper, we handle the trees for which $B_3(T) = B(T)$. Before launching into our main result, we introduce some useful notation and a widely applicable lemma:

Definition 3. *Let T be a tree and let $u, v \in V(T)$. We denote by uTv the unique path from u to v in T, and we denote $u_v := V(uTv) \cap N_T(u)$. This notation applies to all vertices.*

Definition 4. *Given two vertices x and p, the notation $(V(xTp) \cap N_G(x))^-$ refers to all vertices of the form a_x where $a \in V(xTp) \cap N_G(x)$.*

Lemma 1. *If $x \in B(T) \setminus \{r\}$ has degree 3, then the two children of x are adjacent in G.*

Proof. Suppose x_r is adjacent to some child x^* of x. Then $T' := T - \{xx^*\} + \{x_r x^*\}$ violates (T2). Since no claw can be centered at x, this requires that the two children are adjacent. □

Theorem 1. *If G is connected, claw-free, and has some minimal rooted spanning tree (T, r) on $k+1$ branch vertices, all with degree 3, then $X := L(T) \cup B(T) \setminus \{r\}$ is an independent set of size $2k+3$, and $\sum_{x \in X} \deg_G(x) \leq n-3$.*

Proof. Let G be a connected, claw-free graph, and let (T, r) be a minimal rooted spanning tree on $k+1$ branch vertices, all with degree 3. Denote $N_T(r) =: \{r_1, r_2, r_3\}$. Since no claw can be centered at r, we may assume by symmetry that $r_1 r_2 \in E(G)$.

We will first show that X is independent. Suppose $xy \in E(G)$ for some $x, y \in X$. If there is some branch vertex z in the interior of xTy, then $T' := T - \{zz_x\} + \{xy\}$ violates (T1). If there is none, we may assume by symmetry that $x \in V(rTy)$. Define $\{x^*\} := N_T(x) \setminus \{x_y, x_r\}$, so Lemma 1 insures that $x_y x^* \in E(G)$, so $T' := T - \{xx_y, xx^*\} + \{xy, x_y x^*\}$ violates (T1). Therefore X is independent.

To limit the degree sum of X, we consider the spanning tree piecemeal, addressing the neighbors of different vertices of X within each part and showing that certain sets of these neighbors cannot overlap, which in turn will limit their sum total in cardinality, which will equal their degree sum when applied to the entire spanning tree.

For each $x \in X$, define $p(x)$ as the branch vertex on the path xTr nearest to x (other than x itself, but may be r). If $p := p(x) \neq r$, then define $\{p^*\} := N_T(p) \setminus \{p_x, p_r\}$, so Lemma 1 gives that $p_x p^* \in E(G)$. Note that the $2k+3$ sets $(V(xTp) \cap N_G(x))^-$ and $V(xTp_x) \cap N_G(y)$ for each $y \in X \setminus \{x\}$ are all subsets of $V(xTp_x)$. We will show that these $2k+3$ sets are pairwise disjoint. We begin by showing that $(V(xTp) \cap N_G(x))^-$ is disjoint from each of the other $2k+2$ sets. Let $y \in X \setminus \{x\}$ and suppose there is some $a \in V(xTp)$ such that $xa, ya_x \in E(G)$. Consider three cases regarding the location of y:

Case 1: Suppose $y = p$. Then $T' := T - \{aa_x, pp_x, pp^*\} + \{xa, pa_x, p_x p^*\}$ violates (T1) since p is no longer a branch vertex.

Case 2: Suppose $p \in V(xTy)$. Then $T' := T - \{pp_x, aa_x\} + \{xa, ya_x\}$ violates (T1) since p is no longer a branch vertex.

Case 3: Suppose $x \in V(yTp)$. Define $\{x^*\} := N_T(x) \setminus \{x_y, x_r\}$, so Lemma 1, so $T' := T - \{aa_x, xx_y, xx^*\} + \{xa, ya_x, x_y x^*\}$ violates (T1) since x is not longer a branch vertex.

Therefore $(V(xTp) \cap N_G(x))^-$ is disjoint from the other $2k+2$ sets. We will now show that they are pairwise disjoint from each other. Let $y, z \in X \setminus \{x\}$ and suppose some $a \in N_G(y) \cap V(xTp_x) \cap N_G(z)$. Consider six cases regarding the locations of y and z:

Case 1: Suppose $y = p$ (or, symmetrically, $z = p$) and $p_r \in V(xTz)$. Since $\{a, a_x, p, z\}$ is not a claw, either $a_x p \in E(G)$ or $a_x z \in E(G)$. If $a_x p \in E(G)$, then $T' := T - \{aa_x, pp_x, pp^*\} + \{az, a_x p, p_x p^*\}$ violates (T1). Otherwise $a_x z \in E(G)$, so $T' := T - \{aa_x, pp_x, pp^*\} + \{ap, a_x z, p_x p^*\}$ violates (T1) since p is no longer a branch vertex.

Case 2: Suppose $y = p$ (or $z = p$) and $p^* \in V(xTz)$. Since $\{a, a_x, p, z\}$ is not a claw, either $a_x p \in E(G)$ or $a_x z \in E(G)$. If $a_x z \in E(G)$, then $T' := T - \{aa_x, pp_x, pp^*\} + \{ap, a_x z, p_x p^*\}$ violates (T1) since p is no longer a branch vertex. Otherwise $a_x p \in E(G)$, so since $\{p, p_r, p_x, a_x\}$ is not a claw and $p_r p^* \in E(G)$, either $p_r a_x \in E(G)$ or $p_x a_x \in E(G)$. If $p_x a_x \in E(G)$, then $T' := T - \{aa_x, pp_x\} + \{p_x a_x, za\}$ violates (T1) since p is no longer a branch vertex. Otherwise $p_r a_x \in E(G)$, so $T' := T - \{aa_x, pp_x\} + \{p_r a_x, za\}$ violates (T2) by the proximity of p_r to r.

Case 3: Suppose $y = p$ (or $z = p$) and $x \in V(pTz)$. Then $T' := T - \{xx_p, pp_x, pp^*\} + \{pa, za, p_x p^*\}$ violates (T1) since two branch vertices are lost (x and p) while one is gained (a).

Case 4: Suppose $p \in V(xTy)$ and $x \in V(pTz)$ (or vice versa). Then $T' := T - \{pp_x, xx_p\} + \{ya, za\}$ violates (T1) since two branch vertices are lost (x and p) while one is gained (a).

Case 5: Suppose $p \in V(xTy)$ and $p \in V(xTz)$. Since $\{a, a_x, y, z\}$ is not a claw, we may assume by symmetry that $a_x y \in E(G)$, so $T' := T - \{aa_x, pp_x\} + \{a_x y, az\}$ violates (T1) since p is no longer a branch vertex.

Case 6: Suppose $x \in V(pTy)$ and $x \in V(pTz)$. Since $\{a, a_p, y, z\}$ is not a claw, we may assume by symmetry that $a_p y \in E(G)$, so $T' := T - \{aa_p, xx_p\} + \{a_p y, az\}$ violates (T1), since x is no longer a branch vertex.

Therefore all $2k+3$ of these sets are pairwise disjoint. This holds true for all x such that $p(x) \neq r$. To cover the remainder of the spanning tree, we must consider each x such that $p(x) = r$. For each $i \in [3]$, there is exactly one x with $p(x) = r$ and $r_x = r_i$; call it x_i. Note that the $2k+3$ sets $(V(x_3Tr) \cap N_G(x_3))^-$ and $V(x_3Tr) \cap N_G(y)$ with $y \in X \setminus \{x_3\}$ are all subsets of $V(x_3Tr)$; we will show that they are pairwise disjoint and none of them include r. The $-$ superscript ensures that $(V(x_3Tr) \cap N_G(x_3))^-$ does not include r. Suppose $ry \in E(G)$ for some $y \in X \setminus \{x_3\}$. If $x_3 \in V(rTy)$, then $T' := T - \{x_3(x_3)_r\} + \{ry\}$ violates (T1). Otherwise $r \in V(x_3Ty)$, so $T' := T - \{rr_1, rr_2\} + \{ry, r_1r_2\}$ violates (T1). Therefore none of these sets contain r; we will next show that they are disjoint.

We will first show that $(V(x_3Tr) \cap N_G(x_3))^-$ is disjoint from the other $2k+2$ sets. Let $y \in X \setminus \{x_3\}$ and suppose some $a \in V(x_3Tr)$ with $x_3a, ya_{x_3} \in E(G)$. If $r \in V(x_3Ty)$, then $T' := T - \{aa_{x_3}, rr_3\} + \{x_3a, ya_{x_3}\}$ violates (T1). Otherwise $x_3 \in V(rTy)$, so define $\{x_3^*\} := N_T(x_3) \setminus \{(x_3)_y, (x_3)_r\}$. Now Lemma 1 ensures that $x_3^*(x_3)_y \in E(G)$, so $T' := T - \{aa_{x_3}, x_3(x_3)_y, x_3x_3^*\} + \{x_3a, ya_{x_3}, (x_3)_y x_3^*\}$ violates (T1). Therefore $(V(x_3Tp) \cap N_G(x_3))^-$ is disjoint from the other $2k+2$ sets. We will now show that the remaining $2k+2$ sets are pairwise disjoint. Let $y, z \in X \setminus \{x_3\}$ and let $a \in V(x_3Tr)$ such that $ya, za \in E(G)$. Consider three cases regarding the locations of y and z:

Case 1: Suppose $x_3 \in V(rTy)$ and $r \in V(x_3Tz)$ (or vice versa). Then $T' := T - \{rr_3, x_3(x_3)_r\} + \{ya, za\}$ violates (T1) since two branch vertices are lost (r and x_3) while only one is gained (a).

Case 2: Suppose $x_3 \in V(rTy)$ and $x_3 \in V(rTz)$. Since $\{a, a_r, y, z\}$ is not a claw, we may assume by symmetry that $a_r y \in E(G)$, so $T' := T - \{aa_r, x_3(x_3)_r\} + \{a_r y, az\}$ violates (T1), since x_3 is no longer a branch vertex.

Case 3: Suppose $r \in V(x_3Ty)$ and $r \in V(x_3Tz)$. Since $\{a, a_{x_3}, y, z\}$ is not a claw, we may assume by symmetry that $a_{x_3} y \in E(G)$, so $T' := T - \{aa_{x_3}, rr_3\} + \{a_{x_3} y, az\}$ violates (T1), since r is no longer a branch vertex.

Therefore these $2k+3$ sets are all pairwise disjoint. We will now show a few things about r_1 and x_1, which, by symmetry, will also be true of r_2 and x_2. Note that the $2k+3$ sets $(V(x_1Tr) \cap N_G(x_1))^-$ and $V(x_1Tr_1) \cap N_G(y)$ with $y \in X \setminus \{x_1\}$ are all subsets of $V(x_1Tr_1)$; we will show that they are pairwise disjoint and none of them include r_1. If $r_1 \in (V(x_1Tr) \cap N_G(x_1))^-$, then $x_1 r \in E(G)$, so $T' := T - \{rr_1, rr_2\} + \{x_1 r, r_1 r_2\}$ violates (T1). Now suppose $r_1 y \in E(G)$ for some $y \in X \setminus \{x_1\}$. If $r \in V(x_1Ty)$, then $T' := T - \{rr_1\} + \{r_1 y\}$ violates (T1). Otherwise $x_1 \in V(rTy)$, so $T' := T - \{x_1(x_1)_y\} + \{r_1 y\}$ violates either (T1) or (T2). Therefore none of these sets contain r_1; it remains to show that they are disjoint.

We will first show that $(V(x_1Tr) \cap N_G(x_1))^-$ is disjoint from the other $2k+2$ sets. Let $y \in X \setminus \{x_1\}$ and suppose some $a \in V(x_1Tr)$ with $x_1 a, ya_{x_1} \in E(G)$. If $r \in V(x_1Ty)$, then $T' := T - \{aa_{x_1}, rr_1\} + \{x_1 a, ya_{x_1}\}$ violates (T1). Otherwise $x_1 \in V(rTy)$, so define $\{x_1^*\} := N_T(x_1) \setminus \{(x_1)_r, (x_1)_y\}$. Lemma 1 requires that $(x_1)_y x_1^* \in E(G)$, so $T' := T - \{aa_{x_1}, x_1(x_1)_y, x_1 x_1^*\} + \{x_1 a, ya_{x_1}, (x_1)_y x_1^*\}$ violates (T1). Therefore $(V(x_1Tr) \cap N_G(x_1))^-$ is disjoint from the other sets; it remains to show that these other sets are pairwise disjoint. Let $y, z \in X \setminus \{x_1\}$ and let $a \in V(x_1Tr_1)$ such that $ya, za \in E(G)$. Consider three cases:

Case 1: Suppose $x_1 \in V(yTr)$ and $r \in V(x_1Tz)$ (or vice versa). Then $T' := T - \{x_1(x_1)_r, rr_1\} + \{ya, za\}$ violates (T1), since two branch vertices are lost (x and r) and only one is gained (a).

Case 2: Suppose $x_1 \in V(yTr)$ and $x_1 \in V(zTr)$. Since $\{a, a_r, y, z\}$ is not a claw, we may assume by symmetry that $a_r y \in E(G)$, so $T' := T - \{aa_r, x_1(x_1)_r\} + \{ya_r, za\}$ violates (T1), since x_1 is no longer a branch vertex.

Case 3: Suppose $r \in V(x_1Ty)$ and $r \in V(x_1Tz)$. Since $\{a, a_{x_1}, y, z\}$ is not a claw, we may assume by symmetry that $a_{x_1} y \in E(G)$, so $T' := T - \{aa_{x_1}, rr_1\} + \{a_{x_1} y, az\}$ violates (T1), since r is no longer a branch vertex.

Therefore these $2k+3$ sets are all pairwise disjoint. For each $x \in X \setminus \{x_1, x_2, x_3\}$ (again denoting $p := p(x)$), we have:

$$\sum_{y \in X} |V(xTp_x) \cap N_G(y)| = \left|(V(xTp) \cap N_G(x))^-\right| + \sum_{y \in X \setminus \{x\}} |V(xTp_x) \cap N_G(y)| \leq |V(xTp_x)|$$

Meanwhile:

$$
\begin{aligned}
\sum_{y\in X}|V(x_3Tr)\cap N_G(y)| &= \left|(V(x_3Tr)\cap N_G(x_3))^-\right| + \sum_{y\in X\setminus\{x_3\}}|V(x_3Tr)\cap N_G(y)| \\
&\leq |V(x_3Tr)\setminus\{r\}| = |V(x_3Tr)|-1, \\
\sum_{y\in X}|V(x_1Tr_1)\cap N_G(y)| &= \left|(V(x_1Tr_1)\cap N_G(x_1))^-\right| + \sum_{y\in X\setminus\{x_1\}}|V(x_1Tr_1)\cap N_G(y)| \\
&\leq |V(x_1Tr_1)\setminus\{r_1\}| = |V(x_1Tr_1)|-1, \text{and} \\
\sum_{y\in X}|V(x_2Tr_2)\cap N_G(y)| &= \left|(V(x_2Tr_2)\cap N_G(x_2))^-\right| + \sum_{y\in X\setminus\{x_2\}}|V(x_2Tr_2)\cap N_G(y)| \\
&\leq |V(x_2Tr_2)\setminus\{r_2\}| = |V(x_2Tr_2)|-1.
\end{aligned}
$$

Summing all these inequalities gives $\sum_{x\in X}\deg_T(x)\leq n-3$, so the theorem is proven.

□

Concluding remarks

To our knowledge, since the proposal of Conjecture 1, this is the first result whose only shortfall from proving the conjecture is the degrees of the branch vertices. We consider this promising, since the only sharpness examples we are aware of will have the kind of spanning tree we consider here. If a proof of Conjecture 1 is within reach, it is likely to be a generalization of our results here, with some clever choices based on the branch vertices of higher degree.

References

[1] Broersma, H., Tuinstra, H.: Independence trees and Hamilton cycles. J. Graph Theory **29**, 227-237 (1998)

[2] Flandrin, E., Kaiser, T., Kužel, R., Li, H., Ryjáček, Z.: Neighborhood unions and extremal spanning trees. Discrete Math. **308**, 2343-2350 (2008)

[3] Fujisawa, J., Matsumura, H., Yamashita, T.: Degree bounded spanning trees. Graphs and Combinatorics **26**, 695-720 (2010)

[4] Gargano, L., Hammar, M.: There are spanning spiders in dense graphs (and we know how to find them). Lecture Notes Computer Science **2719**, 802-816 (2003)

[5] Gargano, L., Hammar, M., Hell, P., Stacho, L., Vaccaro, U.: Spanning spiders and light-splitting switches. Discrete Math. **285**, 83-95 (2004)

[6] Gargano, L., Hell, P., Stacho, L., Vaccaro, U.: Spanning trees with bounded number of branch vertices. Lecture Notes Computer Science **2380**, 355-365 (2002)

[7] Kano, M., Kyaw, A., Matsuda, H., Ozeki, K., Saito, A., Yamashita, T.: Spanning trees with small number of leaves in a claw-free graph. Ars. Combin. **103**, 137-154 (2012)

[8] Matsuda, H., Matsumura, H.: On a k-tree containing specified leaves in a graph. Graphs and Combinatorics **22**, 371-381 (2006)

[9] Matsuda, H., Ozeki, K., Yamashita, T.: Spanning Trees with a Bounded Number of Branch Vertices in a Claw-Free Graph. Graphs and Combinatorics **30**, 429-437 (2014)

[10] Matthews, M.M., Sumner, D.P.: Longest paths and cycles in $K_{1,3}$-free graphs. J. Graph Theory **9**, 269-277 (1985)

[11] Neumann-Lara, V., Rivera-Campo, E.: Spanning trees with bounded degrees. Combinatorica **11**, 55-61 (1991)

[12] Ozeki, K., Yamashita, T.,: Spanning Trees: A Survey. Graphs and Combinatorics **27**: 1-26 (2011).

[13] Tsugaki, M., Yamashita, T.: Spanning trees with few leaves. Graphs and Combinatorics **23**, 585-598 (2007)

[14] Win, S.: Existenz von Gerüsten mit vorgeschriebenem Maximalgrad in Graphen (German). Abh. Math. Sem. Univ. Hamburg **43**, 263-267 (1975)

[15] Win, S.: On a conjecture of Las Vergnas concerning certain spanning trees in graphs. Result. Math. **2**, 215-224 (1979)