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Structure in sparse k-critical graphs

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ABSTRACT

Recently, Kostochka and Yancey [7] proved that a conjecture of Ore is asymptotically true by showing that every k-critical graph satisfies $|E(G)| \geq \left\lceil \left(\frac{k}{2} - \frac{1}{k-1}\right)|V(G)| - \frac{k(k-3)}{2(k-1)} \right\rceil$. They also characterized [8] the class of graphs that attain this bound and showed that it is equivalent to the set of k-Ore graphs. We show that for any $k \geq 33$ there exists an $\varepsilon > 0$ so that if G is a k-critical graph, then $|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \varepsilon\right)|V(G)| - \frac{k(k-3)}{2(k-1)} - (k-1)\varepsilon T(G)$, where $T(G)$ is a measure of the number of disjoint K_{k-1} and K_{k-2} subgraphs in G. This also proves for $k \geq 33$ the following conjecture of Postle [12] regarding the asymptotic density: For every $k \geq 4$ there exists an $\varepsilon_k > 0$ such that if G is a k-critical K_{k-2}-free graph, then $|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \varepsilon_k\right)|V(G)| - \frac{k(k-3)}{2(k-1)}$. As a corollary, our result shows that the number of disjoint K_{k-2} subgraphs in a k-Ore graph scales linearly with the number of vertices and, further,

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that the same is true for graphs whose number of edges is close to Kostochka and Yancey's bound.

1. Introduction

Given a graph G the *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that there exists a mapping $\phi : V(G) \to \{1, \ldots, k\}$ where $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$. Such a mapping is called a *proper k-coloring* of G. We say that G is k-colorable if G has a proper k-coloring. There is an obvious connection between the number of edges in a graph and the graph's chromatic number. Each edge is a restriction on the vertex labeling, and thus removing edges can lower the chromatic number of a graph. Indeed, the chromatic number of $G - e$ is either $\chi(G)$ or $\chi(G) - 1$. It is natural to study the class of graphs which are as sparse as possible for a given chromatic number.

A graph G is *k-critical* if $\chi(G) = k$ and every proper subgraph is $(k-1)$-colorable. Viewing k-critical graphs as minimal graphs with chromatic number k leads to the question of how small such graphs can be. Let $f_k(n)$ denote the minimum number of edges in a k-critical graph, Ore conjectured [10] the following.

Conjecture 1.1 *(Ore 1967 [10]). If $k \geq 4$, then*

$$f_k(n + k - 1) = f_k(n) + (k - 1)\left(\frac{k}{2} - \frac{1}{k - 1}\right).$$

As $\delta(G) \geq k - 1$ for any k-critical graph, it is clear that $f_k(n) \geq \frac{k-1}{2}n$. Since Dirac's 1957 paper [2], there have been many improvements over the years to the bounds for $f_k(n)$ ([3], [6], [9]) Recently, Kostochka and Yancey [7] made an important breakthrough.

Theorem 1.2 *(Kostochka and Yancey 2014 [7], Theorem 3). If $k \geq 4$ and G is k-critical, then*

$$|E(G)| \geq \left\lceil \left(\frac{k}{2} - \frac{1}{k-1}\right)|V(G)| - \frac{k(k-3)}{2(k-1)} \right\rceil.$$

They also showed in [8] that the class of graphs which attain this bound are *k-Ore graphs*, which are defined below. First, we begin with a definition used to construct k-Ore graphs.

Definition 1. An *Ore composition* of two graphs G_1 and G_2 is a graph obtained by the following procedure: (1) delete an edge xy from G_1, (2) split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree, and (3) identify x with z_1 and y with z_2.

Note that the Ore composition of two graphs is not unique, depending on which edge is deleted from G_1, which vertex z of G_2 is split, and how the neighbors of z are partitioned. Indeed, even the order in which we list the graphs is important; when we say that G is an Ore composition of H and F we mean that G is one of the graphs obtained by an Ore composition where H plays the role of G_1 (called the *edge-side* of the composition) and F plays the role of G_2 (called the *split-side* of the composition). The identified vertices $\underline{xz_1}$ and $\underline{yz_2}$ are called the *overlap vertices* of the composition. Further, we call the edge xy from step (1) the *replaced edge* of H and call the vertex z from step (2) the *split vertex* of F.

Definition 2. A graph G is a *k-Ore graph* if it is in the smallest class of graphs containing K_k which is closed under the Ore composition operation.

Equivalently, this is the class of graphs obtainable by successive Ore compositions of either K_k or other k-Ore graphs.

To prove Theorem 1.2, which shows that Ore's Conjecture is asymptotically true, Kostochka and Yancey established the following result on the density of a k-critical graph using a *potential function*,

$$\rho_{KY}(G) := (k-2)(k+1)|V(G)| - 2(k-1)|E(G)|.$$

Theorem 1.3 *(Kostochka and Yancey 2014 [7], Theorem 5). If $k \geq 4$ and G is k-critical, then $\rho_{KY}(G) \leq k(k-3)$.*

In a later paper, they also showed the following.

Theorem 1.4 *(Kostochka and Yancey 2016+ [8], Theorem 6). If $k \geq 4$ and G is k-critical, then $\rho_{KY}(G) = k(k-3)$ if and only if G is a k-Ore graph.*

The k-Ore graphs are the graphs which attain the bound of Theorem 1.3, and hence it is natural to ask if an increase in edge density is possible when forbidding subgraphs which arise through Ore constructions. In [11], Postle shows an increase in asymptotic density for 4-critical graphs when forbidding both K_3 and C_4 subgraphs. By a construction of Thomas and Walls [13], it is not sufficient to forbid only K_3.

Theorem 1.5 *([11]). There exists $\varepsilon > 0$ such that if G is a 4-critical graph of girth at least five, then*

$$|E(G)| \geq \left(\frac{5}{3} + \varepsilon\right)|V(G)| - \frac{2}{3}.$$

For larger values of k, it is also not sufficient to forbid only K_{k-1}. This leads to the following conjecture.

Conjecture 1.6 *([12]). For every $k \geq 4$, there exists $\varepsilon_k > 0$ such that if G is a k-critical K_{k-2}-free graph, then*

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \varepsilon_k\right)|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

The conjecture has been proven for $k = 5$ [12] and $k = 6$ [4]. In this paper, we prove the conjecture for $k \geq 33$. The method of proof also gives information about the structure of k-Ore graphs; in particular, we also prove that there are linearly many vertex-disjoint K_{k-2} subgraphs in k-Ore graphs. In order to track vertex-disjoint K_{k-2} subgraphs (including those inside K_{k-1} subgraphs), we define the following graph parameter.

Definition 3. When a graph H is a disjoint union of r copies of K_{k-1} and s copies of K_{k-2} subgraphs, define $T(H) := 2r + s$. Let G be an arbitrary graph. If G is K_{k-2}-free, then $T(G) = 0$. Otherwise, define

$$T(G) := \max_{H \subseteq G}\{T(H) \mid H \text{ is a disjoint union of } K_{k-1} \text{ and } K_{k-2} \text{ components}\}.$$

In a k-Ore graph, $T(G)$ can be shown to be lower-bounded by some constant times the number of vertices (Lemma 3.3). Using the subgraph-measuring parameter $T(G)$ we define the following modified potential function.

Definition 4. Let $\varepsilon = \frac{4}{k^3-2k^2+3k}$ and $\delta = (k-1)\varepsilon$. Given a graph G define the *ε-potential* to be

$$\rho(G) := ((k-2)(k+1) + \varepsilon)|V(G)| - 2(k-1)|E(G)| - \delta T(G).$$

For a vertex subset $R \subseteq V(G)$, we define

$$\rho_G(R) := ((k-2)(k+1) + \varepsilon)|R| - 2(k-1)|E(G[R])| - \delta T(G[R]),$$

where $G[R]$ is the induced subgraph of G on R.

One can check that the construction of ε guarantees that $\varepsilon \leq 1$ for all $k \geq 2$ (in particular, it is true for all values of k covered in this paper). With this modified potential function in hand, we are now able to state the main result of this paper.

Theorem 1.7. *If G is a k-critical graph that is not a k-Ore graph and $k \geq 33$, then $\rho(G) \leq k(k-3) - 2(k-1)$.*

We prove this using the potential method of Kostochka and Yancey; however, a limitation in the discharging method used restricts this result to the range where $k \geq 33$. Because reductions used in our proof could possibly create k-Ore graphs as auxiliary graphs, it is important that we also establish bounds for the ε-potential of k-Ore graphs. In Section 3, we prove the following.

Theorem 1.8. *If G is a k-critical graph that is a k-Ore graph and $k \geq 4$, then*

1. $\rho(G) = k(k-3) + k\varepsilon - 2\delta$ *if* $G = K_k$, *and*
2. $\rho(G) \leq k(k-3) + |V(G)|\varepsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta$ *if* $G \neq K_k$.

Note that Theorem 1.8 is proven using similar methods for $k = 5$ in [12]. Removing the notation of ε-potential, Theorem 1.7 and 1.8 give the following corollaries.

Corollary 1.9. *If $k \geq 33$ and G is k-critical, then*

$$|E(G)| \geq \left\lceil \frac{[(k-2)(k+1)+\varepsilon]\,|V(G)| - k(k-3) + 2\delta - k\varepsilon - \delta T(G)}{2(k-1)} \right\rceil,$$

where $\varepsilon = \frac{4}{k^3-2k^2+3k}$, $\delta = (k-1)\varepsilon$, and $T(G)$ is the subgraph-measuring parameter from Definition 3.

Corollary 1.10. *If $k \geq 33$, then there exists some $\varepsilon_k > 0$ such that if G is k-critical and K_{k-2}-free, then*

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \varepsilon_k\right)|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

Corollary 1.10 confirms Conjecture 1.6 for $k \geq 33$. We note that the class of K_s-free k-critical graphs was also studied by Krivelevich [9].

1.1. Outline of paper and notation

The paper is organized as follows. In Section 2, we establish some values for ε-potential. We also prove some results about list colorings which are used in Section 7. In Section 3, we prove Theorem 1.8 and also prove results about subgraphs in k-Ore graphs. These results are needed for our approach to Theorem 1.7. Sections 4–7 address Theorem 1.7. In Section 4, we define an auxiliary graph constructed from a partial $(k-1)$-coloring of a graph, and prove lemmas about the ε-potential of said graph. In Section 5, we work towards an important lemma (Lemma 5.7) which says that subgraphs in a minimal counterexample to Theorem 1.7 must be many edges away from being k-critical. In Section 6, we prepare for discharging by proving results on the structure near vertices of low degree in a minimal counterexample to Theorem 1.7. In Section 7, we complete the proof of Theorem 1.7 using a discharging argument.

Throughout the paper, we make use of the following concepts and notation. Given a graph G, let x, y be vertices of G and R be a proper vertex subset of G. We use

$G/\underline{xy}$ to refer to the graph obtained from G by identifying x and y; that is $G/\underline{xy}$ is obtained by deleting x, y and adding a new vertex $\underline{xy}$ which is adjacent to each vertex in $N_G(x) \cup N_G(y)$. The *boundary vertices* of R (in G) is the set $\partial_G R := \{u \in R \mid N_G(u) - R \neq \emptyset\}$. The *closed neighborhood* of x is the set $N_G[x] := N_G(x) \cup \{x\}$.

The *maximum independent cover number* of G, denoted $\mathrm{mic}(G)$, is the maximum of $\sum_{x \in I} \deg_G(x)$ over all independent sets $I \subseteq V(G)$. For terms not defined here see [14].

2. Preliminaries

When proving bounds on $\rho(G)$, it is important to know the ε-potential of complete graphs.

Observation 2.1.

1. $\rho(K_k) = k^2 - 3k + k\varepsilon - 2\delta$.
2. $\rho(K_1) = k^2 - k - 2 + \varepsilon$.
3. $\rho(K_{k-1}) = 2k^2 - 6k + 4 + (k-1)\varepsilon - 2\delta$.
4. *For $1 < \ell < k - 1$, the ε-potential of K_ℓ is bounded by $\rho(K_\ell) \geq 2k^2 - 4k - 2 + 2\varepsilon$.*

We now establish some edge bounds which will be needed for the final stage of discharging in Section 7. Given a graph G and vertex subsets A, B, we define $e(A, B)$ to be the number of edges from a vertex in A to a vertex in B. That is, let $e(A, B) := \sum_{a \in A} |N_{G[A \cup B]}(a) \cap B|$. We use the following lemma due to Kierstead and Rabern.

Lemma 2.2 *(Kierstead and Rabern 2015 [5], Main Lemma). Let G be a nonempty graph and $f : V \to \mathbb{N}$ with $f(v) \leq \deg_G(v) + 1$ for all $v \in V(G)$. If there is an independent set $A \subseteq V(G)$ such that*

$$e(A, V(G)) \geq \sum_{v \in V(G)} \left[\deg_G(v) + 1 - f(v)\right], \tag{1}$$

then G has a nonempty induced subgraph H that is f_H-choosable where $f_H(v) := f(v) + \deg_H(v) - \deg_G(v)$.

Lemma 2.3. *Let G be a k-critical graph with vertex subsets A, B_0, B_1 such that A is independent,* $\deg_G(a) = k - 1$ *for each $a \in A$, and* $\deg_G(b) = k + i$ *for each $b \in B_i$ where $i \in \{0, 1\}$. Then $e(A, B_0 \cup B_1) < |A| + 2|B_0| + 3|B_1|$.*

Proof. Suppose that G is a k-critical graph with vertex subsets A, B_0, B_1 such that A is independent, $\deg_G(a) = k - 1$ for each $a \in A$, and $\deg_G(b) = k + i$ for each $b \in B_i$ where $i \in \{0, 1\}$. Let $B = B_0 \cup B_1$. Suppose to the contrary that $e(A, B_0 \cup B_1) \geq |A| + 2|B_0| + 3|B_1|$ holds true.

Let $f : A \cup B \to \mathbb{N}$ where $f(v) = \deg_{G[A\cup B]}(v)$ if $v \in A$ and $f(v) = \deg_{G[A\cup B]}(v) - 1 - i$ if $v \in B_i$. Then the right side of Equation (1) becomes

$$\sum_{v\in A} 1 + \sum_{v \in B_0} 2 + \sum_{v\in B_1} 3 = |A| + 2|B_0| + 3|B_1|.$$

It follows from Lemma 2.2 that $G[A\cup B]$, and thus G, has a nonempty induced subgraph H that is f_H-choosable where $f_H(v) := f(v) + \deg_H(v) - \deg_{G[A\cup B]}(v)$.

Since G is k-critical, there exists a $(k-1)$-coloring ϕ of $G - H$. For each vertex $v \in V(H) \cap A$, there are at least $\deg_H(v)$ colors available and we see that $f_H(v) = \deg_H(v)$. Similarly, for each $v \in V(H) \cap B$, there are at least $\deg_H(v) - 1 - i$ colors available and $f_H(v) = \deg_H(v) - 1 - i$. Therefore, we can use f_H-choosability to extend ϕ to all of G, which is a contradiction. ■

3. k-Ore graphs

Here, we build up results regarding k-Ore graphs, which will be needed in order to bound the ε-potential for the reductions of general k-critical graphs that we will be using in subsequent sections.

Proposition 3.1. *Given a k-Ore graph G, there is a sequence of k-Ore graphs $G_1, G_2, \dots, G_s$ where $G_1 = K_k$, $G_s = G$, and for each $2 \le i \le s$, the graph G_i is an Ore composition of G_{i-1} and a k-Ore graph.*

Proof. Let G be a k-Ore graph. We will prove this by induction on $|V(G)|$. If G is K_k the result is trivial, so we may assume that G is an Ore composition of two k-Ore graphs G_1 and G_2 with overlap vertices $\{x, y\}$. By induction, there is a sequence $\mathcal{H} = H_1, H_2, ..., H_r$ where $H_1 = K_k$ and $H_r = G_1$ and each H_i is an Ore composition of H_{i-1} and a k-Ore graph. Then the desired sequence for G is $\mathcal{H}, G$. ■

Using this proposition, one can picture each k-Ore graph as a copy of K_k where some number of edges are replaced by split k-Ore graphs. In fact, any k-Ore graph can be obtained by simultaneously replacing some edges of a K_k with suitable split k-Ore graphs. Before examining ε-potential, we establish bounds on the subgraph-measuring parameter $T(G)$.

Lemma 3.2. *If G is an Ore composition of G_1 and G_2, then $T(G) \ge T(G_1) + T(G_2) - 2$. Moreover, if $G_1 = K_k$ or $G_2 = K_k$, then $T(G) \ge T(G_1) + T(G_2) - 1$. Further, if both G_1 and G_2 are K_k, then $T(G) = 4$.*

Proof. Suppose that G is an Ore composition of G_1 and G_2. Let e be the replaced edge of G_1 and z be the split vertex of G_2. From the definition of an Ore composition, it follows that $T(G) \ge T(G_1 - e) + T(G_2 - \{z\})$, and hence $T(G) \ge T(G_1) + T(G_2) - 2$.

If $G_1 = K_k$, then $T(K_k - e) = 2 = T(K_k)$ so we get $T(G) \geq T(G_1) + T(G_2) - 1$. We obtain a similar result if $G_2 = K_k$ as $T(K_k - z) = 2$. Further, if both G_1 and G_2 are K_k, then $T(G) = 4$. ■

Note that the conclusion of Lemma 3.2 is symmetric.

Lemma 3.3. *If G is a k-Ore graph and $G \neq K_k$, then $T(G) \geq 2 + \frac{|V(G)|-1}{k-1}$.*

Proof. We proceed by induction on $|V(G)|$. Let G be an Ore composition of two k-Ore graphs G_1 and G_2. If both G_1 and G_2 are K_k, then $|V(G)| = 2k - 1$; in this case, $T(G) = 4$ as desired. Suppose instead that exactly one of G_1, G_2 is K_k. Because the conclusion of Lemma 3.2 is symmetric and any Ore composition of a graph with K_k adds $k - 1$ vertices, we may assume without loss of generality that $G_1 = K_k$. It follows that

$$T(G) \geq T(G_2) + 1 \geq \left(2 + \frac{|V(G_2)| - 1}{k - 1}\right) + 1 = 2 + \frac{|V(G)| - 1}{k - 1},$$

as desired. Finally, suppose that neither G_1 nor G_2 is K_k. Then as $|V(G)| = |V(G_1)| + |V(G_2)| - 1$, it follows from Lemma 3.2 and induction that

$$T(G) \geq \left(2 + \frac{|V(G_1)| - 1}{k - 1}\right) + \left(2 + \frac{|V(G_2)| - 1}{k - 1}\right) - 2 = 2 + \frac{|V(G)| - 1}{k - 1}. \quad ■$$

Using Lemma 3.3, we now prove Theorem 1.8.

Proof of Theorem 1.8. By the definition of ε-potential, it follows that $\rho(K_k) = k(k - 3) + k\varepsilon - 2\delta$. Now suppose that G is a k-Ore graph which is not K_k. Then G has $k+\ell(k-1)$ vertices and $\frac{(\ell+1)k(k-1)}{2} - \ell$ edges for some $\ell \geq 1$. Using Lemma 3.3, it is again a straightforward calculation to show that $\rho(G) \leq k(k-3) + |V(G)|\varepsilon - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta$. ■

It is essential for the proof of Theorem 1.7 to understand the behavior of certain subgraphs of k-Ore graphs. Two useful subgraphs are defined below.

Definition 5. A subgraph $D \subseteq G$ is a *diamond of G* if $D = K_k - uv$ and $\deg_G(x) = k - 1$ for each $x \in V(D) - \{u, v\}$. The vertices u and v are the *endpoints* of the diamond. A subgraph $D' \subseteq G$ is an *emerald of G* if $D' = K_{k-1}$ and $\deg_G(x) = k - 1$ for each $x \in V(D')$.

Lemma 3.4. *If G is a k-Ore graph and $v \in V(G)$, then there exists a diamond or emerald of G in $G - v$.*

Proof. We prove this by induction on $|V(G)|$. Suppose that G is a k-Ore graph and let $v \in V(G)$ be an arbitrary vertex. If $G = K_k$, then $G - v$ is an emerald of G, as desired.

Therefore we may assume that G is an Ore composition of two k-Ore graphs G_1 and G_2 with overlap vertices $\{a, b\}$. We choose this composition to minimize $|V(G_1)|$, the order of the edge-side. By induction, there is an emerald or diamond D of G_2 not containing $\underline{ab}$. Hence, if $v \in V(G_1)$, then D is as desired. So we may assume that $v \in V(G_2) - \{\underline{ab}\}$.

Now if $G_1 = K_k$, then $G_1 - ab$ is a diamond of G not containing v as desired. Therefore, we may assume that G_1 is a composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{x, y\}$. By our choice of G_1 it follows that $ab \in E(H_1)$. Thus there is an emerald or diamond subgraph D of H_2 not containing $\underline{xy}$. Note that D is also an emerald or diamond of G and $v \notin V(D)$, as desired. ■

Lemma 3.5. *If G is a k-Ore graph and $D = K_{k-1}$ is a subgraph of G, then either $G = K_k$ or there exists a diamond or emerald of G disjoint from D.*

Proof. We prove this by induction on $|V(G)|$. Suppose that G is a k-Ore graph and let $D = K_{k-1}$ be a subgraph of G. When $G = K_k$, the lemma is trivial. So we may assume that G is an Ore composition of two k-Ore graphs G_1 and G_2 with overlap vertices $\{a, b\}$. Choose this composition to minimize the order of the edge-side, $|V(G_1)|$. As $\{a, b\}$ is an independent cutset in G, it follows that either $D \subseteq G_1 - ab$ or $D \subseteq G_2$. If $D \subseteq G_1 - ab$, then by Lemma 3.4 there exists a diamond or emerald D' of $G_2 - \underline{ab}$ and D' is disjoint from D as desired.

Thus we may assume that $V(D) \subseteq V(G_2) \cup \{a, b\}$. We examine two cases based on whether $V(D)$ contains any of the overlap vertices $\{a, b\}$ or not.

Since a is not adjacent to b in G_2, they cannot both be in D. So first, suppose that $|V(D) \cap \{a, b\}| = 1$ and without loss of generality, we assume that $a \in V(D)$. If $G_2 \neq K_k$, then by induction, there is a diamond or an emerald of G_2 disjoint from D and this is also a diamond or an emerald of G, as desired. Therefore we may assume that $G_2 = K_k$ and thus b has one neighbor on the split-side of G. It follows that $\deg_{G_1}(b) = \deg_G(b)$. By Lemma 3.4 there is a diamond or emerald D' of G_1 in $G_1 - a$. If D' is a diamond, then D' is also a diamond of G. If D' is an emerald, then because $\deg_{G_1}(b) = \deg_G(b)$, it follows that D' is an emerald of G. In either case, $D' \cap D = \emptyset$ as desired.

Second, suppose that $V(D)$ contains neither a nor b. If $G_1 = K_k$, then $G_1 - ab$ is a diamond that is disjoint from D. Otherwise, G_1 is a composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{x, y\}$. By our choice of G_1 it follows that $ab \in E(H_1)$. By Lemma 3.4 there is a diamond or emerald D' of $H_2 - \underline{xy}$, which then contains no vertices of D. Thus, D' is also a diamond or emerald of G, as desired. ■

4. Critical extensions

We now turn towards proving the main result, Theorem 1.7. We do this by discharging on a minimal counterexample; therefore we need to precisely define what makes a graph minimal.

Definition 6. A graph H is *smaller* than a graph G if $|V(G)| > |V(H)|$ or, if $|V(G)| = |V(H)|$, then H is smaller if either $|E(G)| > |E(H)|$ or if $|E(G)| = |E(H)|$ and G has fewer pairs of vertices with the same closed neighborhood.

Given a k-critical graph G, we have a particular method of examining what subgraphs exist in G. Note that if R is a proper vertex subset of G, then we can properly $(k-1)$-color $G[R]$. Such a coloring is used to create the following auxiliary graph.

Definition 7. Given a k-critical graph G and a proper $(k-1)$-coloring ϕ on a vertex subset R, we define the graph $G_{R,\phi}$ to be the graph obtained from G by identifying all vertices in $\phi^{-1}(i)$ to a single vertex x_i for $1 \leq i \leq k-1$, adding the edge $x_i x_j$ for each $1 \leq i < j \leq k-1$, and then deleting any parallel edges so that the new vertices form a complete subgraph with no parallel edges.

Note that if $uv \in E(G)$ for $u \in R$ and $v \in V(G) - R$, then $vx_{\phi(u)} \in E(G_{R,\phi})$. Further, we will always color R with as few colors as possible, so then it follows that $G_{R,\phi}$ is a smaller graph than G, or possibly $G_{R,\phi} = G$ if R is a clique. One can observe that $G_{R,\phi}$ is not $(k-1)$-colorable; a proof of this is in [7] (Claim 14). Therefore, there is a k-critical subgraph $W \subseteq G_{R,\phi}$. Because G is k-critical, W must contain at least one vertex in $\{x_1, \ldots, x_{k-1}\}$. The fact that W is smaller than G when R is not a clique is used frequently in subsequent ε-potential calculations.

Definition 8. Given a graph $G_{R,\phi}$ obtained via Definition 7 and a k-critical subgraph W, we define $R' := (R \cup V(W)) - X$ to be a *W-critical extension of* R where $X := V(W) \cap \{x_1, \ldots, x_{k-1}\}$ is called the *core* of the W-critical extension. If $R' = V(G)$, then we say that R' is a *spanning* W-critical extension. Lastly, the W-critical extension R' is *complete* if

$$|E(G[R'])| = |E(G[R])| + |E(W)| - |E(K_{|X|})|. \tag{2}$$

For a general W-critical extension R', it is possible that the left side of Equation (2) is larger. If we have $|E(G[R'])| = |E(G[R])| + |E(W)| - |E(K_{|X|})| + i$, then we say that the W-critical extension is *i-incomplete*.

Thus a W-critical extension is complete if the edges from R to $V(W)$ in $G[R']$ correspond to the edges from X to $V(W) - X$ in W, and incompleteness comes from three sources. First, edges from R to $V(W)$ in $G[R']$ that are not needed in W get counted on the left but never on the right. Second, if $N_R(w) \cap (\text{color } \ell)$ is larger than 1 for some $w \in V(W) - R$ and color ℓ, then $|E(G[R'])|$ counts all of these edges but $|E(W)|$ counts at most one. Third, if an edge $x_i x_j$ is not used in W, then it is not counted by $|E(W)|$ but is subtracted by $|E(K_{|X|})|$.

Lemma 4.1. *Suppose that G is a k-critical graph. If R' is a W-critical extension of $R \subsetneq V(G)$ with core X, then*

$$\rho_G(R') \leq \rho_G(R) + \rho(W) - \left(\rho(K_{|X|}) + \delta T(K_{|X|}) - \delta|X|\right). \tag{3}$$

Proof. Suppose that G is a k-critical graph with proper vertex subset R and that $G[R]$ is properly $(k-1)$-colored by ϕ. Let R' be any W-critical extension. The three elements of a graph that contribute to ε-potential are the vertices, the edges, and T. We note that each side of the inequality in Equation (3) counts the same number of vertices. For the edges, each side of Equation (3) counts some edges that the other side does not. Note that only $\rho_G(R')$ includes edges in G from R to $V(W) - X$, only the right side includes edges in $G_{R,\phi}$ from X to $V(W) - X$, and all other edges are accounted for by both sides. However, each edge from X to $V(W) - X$ corresponds to at least one distinct edge from R to $V(W) - X$, so the negative contribution of edges to the ε-potential is always greater on the left side. In fact, if the W-critical extension is i-incomplete, then the left side counts exactly i edges more than the left.

Therefore, if Equation (3) is not satisfied, it can only be because of the contribution of the subgraph-measuring parameter T. We observe that $T(G[R']) \geq T(G[R]) + T(W - X)$ and that $T(W - X) \geq T(W) - |X|$ because each $x_i \in X$ could be in at most one subgraph counted by $T(W)$. Therefore, the desired inequality holds. ■

Corollary 4.2. *Suppose G is a minimal counterexample to Theorem 1.7. If R' is a W-critical extension of $R \subsetneq V(G)$ and R is not a clique, then $\rho_G(R') \leq \rho_G(R) - 2(k-1) - \delta$.*

Proof. Let G be a minimal counterexample to Theorem 1.7. We aim to maximize the right side of Equation (3). Because R is not a clique we may assume that W is smaller than G. Therefore $\rho(W)$ follows Theorems 1.7 and 1.8, depending on whether W is a k-Ore graph or not. It follows that the right side is maximized when W is a k-Ore graph and $|X| = 1$, so we make those two assumptions as well. If $W = K_k$, then because $T(W) = T(W - x)$ for $x \in X$ we can ignore the contribution of $\delta|X|$ in Equation (3). It follows in this case that

$$\begin{aligned}
\rho_G(R') &\leq \rho_G(R) + (k^2 - 3k + k\varepsilon - 2\delta) - (k^2 - k - 2 + \varepsilon) \\
&= \rho_G(R) - 2(k-1) + (k-1)\varepsilon - 2\delta.
\end{aligned}$$

But recall that $\delta = (k-1)\varepsilon$, so the corollary holds when $W = K_k$.

If W is not K_k, then it follows from Theorem 1.8 that

$$\rho_G(R') \leq \rho_G(R) - 2(k-1) - \varepsilon + \delta + |V(G)|\varepsilon - \left(2 + \frac{|V(G)| - 1}{k-1}\right)\delta.$$

Again, because $\delta = (k-1)\varepsilon$ the corollary is proven. ■

5. Edge-additions

The goal of this section is to establish Lemma 5.7 which says that a subgraph of a minimal counterexample to Theorem 1.7 cannot be within $\frac{k-4}{2}$ edges of being a smaller k-critical graph. This will be used to establish structural results in Section 6.

Definition 9. A proper vertex subset $R \subsetneq V(G)$ is *i-collapsible in G* if for all proper $(k-1)$-colorings ϕ of $G[R]$ using color set C

$$\min_{c \in C} \left|\{uv \in E(G) \mid u \in \phi^{-1}(C-c) \cap R \text{ and } v \in V(G)-R\}\right| \leq i. \tag{4}$$

That is, a proper vertex subset R is i-collapsible if there is a "majority" color class in $\phi(\partial_G R)$ which covers all but at most i edges from R into $V(G)-R$. Note that the boundary vertices $\partial_G R$ of a 0-collapsible set receive the same color in every proper $(k-1)$-coloring of R.

Definition 10. Let G be a k-critical graph. An $(i+1)$-*edge-addition in G* is a set S of at most $(i+1)$ edges such that there exists a k-critical graph H with $S \subseteq E(H)$, $H-S \subseteq G$, and $V(H) \subsetneq V(G)$.

Thus, a 1-edge-addition is a single edge that, when added to G, forms a k-critical subgraph on fewer vertices than $|V(G)|$. For i-edge-additions with $i > 1$, the size of S is more flexible; this is important for making the subsequent arguments efficiently. In the proof of Lemma 5.7 we do specify the number of edges in S, but this will be controlled inductively rather than semantically.

Lemma 5.1. *A minimal counterexample to Theorem 1.7 does not contain a 2-vertex cutset.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and suppose that there exists a 2-vertex cutset $\{x, y\}$. Because G is k-critical, by Dirac [1], deleting $\{x, y\}$ leaves behind two components H_1 and H_2 such that $\tilde{G}_1 = G - H_2$ is $(k-1)$-colorable by ϕ where $\phi(x) = \phi(y)$ and $\tilde{G}_2 = G - H_1$ is $(k-1)$-colorable by ψ where $\psi(x) \neq \psi(y)$. Moreover, because G is k-critical there does not exist a proper $(k-1)$-coloring of $\tilde{G}_1$ where x and y receive different colors. This fact prevents x and y from having a common neighbor z in $\tilde{G}_2$, as a proper $(k-1)$-coloring of $G - xz$ would be a contradiction. Therefore x and y have no common neighbors in $\tilde{G}_2$, which implies that G is an Ore composition of $\tilde{G}_1 + xy$ and $\tilde{G}_2/\underline{xy}$, which we rename G_1 and G_2 respectively.

Because G is not a k-Ore graph, at most one of G_1 and G_2 is a k-Ore graph. From the definition of an Ore composition, it follows that $\rho(G) = \rho(G_1) + \rho(G_2) - k^2 - 3k - \varepsilon + \delta\,(T(G_1) + T(G_2) - T(G))$. Because the following argument does not rely on the distinction between edge-side or split-side, we may assume without loss of generality that G_1 is not a k-Ore graph. Using Lemma 3.2 and the fact that G_1 is smaller than G, we have

$$\rho(G) \leq \rho(G_2) - 2(k-1) - \varepsilon + 2\delta.$$

Thus G_2 has higher ε-potential than G. As G is a minimal counterexample to Theorem 1.7 and G_2 is smaller than G, it follows that G_2 must be a k-Ore graph.

If $G_2 \neq K_k$, then, it follows from Theorem 1.8 that $\rho(G) \leq k(k-3) - 2(k-1) + (n-1)\left(\varepsilon - \frac{\delta}{k-1}\right)$ where $n = |V(G_2)|$. If $G_2 = K_k$, then Lemma 3.2 gives $T(G_1) + T(G_2) - T(G) \leq 1$, so it follows that $\rho(G) \leq k(k-3) + k\varepsilon - 2\delta - 2(k-1) - \varepsilon + \delta$. Because $\delta = (k-1)\varepsilon$ both of these inequalities show that $\rho(G) \leq k(k-3) - 2(k-1)$, contradicting that G is a minimal counterexample to Theorem 1.7. ■

Proposition 5.2. *Let G be a k-critical graph. If $R \subsetneq V(G)$ is a proper vertex subset where all W-critical extensions of R are spanning, have core size 1, and are at most i-incomplete, then R is i-collapsible in G.*

Proof. Let G be a k-critical graph and suppose that we have a proper vertex subset R such that all W-critical extensions of R are spanning, have core size 1, and are at most i-incomplete. Then let ϕ be an arbitrary proper coloring of R using color set $[k-1]$ and let R' be a W-critical extension using ϕ. By hypothesis, $R' = V(G)$. If we permute the colors of ϕ so that the vertex in X corresponds to color class 1, then each edge from $\phi^{-1}(\{2, 3, \ldots, k-1\}) \cap R$ to $V(G) - R$ contributes to the incompleteness of the W-critical extension. There are at most i such edges so, by definition, R is i-collapsible. ■

Lemma 5.3. *If G is a minimal counterexample to Theorem 1.7 with an i-collapsible subset $R \subsetneq V(G)$ for $i \leq (k-3)/2$, then there is an $(i+1)$-edge-addition in G.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and let $R \subsetneq V(G)$ be an i-collapsible subset for $i \leq (k-3)/2$. Suppose, for the sake of contradiction, that there is no $(i+1)$-edge-addition in G. For each $u \in \partial_G R$ let $w(u) = |\{uv \in E(G) \mid v \in V(G) - R\}|$. Because G is a k-critical graph, G is $(k-1)$-edge-connected and thus $\sum_{u \in \partial_G R} w(u) \geq k-1$. Let $\partial_G R = \{u_1, \ldots, u_s\}$ and, without loss of generality, assume that $w(u_1) \geq w(u_2) \geq \cdots \geq w(u_s) \geq 1$. ■

Case 1. Suppose $w(u_2) + \cdots + w(u_s) \geq i + 2$.

This case is the same as Case 2 of Lemma 16 in [7], which shows that, for all proper $(k-1)$-colorings ϕ of $G[R]$ using color set C and for any color class $\ell \in C$

$$\sum_{u \in \partial_G R - \phi^{-1}(\ell)} w(u) \geq i + 1.$$

However, R is i-collapsible so this is a contradiction.

Case 2. Suppose $w(u_2) + \cdots + w(u_s) \leq i + 1$.

For $i = 0$, this implies that $\{u_1, u_2\}$ is a 2-vertex cutset in G so, by Lemma 5.1, we may assume that $i \geq 1$. Let $S = \{u_1 u_j \mid 2 \leq j \leq s\}$. Because we have assumed that there is no $(i + 1)$-edge-addition and because S is a set of at most $i + 1$ edges, there is a proper coloring ϕ of $G[R] + S$ using color set $[k - 1]$, and u_1 is the unique vertex of $\partial_G R$ in its color class. Without loss of generality, let $\phi(u_1) = 1$. Because $i \leq \frac{k-3}{2}$ it follows that $w(u_1) \geq i + 1$. Therefore Equation (4) in the definition of i-collapsible can only be witnessed by color 1. Because R is i-collapsible by hypothesis it follows that $w(u_2) + \cdots + w(u_s) \leq i$.

Let ψ be a proper $(k - 1)$-coloring of $G[(V(G) - R) \cup \{u_1\}]$ which uses the same colors as ϕ such that $\psi(u_1) = 1$ and choose ψ so that the number of edges from $\partial_G R$ to $V(G) - R$ which have endpoints colored the same by $\overline{\psi} := \psi|_{V(G)-R} \cup \phi|_R$ is minimized. Since $\overline{\psi}$ is not a proper $(k - 1)$-coloring of G, we may assume that $\phi(u_p) = 2$ and one of its neighbors x in $V(G) - R$ also receives color 2.

We will reach a contradiction by relabeling the colors of ψ to interchange 2 with another color ℓ in such a way that $\overline{\psi}$ now gives $u_p x$ differently colored endpoints, and so that no edge from $\partial_G R$ to $V(G) - R$ which previously had differently colored endpoints now has endpoints colored the same. By showing that such an ℓ exists, we contradict our initial choice of ψ.

Initially, we consider $k - 2$ color candidates for ℓ, obviously needing to remove color 2 as an option. We also remove color 1 from consideration, so that $\psi(u_1) = \phi(u_1)$ does not change. Finally, for each of the at most i edges $u_j v$ from $\partial_G R - \{u_1\}$ to $V(G) - R$ we remove $\phi(u_j)$ if $\phi(u_j) \neq 2$ and remove $\psi(v)$ if $\phi(u_j) = 2$. This leaves at least $(k - 2) - 1 - i \geq \frac{k-3}{2} \geq i$ choices. Recall that $i \geq 1$, so there does exist a color ℓ which contradicts our initial choice of ψ, and completes the proof.

Proposition 5.4. *Let G be a minimal counterexample to Theorem 1.7. If $R \subsetneq V(G)$ is a proper vertex subset that is not a clique and $\rho_G(R) < \rho(G) + k^2 - 3k + 4 - \varepsilon$, then every W-critical extension of R has core size 1.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and let $R \subsetneq V(G)$ be a proper vertex subset that is not a clique such that $\rho_G(R) < \rho(G) + k^2 - 3k + 4 - \varepsilon$. Suppose that R' is a W-critical extension with core X where $|X| > 1$. The computation in Corollary 4.2 maximized the right side of Equation (3) by assuming that $|X| = 1$. But if $|X| > 1$, then that computation is maximized by assuming $|X| = k - 1$ which yields

$$\rho_G(R') \leq \rho_G(R) + \rho(W) - (2k^2 - 6k + 4 + (k - 1)\varepsilon - 2\delta).$$

Because $\rho(G) \leq \rho_G(R')$ and using the hypothesis, we get

$$\rho(G) < \rho(G) + k^2 - 3k + 4 - \varepsilon + \rho(W) - (2k^2 - 6k + 4 + (k - 1)\varepsilon - 2\delta).$$

This simplifies to $\rho(W) > k^2 - 3k + k\varepsilon - 2\delta$. By Theorem 1.8, this ε-potential is too high for W to be a k-Ore graph. And because W is smaller than G, we reach a contradiction with the minimality of G. ■

Lemma 5.5. *Let G be a minimal counterexample to Theorem 1.7. There is no 1-edge-addition in G.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and suppose that there is a 1-edge-addition in G. Among all 1-edge-additions S, pick one that minimizes the order of the k-critical graph $H \subseteq G + S$. Let $R = V(H)$ and let R' be a W-critical extension of R. Now $\rho_G(R) \leq \rho(H) + 2(k-1) + \delta$ and, because R is not a clique, Corollary 4.2 implies that $\rho_G(R') \leq \rho(H)$ It follows that H must be a k-Ore graph, as otherwise H is smaller than G and $\rho_G(R') < \rho(G)$ which is not possible.

The k-Ore graph with largest ε-potential is K_k so we have

$$\rho_G(R) \leq [k(k-3) + k\varepsilon - 2\delta] + 2(k-1) + \delta < \rho(G) + 2(k-1) + k\varepsilon - \delta + 2(k-1).$$

By Proposition 5.4, and because $4(k-1) + k\varepsilon - \delta < k^2 - 3k + 4 - \varepsilon$ for all $k \geq 6$, the core of R' has size 1. Corollary 4.2 implies that $\rho_G(R') < \rho(G) + 2(k-1) + k\varepsilon - 2\delta$. Note that R' must be complete because otherwise the right side of this inequality would be at least $2(k-1)$ lower, and we would again have $\rho_G(R') < \rho(G)$. Further, R' must be spanning because otherwise there exists a vertex subset R'' such that $\rho_G(R'') < \rho(G)$. Therefore, R is 0-collapsible in G by Proposition 5.2.

By definition, in every proper $(k-1)$-coloring of $G[R]$, each vertex in $\partial_G R$ receives the same color. If H is K_k, then $R = \{u_1, u_2, \ldots, u_k\}$ and we can assume that $\{u_1 u_k\} = S$. We properly $(k-1)$-color $G[R]$ with ϕ so that $\phi(u_j) = j$ for $1 \leq j \leq k-1$ and $\phi(u_k) = 1$. Because each vertex in $\partial_G R$ receives the same color, this means that $\{u_1, u_k\}$ is a 2-vertex cutset in G which contradicts Lemma 5.1.

Therefore H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. Note that S must be on the edge-side of the composition—that is $S \subseteq E(H_1)$—because otherwise $\{ab\}$ is a 1-edge-addition that contradicts our choice of S. By Lemma 5.1 the set $\{a, b\}$ cannot be a cutset in G so there must be $u, v \in \partial_G R - \{a, b\}$ such that $u \in V(H_1)$ and $v \in V(H_2) \cap G$. If any proper $(k-1)$-coloring ϕ of $G[R]$ has $\phi(u) \notin \{\phi(a), \phi(b)\}$, then we can relabel the colors on H_1 so that $\phi(u) \neq \phi(v)$. This contradicts the fact that R is 0-collapsible. So without loss of generality, we may assume that $\phi(u) = \phi(a)$. Let $P = (V(H_2) \cap G) \cup \{a, b\}$. Now either $\psi(v) = \psi(a)$ in every proper $(k-1)$-coloring ψ of $G[P]$ or we can produce a proper $(k-1)$-coloring of R where u and v receive different colors. Thus av is a 1-edge-addition that yields a k-critical subgraph of order at most $|V(H_2)| + 1$ which contradicts our choice of S. ■

Corollary 5.6. *Let G be a minimal counterexample to Theorem 1.7. For any subgraph $H \subseteq G$, there is no diamond of H. Further, if there is an emerald D of H, then there* *exists a vertex $z \in V(G)-V(D)$ such that $xz \in E(G)$ for each $x \in V(D)$ with* $\deg_G(x) = k-1$. *Therefore, there is no emerald of G.*

Proof. Let G be a minimal counterexample to Theorem 1.7, and let H be a subgraph of G. If D is a diamond of H with endpoints $\{u, v\}$, then $\{uv\}$ is a 1-edge-addition in G which contradicts Lemma 5.5. So we may assume that D is an emerald of H.

Note that $\deg_D(x) = k-2$ for each $x \in V(D)$ so each such x is adjacent in G to at least one vertex $V(G) - V(D)$. If there is at most one $x \in V(D)$ with $\deg_G(x) = k-1$, then the corollary is trivially true. Suppose then, for the sake of contradiction, that x, y are vertices in D with $\deg_G(x) = \deg_G(y) = k-1$ and a, b are vertices in $V(G) - V(D)$ such that $\{ax, by\} \subseteq E(G)$ and $a \neq b$. For any proper $(k-1)$-coloring ϕ of $G - \{x\}$ it must be the case that the neighbors of x all receive distinct colors. If we could recolor y using $\phi(a)$, then ϕ would extend to all of G which is a contradiction. Therefore, $\phi(b)$ must be the same color as $\phi(a)$. But now $\{ab\}$ is a 1-edge-addition in G which contradicts Lemma 5.5.

Lastly, if D is an emerald of G, then the vertex z guaranteed by the above argument makes a K_k subgraph in G which is not possible in a minimal counterexample to Theorem 1.7. ■

Lemma 5.7. *In a minimal counterexample G to Theorem 1.7, there is no proper vertex subset R where R is not a clique and $\rho_G(R) < \rho(G) + 2(i+1)(k-1) + \delta$ for $1 \leq i \leq \frac{k-4}{2}$. Further, G does not have an i-edge-addition for $1 \leq i \leq \frac{k-4}{2}$.*

Proof. Let G be a minimal counterexample to Theorem 1.7. We will show first that a subset of the given ε-potential implies that there is an i-edge-addition in G, and then prove inductively that there are no $\frac{k-4}{2}$-edge-additions in G. First note that, by Corollary 4.2, there is no proper subset that is not a clique and has ε-potential less than $\rho(G) + 2(k-1) + \delta$.

Claim 5.7.1. *For each i with $1 \leq i \leq \frac{k-4}{2}$ if G has no proper vertex subset that is not a clique with ε-potential less than $\rho(G) + 2i(k-1) + \delta$, R is a proper vertex subset that is not a clique, and $\rho_G(R) < \rho(G) + 2(i+1)(k-1) + \delta$, then every W-critical extension of R is spanning, has core size 1, and is at most $(i-1)$-incomplete. Further, there is an i-edge-addition in G.*

Proof of Claim. Given i, where $1 \leq i \leq \frac{k-4}{2}$, suppose that G has no proper vertex subset that is not a clique with ε-potential less than $\rho(G) + 2i(k-1) + \delta$ and let R be a proper vertex subset that is not a clique and $\rho_G(R) < \rho(G) + 2(i+1)(k-1) + \delta$. For $i \leq \frac{k-4}{2}$, this implies that $\rho_G(R) < \rho(G) + k^2 - 3k + 2 + \delta$ so every W-critical extension R' has core size 1 by Proposition 5.4.

By Corollary 4.2 we also have ε-potential $\rho_G(R') \leq \rho_G(R) - 2(k-1) - \delta < \rho(G) + 2i(k-1)$. By the hypothesis of the claim, R' must be all of $V(G)$. Also, R' can be at most

$(i-1)$-incomplete as otherwise the right side of the inequality would be at least $2i(k-1)$ lower and we would have $\rho_G(R') < \rho(G)$, which is not possible. By Proposition 5.2 and Lemma 5.3, R is $(i-1)$-collapsible in G and hence there is an i-edge-addition in G. ∎

Suppose now that there is an i-edge-addition in G. We will prove inductively that any i with $1 \leq i \leq \frac{k-4}{2}$ gives a contradiction. Lemma 5.5 shows that $i \neq 1$. We may assume that there is no $(i-1)$-edge-addition, so by Claim 5.7.1 there is no proper vertex subset R with $\rho_G(R) < \rho(G) + 2i(k-1) + \delta$. Note that this inductive hypothesis guarantees that $|S| = i$. Because each edge of S might contribute to $T(H)$, we have $\rho_G(R) \leq \rho(H) + 2i(k-1) + i\delta$. Among all i-edge-additions S, we will choose one that minimizes the order of the k-critical graph $H \subseteq G + S$. ∎

Case 1. *H is not a k-Ore graph.*

Because H is smaller than the minimal counterexample G, we have $\rho(H) < \rho(G)$. Thus, we bound the ε-potential of R by $\rho_G(R) < \rho(G) + 2i(k-1) + i\delta < \rho(G) + 2(i+1)(k-1) + \delta$. By Claim 5.7.1, every W-critical extension of R is spanning, has core size 1, and is at most $(i-1)$-incomplete. Further, there must be some W-critical extension R' that is exactly $(i-1)$-incomplete. Otherwise, Proposition 5.2 implies that R is $(i-2)$-collapsible and there is an $(i-1)$-edge-addition in G by Lemma 5.3.

Choose such a $(i-1)$-incomplete W-critical extension R'. Using Lemma 4.1 and the $(i-1)$-incompleteness of R' we bound the ε-potential as follows:

$$\rho_G(R') < [\rho(G) + 2i(k-1) + i\delta] + \rho(W) - 2(i-1)(k-1) - (k^2 - k - 2 + \varepsilon - \delta)$$

But $R' = V(G)$, so this implies that $\rho(W) > k^2 - 3k - (i+1)\delta + \varepsilon$. Because W is smaller than G, this contradicts the minimality of G unless W is a k-Ore graph. By Lemma 3.4, there is a subgraph $D \subseteq W - X \subseteq G$ which is an emerald of W. Corollary 5.6 gives a vertex $z \in V(G) - V(D)$ such that $xz \in E(G)$ for each $x \in V(D)$ with $\deg_G(x) = k-1$. Because R' is spanning, the only edges in G that can cause $\deg_G(x) > k-1$ for $x \in V(D)$ are edges from x to R which do not correspond to an edge used in W. These edges contribute to the incompleteness of a W-critical extension, so z has at most $(i-1)$ non-neighbors in D. Adding these edges yields a K_k, which contradicts the inductive hypothesis.

Case 2. *H is a k-Ore graph but is not K_k.*

If H is a k-Ore graph that is not K_k, then $\rho(H) \leq k(k-3) + \varepsilon - 2\delta$, and because $k(k-3) < \rho(G) + 2(k-1)$ we also have $\rho(R) < \rho(G) + 2(i+1)(k-1) + (i-2)\delta + \varepsilon$. For $i \leq \frac{k-4}{2}$, this upper bound satisfies the hypothesis of Proposition 5.4 so every W-critical extension R' has core size 1. Corollary 4.2 implies that $\rho_G(R') < \rho(G) + 2i(k-1) + (i-3)\delta + \varepsilon$. For $i = 2$, the W-critical extension R' is at most 1-incomplete because otherwise the right side is lowered by at least $4(k-1)$ and we get $\rho_G(R') < \rho(G) - \delta + \varepsilon$. This implies that $\rho_G(R') < \rho(G)$ which is not possible. Note that for $i > 3$ it is possible that R' is i-incomplete according to this bound, but cannot be j-incomplete for $j \geq i+1$.

First, suppose that H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. Note that all edges of S must be on the edge-side of the composition H_1 as otherwise adding ab to $S \cap E(H_1)$ is an i-edge-addition that contradicts our choice of S. Thus $H_2 - \underline{ab} \subseteq G$. By Lemma 3.4, there is a subgraph $D \subseteq H_2 - \underline{ab} \subseteq G$ which is an emerald of H_2. Corollary 5.6 gives a vertex $z \in V(G) - V(D)$ such that $xz \in E(G)$ for each $x \in V(D)$ with $\deg_G(x) = k - 1$. For each $x \in V(D)$, we have $\deg_{H_2}(x) = \deg_H(x) = k - 1$, so $z \in V(H)$ and either $x \in N_G(z)$ or $x \in \partial_G R$. But adding the edges $\{yz \mid y \in V(D) \cap \partial_G R\}$ creates a K_k subgraph so by the inductive hypothesis and our choice of S it follows that $|V(D) \cap \partial_G R| \geq i + 1$.

By Lemma 5.1, $\{a, b\}$ is not a cutset so there is some $u \in \partial_G R - \{a, b\}$ such that $u \in V(H_1)$. Let ϕ be a proper $(k-1)$-coloring of $G[R]$, with the colors permuted so that the vertex in the core X of the W-critical extension corresponds to color class 1. Thus each edge from $\phi^{-1}(\{2, 3, \ldots, k-1\}) \cap R$ to $V(G) - R$ contributes to the incompleteness of the W-critical extension. In the case where $i = 2$, R' is at most 1-incomplete so $|V(D) \cap \partial_G R| \leq 2$. This contradicts our earlier bound on this set. Therefore we may assume $i > 3$ for the rest of this case. Because R' is at most i-incomplete $|V(D) \cap \partial_G R| \leq i+1$. This implies that $|V(D) \cap \partial_G R| = i+1$, R' is exactly i-incomplete, and that $\phi(u) = 1$.

If $\phi(u) \notin \{\phi(a), \phi(b)\}$, then we can relabel the colors on H_1 only so that $\phi(u)$ is not given to any vertex in $V(D) \cap \partial_G R$. Because all W-critical extensions of R have a core of size 1, this new coloring would give a W-critical extension that is $i + 1$ incomplete which is a contradiction. Therefore it must be the case that, for every proper $(k-1)$-coloring of $G[R]$, $\phi(u) \in \{\phi(a), \phi(b)\}$. This means that $\{ua, ub\}$ is a 2-edge-addition which contradicts the fact that $i > 3$.

Case 3. H is K_k.

For this case, we further refine our bound $\rho_G(R) \leq \rho(H) + 2i(k-1) + t\delta$. We do not know how many edges of S contribute to $T(H)$, but $t \leq 2$. The ε-potential of H is $\rho(H) = k(k-3) + k\varepsilon - 2\delta$ and so $\rho(R) < \rho(G) + 2(i+1)(k-1) + (t-2)\delta + k\varepsilon$. For $i \leq \frac{k-4}{2}$, this upper bound satisfies the hypothesis of Proposition 5.4 so every W-critical extension R' has core size 1. Corollary 4.2 implies that $\rho_G(R') < \rho(G) + 2i(k-1) + (t-3)\delta + k\varepsilon$. Note that R' is at most i-incomplete, as otherwise $\rho_G(R') < \rho(G)$.

We label $R = V(H) = \{u_1, \ldots, u_k\}$ so that $u_1u_k \in S$ and properly $(k-1)$-color $G[R]$ with ϕ so that $\phi(u_j) = j$ for $1 \leq j \leq k-1$ and $\phi(u_k) = 1$. Because $\deg_G(u_j) \geq k-1$ each vertex $u_j \in R$ has at least as many edges in G from u_j to $V(G) - R$ as the number of edges of S incident with u_j. Any color class that not incident to an edge in S will miss at least $i + 1$ endpoints of S. So for R' to be at most i-incomplete, the vertex in the core X corresponds to color class 1 and every edge in S must incident to at least one of u_1 or u_k. If u_1u_2 and u_ku_3 are both in S, then switching the colors on u_2 and u_k give a proper $(k-1)$-coloring of $G[R]$ where every color class is not incident to at least one edge in S, which is a contradiction. Thus we may assume that, without loss of generality, either $|S| = 3$ and S forms a triangle subgraph or S forms a star subgraph with u_1 as the center. In either case, $t \leq 1$ because $G[R]$ has a K_{k-2} subgraph. Thus the bound given by Corollary 4.2 is $\rho_G(R') < \rho(G) + 2i(k-1) - 2\delta + k\varepsilon$. With this bound, R' cannot be i-incomplete because $2\delta > k\varepsilon$.

If S is a triangle, let $S = \{u_1u_k, u_1u_2, u_ku_2\}$. Because R' is at most 2-incomplete, by changing which two vertices of $\{u_1, u_2, u_k\}$ have the same color in a proper $(k-1)$-coloring of $G[R]$, it follows that $\partial_G R = \{u_1, u_2, u_k\}$ and each of these vertices has exactly two edges to $V(G) - R$. Thus there are 6 edges from R to $V(G) - R$. However, $i \leq \frac{k-4}{2}$ and $i = 3$ imply that $k \geq 10$, which is a contradiction as k-critical graphs are $(k-1)$-edge-connected.

Suppose instead that S is a star with u_1 as the center. Because R' is at most $(i-1)$-incomplete, every leaf u_j of the star has exactly one neighbor in $V(G) - R$, say y_j. Consider the graph $F = G - \{u_2, \dots, u_k\}$. No proper $(k-1)$-coloring ψ of F can be extended to all of G, so it follows that $\psi(u_1) = \psi(y_j)$ for each j where u_j is a leaf of S. Thus u_1y_j is a 1-edge-addition in G, which contradicts Lemma 5.5. ■

6. Cloning

Cloning is a reduction operation that will help us understand the structures that exist near vertices of degree $k-1$ in a minimal counterexample to Theorem 1.7.

Definition 11. Let G be a k-critical graph with $xy \in E(G)$ such that $\deg_G(x) = k-1$. We define *cloning x with y* to mean constructing a new graph $G_{y\to x}$ such that $V(G_{y\to x}) = V(G) \cup \{\tilde{x}\} - \{y\}$ and $E(G_{y\to x}) = E(G - y) \cup \{\tilde{x}v \mid v \in N_G(x)\} \cup \{\tilde{x}x\}$.

Thus the vertex y is replaced with the new vertex $\tilde{x}$, which is a copy of x. Below we define the notion of a cluster, which was introduced in [7].

Definition 12. A *cluster* is a maximal set $R \subseteq V(G)$ such that $\deg_G(x) = k-1$ for every $x \in R$ and $N_G[x] = N_G[y]$ for every pair $x, y \in R$.

Note that if $x \in V(G)$ is in a cluster C_x and $xy \in E(G)$, then in $G_{y\to x}$ the new vertex $\tilde{x}$ is added to the cluster C_x. Further, if x' is a second vertex in C_x, then $G_{y\to x'} = G_{y\to x}$. If y is already in C_x, then $G_{y\to x} = G$. If y is not in C_x, then $G_{y\to x}$ is smaller than G except in the case where $\deg(y) = k-1$ and $G_{y\to x}$ is k-critical. In this case, we further need y to be in a cluster of size at most $|C_x|$ for $G_{y\to x}$ to be smaller than G.

Lemma 6.1. *If G is a k-critical graph where $xy \in E(G)$, x is in a cluster of size s, and $\deg_G(y) \leq k-2+s$, then $G_{y\to x}$ is not $(k-1)$-colorable.*

Proof. Let G be a k-critical graph and let $xy \in E(G)$ such that x is in a cluster C_x of size s and $\deg_G(y) \leq k-2+s$. Suppose, for the sake of contradiction, that ϕ is a proper $(k-1)$-coloring of $G_{y\to x}$. Let ψ be the partial proper coloring of G obtained by copying $\phi(u)$ for every $u \in V(G) - \{y\}$. Because y has at most $k-2$ neighbors outside of C_x we can choose $\psi(y)$ to be a color distinct from these neighbors. But now $\psi(y) = \psi(z)$ for some vertex $z \in C_x$ because G is k-critical. Without loss of generality, we can assume that $z = x$. We recolor x so that $\psi(x) := \phi(\tilde{x})$ and now ψ is a proper $(k-1)$-coloring of G, which is a contradiction. ■

Lemma 6.2. *Suppose that G is a minimal counterexample to Theorem 1.7 and $xy \in E(G)$ such that (1) x is in a cluster C_x of size s, (2) $\deg_G(y) \le k-2+s$, and (3) if y is in a cluster C_y, then $C_y \ne C_x$ and $|C_y| = t \le s$. Then for any k-critical subgraph $H \subseteq G_{y\to x}$ either H is a k-Ore graph or $H = G_{y\to x}$. Moreover, $H = G_{y\to x}$ is only possible if $\deg_G(y) = k-1$.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and let $xy \in E(G)$ such that (1) x is in a cluster C_x of size s, (2) $\deg_G(y) \le k-2+s$, and (3) if y is in a cluster C_y, then $C_y \ne C_x$ and $|C_y| = t \le s$. Let $G_{y\to x}$ be the graph obtained by cloning x with y. By Lemma 6.1 $G_{y\to x}$ is not $(k-1)$-colorable, so there exists a k-critical subgraph $H \subseteq G_{y\to x}$. Note that condition (3) ensures that H is smaller than G. Suppose that H is not a k-Ore graph; we will see that this either leads to contradiction, or implies that $\deg_G(y) = k-1$ and $H = G_{y\to x}$.

We let $R = V(H) - \{\tilde{x}\}$ and note that R is not a clique because H is not a k-Ore graph. One can compute that $\rho_G(R) \le \rho(H) + k^2 - 3k + 4 - \varepsilon + \delta$. Let R' be a W-critical extension of R with core X. Because $\rho(G) \le \rho_G(R')$ and because H is smaller than G but is not a k-Ore graph, Lemma 4.1 yields the inequality

$$[\rho(K_{|X|}) + \delta T(K_{|X|}) - \delta|X|] - k^2 + 3k - 4 + \varepsilon - \delta < \rho(W). \tag{5}$$

For $1 < |X| < k-1$, this gives W an ε-potential that is too high for W to be a k-Ore graph by Theorem 1.8. Because W is smaller than G, this contradicts the minimality of G.

Suppose now that $|X| = k-1$. Then Observation 2.1 implies that $k^2 - 3k + k\varepsilon - k\delta < \rho(W)$, which is a contradiction unless W is a k-Ore graph. When W is a k-Ore graph, Equation (5) is almost tight; more specifically, the difference between the two sides is less than $2(k-1)$. Therefore, it follows that R' is a spanning and complete W-critical extension, because otherwise the right side is lowered by at least $2(k-1)$.

If W is not K_k, Lemma 3.5 implies that $D \subseteq W$ is a diamond or emerald of W disjoint from X. Because $W - X \subseteq G$, Corollary 5.6 implies that D is an emerald of W. But R' is a spanning and complete extension, so $\deg_G(x) = k-1$ for each $x \in V(D)$. Thus D is an emerald of G, which contradicts Corollary 5.6. Therefore we may assume that W is K_k, and it follows that $V(G) = R \cup \{y\}$. Thus $T(H)$ and $T(G)$ can differ by at most 1, and it must be that $|E(H)| = |E(G)|$. This implies that $\deg_G(y) = k-1$ and $H = G_{y\to x}$.

Suppose instead that $|X| = 1$. We claim that R' must be a spanning W-critical extension that is at most $\frac{k-4}{2}$-incomplete. For an i-incomplete W-critical extension, we have

$$\rho(G) \le \rho_G(R') \le \rho(H) + \rho(W) - 2k + 6 - 2i(k-1) - 2\varepsilon + 2\delta, \tag{6}$$

which because H is smaller than G yields

$$2k - 6 + 2i(k-1) + 2\varepsilon - 2\delta < \rho(W). \tag{7}$$

Lemma 5.7 implies that any proper vertex subset that is not a clique must have ε-potential at least $\rho(G)+k^2-3k+2+\delta$. If R' is not spanning, the left side of Equation (6), and subsequently Equation (7), can be increased by $k^2-3k+2+\delta$. Thus $k^2-k-4+2\varepsilon-\delta > \rho(W)$, which contradicts either Theorem 1.8 or the minimality of G. So we may assume that R' is spanning. If $i \ge \frac{k-3}{2}$, then we get $k^2-2k-3+2\varepsilon-2\delta < \rho(W)$ which also contradicts either Theorem 1.8 or the minimality of G. Therefore R' is spanning and is at most $\frac{k-4}{2}$-incomplete. In fact, there must be a particular W-critical extension R' that is $\frac{k-4}{2}$-incomplete or $\frac{k-5}{2}$-incomplete, as otherwise R is $\frac{k-6}{2}$-collapsible and then there exists a $\frac{k-4}{2}$-edge-addition in G by Proposition 5.2 and Lemma 5.3, which contradicts Lemma 5.7.

We choose such an i-incomplete W-critical extension R' for $i \in \left\{\frac{k-4}{2}, \frac{k-5}{2}\right\}$. Now Equation (7) becomes $k^2-4k-1+2\varepsilon-2\delta < \rho(W)$. This ε-potential does not match the conclusion of Theorem 1.7 so W must be a k-Ore graph by the minimality of G. As W is a k-Ore graph, Lemma 3.4 implies that $D \subseteq W$ is a diamond or emerald of W disjoint from X. Corollary implies that 5.6 D is an emerald of W and there must exist a vertex z in $V(G)-V(D)$ such that $xz \in E(G)$ for each $x \in V(D)$ with $\deg_G(x) = k-1$. However, R' is at most $\frac{k-4}{2}$-incomplete, so there are at most $\frac{k-4}{2}$ vertices of D that are not adjacent to z. The set of edges from these vertices to z is a $\frac{k-4}{2}$-edge-addition, which contradicts Lemma 5.7. ■

To talk about the different outcomes of a cloning operation, we introduce the following terminology.

Definition 13. A *gadget*, H°, is a graph obtained from a k-Ore graph H by deleting a vertex x of degree $k-1$ in a cluster of size at least 2. Note that the requirement of cluster size prevents x from being an overlap vertex of an Ore composition. A *gadget of* G is a subgraph of G that is a gadget.

Definition 14. A *key vertex* of a k-Ore graph H is a vertex x such that, whenever H is an Ore composition of two graphs H_1 and H_2 with overlap vertices $\{a,b\}$, $x \in V(H_1)-\{a,b\}$. That is, x is on the edge-side of the composition and is not an overlap vertex. A *key vertex* of a gadget is a vertex which is a key vertex of the corresponding k-Ore graph.

Corollary 6.3. *Suppose that G is a minimal counterexample to Theorem 1.7 and $xy \in E(G)$ such that (1) x is in a cluster C_x of size s, (2) $\deg_G(y) \le k-2+s$, and (3) if y is in a cluster C_y, then $C_y \neq C_x$ and $|C_y| = t \le s$. Then x is a key vertex of a gadget of G,* *or x is in a K_{k-3} subgraph of G. Moreover, the latter is only possible if $\deg_G(y) = k - 1$ and y is not in the K_{k-3} subgraph.*

Proof. By Lemma 6.2 there is a k-critical graph $H \subseteq G_{y \to x}$. If H is a k-Ore graph, then $H - \tilde{x}$ is a gadget of G. Suppose that H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. If $x \in V(H_2)$ or if $x \in \{a, b\}$, then ab is a 1-edge-addition in G, which contradicts Lemma 5.5. Because every vertex of K_k is trivially a key vertex, it follows that x is a key vertex of $H - \tilde{x}$.

If H is not a k-Ore graph, then $\deg_G(y) = k - 1$ by Lemma 6.2 and thus H and G have the same number of edges. However, H is smaller than G because $s \geq t$. Thus $\rho(H) < \rho(G)$ which is only possible if adding $\tilde{x}$ creates either a K_{k-2} or K_{k-1} subgraph of H that doesn't exist in G. In either case, x is in a K_{k-3} subgraph of G that does not contain y. ■

To aid with discharging, it is useful to classify the vertices of degree $k-1$ in a minimal counterexample to Theorem 1.7 into three distinct groups.

Definition 15. Let G be a minimal counterexample to Theorem 1.7 and suppose that $x \in V(G)$ is a vertex of degree $k - 1$. Let C_x be the cluster containing x; note that every vertex withing a given cluster is classified into the same group.

- If x is a key vertex of a gadget or is in a K_{k-3} subgraph, then we call x a *structure-vertex*.
- If x is not a structure-vertex and is adjacent to a vertex y which belongs to a distinct cluster C_y, then we call x a *near-vertex*. Note that Corollary 6.3 implies that y is necessarily a structure-vertex and that $|C_x| < |C_y|$.
- If x is not a structure-vertex and every neighbor of x with degree $k-1$ is in C_x, then we call x a *lone-vertex*. Note that $|C_x| \leq k - 4$, or x would be a structure-vertex.

Lemma 6.4. *Suppose that G is a minimal counterexample to Theorem 1.7 and that x is a structure-vertex in G. Then x cannot be adjacent to two near-vertices y and z with $C_y \neq C_z$.*

Proof. Let G be minimal counterexample to Theorem 1.7 and suppose that x is a structure-vertex with two near-vertex neighbors y and z such that $C_y \neq C_z$. If $yz \in E(G)$, then Corollary 6.3 implies that either y or z is a structure-vertex, which is a contradiction. Therefore we conclude that $yz \notin E(G)$ and consider $G_{x \to z}$. By Lemma 6.1 there is a k-critical subgraph $H \subseteq G_{x \to z}$, and H cannot include the vertex y. Therefore $|V(H)| < |V(G_{x \to z})| = |V(G)|$ and we know that H is smaller than G. This replaces the need for condition (3) of Lemma 6.2 and Corollary 6.3 and so it follows that z is a structure-vertex, which contradicts the fact that it is a near-vertex. ■

Lemma 6.5. *In a minimal counterexample G to Theorem 1.7, let x and y be adjacent vertices such that $\deg_G(x) = k-1$ and $N_G[x]$ is not a subset of $N_G[y]$. Then $\deg_G(y) \geq |N_G(x) \cap N_G(y)| + 1 + \frac{k-3}{2}$.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and let x and y be adjacent vertices such that $\deg_G(x) = k-1$ and $w \in N_G[x] - N_G[y]$. In any proper $(k-1)$-coloring ϕ of $G - x$, the vertices of $N_G(x)$ all receive distinct colors. Therefore, some vertex of $N_G[y] - N_G[x]$ must be in the same color class as w and adding the edge set $S = \{wu_i \mid u_i \in N_G[y] - N_G[x]\}$ to $G - x$ creates a k-critical subgraph. Using Lemma 5.7 we get $|S| \geq \frac{k-3}{2}$, and this gives the desired bound on $\deg_G(y)$. ■

Lemma 6.6. *Let G be a minimal counterexample to Theorem 1.7 and suppose that x is a key vertex in a gadget of G such that $\deg_G(x) = k-1$. Then x has at least $\frac{k-3}{2}$ neighbors of degree at least $\frac{3(k-3)}{2}$.*

Proof. Let G be a minimal counterexample to Theorem 1.7 and let x be a vertex of degree $k-1$ which is a key vertex of a gadget H° of G. Let H be the k-Ore graph where $H^\circ = H - w$. If H is an Ore composition of two graphs H_1 and H_2 with overlap vertices $\{a, b\}$, then $w \notin \{a, b\}$ and $\deg_H(w) = k-1$ by the definition of gadget. Further, we must have $w \in V(H_1)$ because otherwise $\{ab\}$ is a 1-edge-addition in G which contradicts Lemma 5.5. Therefore if H_2' is the split-side of the composition after separating the split vertex into a and b, then $H_2' \subseteq G$.

Proposition 3.1 gives a sequence of k-Ore graphs such that H can be viewed as a K_k graph with some edges replaced by suitable split k-Ore graphs. The same sequence of Ore compositions lets us view H° as a K_{k-1} graph H' with some edges replaced by the same split k-Ore graphs. Because each step in the sequence is the edge-side of the subsequent Ore composition, $V(H') \subseteq V(G)$. The key vertex x is not an overlap vertex for any Ore composition, so $xu \in E(G)$ for each $u \in V(H') - \{x\}$. Therefore x has one neighbor $z \in V(G) - V(H^\circ)$. We partition the vertices of H' into two sets $A := \{u \in V(H') \mid uz \in E(G)\}$ and $B := V(H') - A$. Note that in any proper $(k-1)$-coloring of H°, each vertex of H' gets a distinct color.

First, we show that $V(G) = V(H^\circ) \cup \{z\}$ is not possible. Suppose, for sake of contradiction that $V(G) = V(H^\circ) \cup \{z\}$. If $H = K_k$, then this implies that G is also K_k, which is a contradiction. Therefore H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$. Because $\rho(G) > k(k-3) - 2(k-1)$ by hypothesis and $\rho(H) \leq k(k-3) - 2\delta + \varepsilon$ by Theorem 1.8, it follows that $|E(G)| = |E(H)|$ and therefore $\deg_G(z) = k-1$. By Lemma 5.1, $\{a, b\}$ is not a cutset, so there exists some zv with $v \in V(H_2)$.

Let ϕ be a proper $(k-1)$-coloring of H°. If $\phi(v) \notin \{\phi(a), \phi(b)\}$, then it is possible to relabel the colors on split-side vertices only so that $\phi(v) = \phi(x)$. But this updated coloring would then extend to z, as two of z's neighbors share a color. Therefore, where H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{a, b\}$, any neighbor v of z with $v \in V(H_2)$ is colored the same as either a or b by any proper $(k-1)$-coloring of H°. This implies that $\{za, zb, ab\}$ is a 3-edge-addition which contradicts Lemma 5.7 because $k \geq 10$.

Therefore $V(H^\circ) \cup \{z\}$ is a proper subset of $V(G)$. But it follows from this that $\{zb \mid b \in B\}$ is a $|B|$-edge-addition in G. By Lemma 5.7, we have $|B| \geq \frac{k-3}{2}$. By Lemma 6.5, $\deg_G(b) \geq |N_G(x) \cap N_G(b)| + 1 + \frac{k-3}{2}$ for each $b \in B$. If b is adjacent to each vertex in $V(H')$, then $|N_G(x) \cap N_G(b)| = k-3$ and we get one more than the desired bound. For any $u \in V(H')$ that is not in $N_G(b)$, H is an Ore composition of two k-Ore graphs H_1 and H_2 with overlap vertices $\{u, b\}$. Let H_2' be the split side of the composition after separating the split vertex into u and b; note that $H_2' \subseteq G$. In any proper $(k-1)$-coloring of H_2', different colors are given to u and b and thus $\{uv \mid v \in N_{H_2'}(b)\}$ is a $|N_{H_2'}(b)|$-edge-addition in G. By Lemma 5.7, it follows that $|N_{H_2'}(b)| \geq \frac{k-3}{2}$. However, the vertices in $|N_{H_2'}(b)|$ may also include the $\frac{k-3}{2}$ vertices in $N_G(b) - N_G(x)$ counted by Lemma 6.5. Therefore, we conclude that $\deg_G(b) \geq \frac{3(k-3)}{2}$. ■

Lemma 6.7. *If x is in a K_{k-3} subgraph $D \subseteq G$, where G is a minimal counterexample to Theorem 1.7 and $\deg_G(x) = k-1$, then x has at least $\frac{k-9}{6}$ neighbors of degree at least $\frac{3(k-3)}{2} - 1$. Furthermore, if x has a neighbor $y \in V(G) - V(D)$ which is in a different cluster, then x has at least $\frac{k-7}{2}$ neighbors of degree at least $\frac{3(k-3)}{2} - 1$.*

Proof. Let G be a minimal counterexample to Theorem 1.7 such that x is a vertex of degree $k-1$ in a K_{k-3} subgraph D. Let z_1, z_2, z_3 be the three neighbors of x in $V(G) - V(D)$. We partition the vertices of D into two sets $A := \{u \in V(D) \mid uz_i \in E(G) \text{ for each } i \in \{1,2,3\}\}$, and $B := V(D) - A$. By Lemma 6.5, each $b \in B$ has degree at least $(k-5) + 1 + \frac{k-3}{2} = \frac{3(k-3)}{2} - 1$. It remains to show that B is a large enough set.

The edges $\{z_1z_2, z_1z_3, z_2z_3\}$ and bz_i for each pair $b \in B$, $i \in \{1,2,3\}$ form a $(3+3|B|)$-edge-addition in G, so it follows from Lemma 5.7 that $|B| \geq \frac{k-9}{6}$. Now suppose without loss of generality that $z_1 = y$ is a vertex of degree $k-1$ that is in a different cluster than x. Because there is at least one vertex in B, Lemma 6.5 implies that $\deg_G(z_1) \geq (|A|-1) + 1 + \frac{k-3}{2}$. But $\deg_G(z_1) = k-1$, so it follows that $|A| \leq \frac{k+1}{2}$. As $A \cup B = V(D)$, this implies that $|B| \geq \frac{k-7}{2}$. ■

7. Discharging

We start by analyzing the local structure of a minimal counterexample to Theorem 1.7. Then we complete the discharging argument in two stages; in the first stage we send charge along edges according to established rules, and in the second stage we average charge across the graph. We define a charge function $w : V(G) \to \mathbb{R}$ so that for all $v \in V(G)$

$$w(v) := (k-2)(k+1) + \varepsilon - \deg_G(v)(k-1).$$

Note that the total initial charge across G is $\rho(G) + \delta T(G)$, and that the charge of a vertex x with degree d is $w(x) = (k-d)(k-1) - 2 + \varepsilon$.

We now define the four sets we need to address in the second stage of discharging.

$$L := \{v \in V(G) \mid v \text{ is a lone-vertex in a cluster of size } 1\},$$
$$M := \{v \in V(G) \mid v \text{ is a lone-vertex in a cluster of size } 2\},$$
$$P := \{v \in V(G) \mid \deg_G(v) = k\},$$
$$Q := \{v \in V(G) \mid \deg_G(v) = k+1\}.$$

Let R be the set $V(G) - (L \cup M \cup P \cup Q)$ which contains the remaining vertices of G.

Discharging Rule #1 (R1): Every vertex of degree at least $k+2$ reserves charge of $-2+\varepsilon$ and sends the remaining charge equally to all neighbors.

Discharging Rule #2 (R2): Every structure-vertex sends total charge $-(k-1)$ spread equally among all neighbors that are near-vertices.

For each vertex v, define $w'(v)$ to be the charge after applying (R1) and (R2) to G. Note that a vertex of degree d which follows (R1) sends out charge $(\frac{k}{d}-1)(k-1)$ to each of its neighbors. Also note that if a structure-vertex x sends charge to a near-vertex y, then $|C_x| > |C_y|$.

Lemma 7.1. *Apply (R1) and (R2) to a minimal counterexample to Theorem 1.7 G with charge function w as above. For every vertex $v \in V(G) - (L \cup M \cup P \cup Q)$, the new charge $w'(v)$ is at most $-2+\varepsilon$.*

Proof. Let G be a minimal counterexample to Theorem 1.7 with charge function $w : V(G) \to \mathbb{R}$ as above, and apply (R1) and (R2). If v is a vertex with $\deg_G(v) \geq k+2$, then by (R1) it follows that $w'(v) \leq -2+\varepsilon$. The cases that we need to check are when v has degree $k-1$ and is either a structure-vertex, near-vertex, or lone-vertex in a cluster of size at least 3.

Case 1a. Suppose that v is a structure-vertex that is a key vertex of a gadget of G.

By Lemma 6.6, the vertex v has at least $\frac{k-3}{2}$ neighbors of degree at least $\frac{3(k-3)}{2}$; we will call these *high-degree* neighbors. For $k \geq 27$, high-degree neighbors have degree at least $\frac{4}{3}k$. Therefore v receives charge of $\frac{-1}{4}(k-1)$ or less from each high-degree neighbor by (R1). The vertex v possibly sends charge $-(k-1)$ by (R2) as well. Therefore it follows that

$$w'(v) \leq k-3+\varepsilon - \left(\frac{k-1}{4}\right)\left(\frac{k-3}{2}\right) + (k-1) = \frac{-1}{8}(k-1)(k-19) - 2 + \varepsilon.$$

Because $k \geq 19$, we have $w'(v) \leq -2+\varepsilon$ as desired.

Case 1b. Suppose that v is a structure-vertex that is in a K_{k-3} subgraph of G.

By Lemma 6.7, the vertex v has at least $\frac{k-9}{6}$ neighbors of degree at least $\frac{3(k-3)}{2} - 1$; we will call these *high-degree* neighbors. For $k \geq 33$, high-degree neighbors have degree at least $\frac{4}{3}k$. As long as v is not affected by (R2) we have

$$w'(v) \leq k - 3 + \varepsilon - \left(\frac{k-1}{4}\right)\left(\frac{k-9}{6}\right) = \frac{-1}{24}(k^2 - 34k + 33) - 2 + \varepsilon.$$

Because $k \geq 33$, we have $w'(v) \leq -2 + \varepsilon$ as desired.

If v is affected by (R2), then v has a neighbor outside of the K_{k-3} which is in a different cluster, and by Lemma 6.7 there are at least $\frac{k-7}{2}$ high-degree neighbors. In this case we have

$$w'(v) \leq k - 3 + \varepsilon - \left(\frac{k-1}{4}\right)\left(\frac{k-7}{2}\right) + (k-1) = \frac{-1}{8}(k^2 - 24k + 23) - 2 + \varepsilon.$$

Because $k \geq 23$, we have $w'(v) \leq -2 + \varepsilon$ as desired.

Case 2. Suppose that v is a near-vertex.

Let v be in a cluster C_v of size t and let u be an adjacent structure-vertex in a cluster C_u of size s. By (R2) each vertex of C_v, including v, receives a charge of $\frac{-(k-1)}{r}$ from each vertex of C_u. Because $s > r$, the final charge on v is

$$w'(v) \leq k - 3 + \varepsilon - \frac{s(k-1)}{r} < -2 + \varepsilon.$$

Case 3. Suppose that v is a lone-vertex in a cluster C_v of size r, where $r \geq 3$.

By definition of lone-vertex, v does not have any neighbors y in a cluster C_y with $C_y \neq C_v$. Let $y_1, y_2, \ldots y_{k-r}$ be the neighbors of v in $V(G) - C_v$. By Corollary 6.3 no y_i has degree less than $k-1+r$, as this would imply that v is a structure-vertex. Therefore, by (R1), each y_i sends charge at most $\left(\frac{k}{k-1+r} - 1\right)(k-1)$ to v. It follows that the upper bound on $w'(v)$ is

$$\hat{w}'(v) \leq k - 3 + \varepsilon + \left(\frac{1-r}{k-1+r}\right)(k-1)(k-r).$$

The second derivative of $\hat{w}'(v)$ with respect to r is positive for all $k > 1$, so we only need to check that $\hat{w}'(v) := -2 + \varepsilon$ for $r = 3$ and $r = k - 4$. For $r = 3$ we have

$$\hat{w}'(v) = k - 3 + \varepsilon + \frac{-2(k-1)(k-3)}{k+2} = -2 + \varepsilon + \frac{(k-1)(8-k)}{k+2}$$

and for $r = k - 4$ we have

$$\hat{w}'(v) = k - 3 + \varepsilon + \frac{(5-k)(k-1)4}{2k-5} = -2 + \varepsilon + \frac{(k-1)(15-2k)}{2k-5}.$$

Because $k \geq 8$, we get $w'(v) \leq -2 + \varepsilon$ as desired. ■

Lemma 7.1, specifically Case 1b, restricts our main result to $k \geq 33$. Although there is approximation in the proof of this case, using a computer algebra system one can check that the result only holds for $k \geq 33$; we paid no penalty in strength of argument by using simplified calculations. Now that we have verified the charge for vertices in R, we need to examine the charge on L, M, P, Q to calculate the total charge. This gives us a lower bound on the combined size of L and P.

Lemma 7.2. *In a minimal counterexample G to Theorem 1.7, let L be the set of lone-vertices in clusters of size 1 and let P be the set of vertices of degree k. Then $|L|+|P| > |V(G)|\left(1-\frac{\varepsilon}{2}\right)$.*

Proof. For each $x \in L \cup M$, every vertex in $N_G(x) \cap R$ has degree at least $k+2$, so a charge of $-\frac{2(k-1)}{k+2}$ or less is sent along each edge from R to $L \cup M$. Let $e(L \cup M, R)$ denote the number of such edges. The total charge on G is bounded by

$$\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} w'(v) \leq \varepsilon|V(G)| - 2|R| + (k-3)(|L|+|M|) - 2|P| \\ - (k+1)|Q| - \frac{2(k-1)}{k+2} e(L \cup M, R). \tag{8}$$

Note that no vertex in M is adjacent to any vertex in P by Corollary 6.3. Thus it follows that

$$e(L \cup M, R) = (k-1)|L| - e(L, P \cup Q) + (k-2)|M| - e(M, Q). \tag{9}$$

It is clear that $e(L, P \cup Q) \leq (k-1)|L|$, and Lemma 2.2 shows that $e(L, P \cup Q) \leq 2|L| + 2|P| + 3|Q|$ (calculations are simpler when we increase the contribution of the independent set L). Then it follows that

$$e(L, P \cup Q) \leq \frac{k-4}{2(k-1)}(k-1)|L| + \left(1 - \frac{k-4}{2(k-1)}\right)(2|L| + 2|P| + 3|Q|). \tag{10}$$

Using Lemma 2.2 we can also bound $e(M, Q) \leq |M| + 3|Q|$. With this, Equation (9), and Equation (10) we can rewrite Equation (8) as

$$\sum_{v \in V(G)} w(v) \leq \varepsilon|V(G)| - 2|R| + (k-3)(|L|+|M|) - 2|P| - (k+1)|Q| \\ - (k-3)|L| + 2|P| - \frac{2(k^2-4k+3)}{k+2}|M| + \frac{9k}{k+2}|Q| \\ = \varepsilon|V(G)| - 2|R| - \frac{k^2-7k+12}{k+2}|M| - \frac{k^2-6k-2}{k+2}|Q|. \tag{11}$$

As the coefficients of $|M|$ and $|Q|$ are both at most -2 for $k > 8$, we have

$$\varepsilon|V(G)| - 2(|V(G)| - |L| - |P|) \geq \sum_{v \in V(G)} w(v) = \rho(G) + \delta T(G) > 0.$$

From this, it follows that $|L| + |P| > |V(G)| \left(1 - \frac{\varepsilon}{2}\right)$. ■

We are now prepared to prove Theorem 1.7.

Proof of Theorem 1.7. We first get a bound on the set L. It is clear that $2|E(G)| \geq k|P| + (k-1)|L|$, so by Lemma 7.2 it follows that

$$2|E(G)| > k|V(G)| \left(1 - \frac{\varepsilon}{2}\right) - |L|. \tag{12}$$

By assumption $\rho(G) > 0$, so $2|E(G)| < \left(k + \frac{\varepsilon-2}{k-1}\right)|V(G)|$ by the definition of ε-potential. Combining this with Equation (12), we have

$$|L| > \frac{|V(G)|}{k-1}\left(2 - \varepsilon - \frac{\varepsilon(k^2-k)}{2}\right).$$

Recall that $\mathrm{mic}(G)$ is the maximum of $\sum_{v \in I} \deg_G(v)$ over all independent vertex subsets I, so $\mathrm{mic}(G) \geq (k-1)|L|$. Kierstead and Rabern (Theorem 2.4 in [5]) show that $2|E(G)| > (k-2)|V(G)| + \mathrm{mic}(G)$. Therefore we can improve Equation (12) to

$$2|E(G)| > |V(G)|\left((k-2) + 2 - \varepsilon - \frac{\varepsilon(k^2-k)}{2}\right).$$

Again, because $\rho(G) > 0$ the definition of ε-potential shows that

$$\left(k + \frac{\varepsilon-2}{k-1}\right)|V(G)| > |V(G)|\left((k-2) + 2 - \varepsilon - \frac{\varepsilon(k^2-k)}{2}\right)$$

and hence

$$\frac{\varepsilon-2}{k-1} > -\varepsilon - \frac{\varepsilon(k^2-k)}{2}.$$

This is equivalent to

$$\frac{4}{k^3 - 2k^2 + 3k} < \varepsilon,$$

which is a contradiction to our choice of ε. ■

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