On Chorded Cycles

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Erdös 101 Meeting

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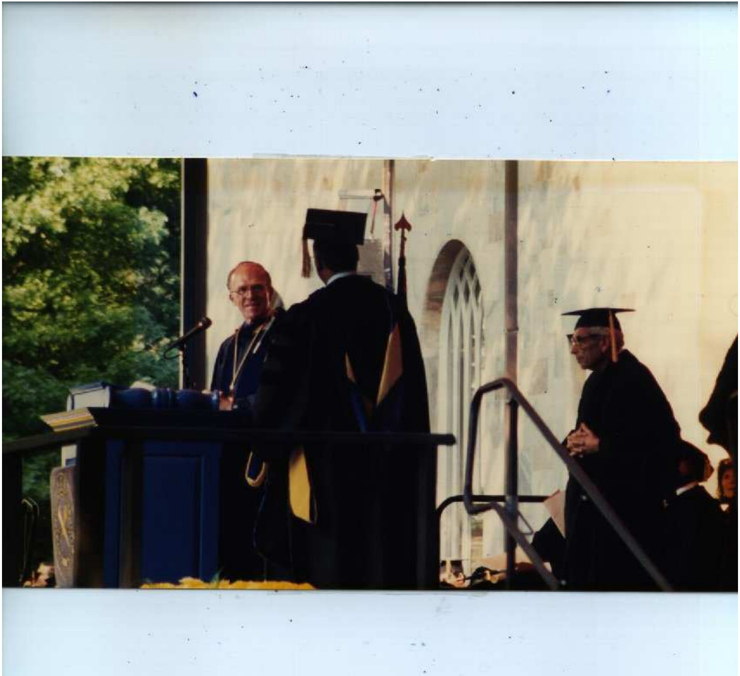

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Theorem (Corradi and Hajnal)

If $\delta(G) \geq 2k$ and $|G| \geq 3k$ then G contains k vertex disjoint cycles.

Over the years there have been many results that find conditions sufficient for cycles (often with various properties like containing a set of vertices, or a set of edges, etc.).

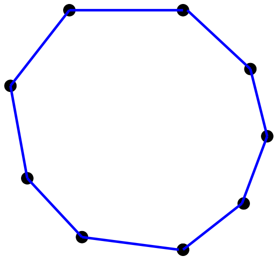
But the one property that was greatly ignored was the following:

An Old Question by Posa, 1960

Question

What conditions imply a graph contains a cycle with a chord?

Here a **chord** is an edge between two vertices on the cycle that is not an edge of the cycle.



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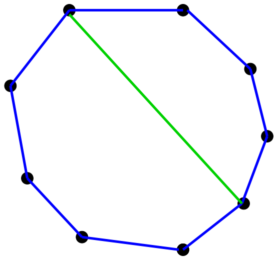

First answer by J. Czipzer 1963 - using min deg $\delta(G)$

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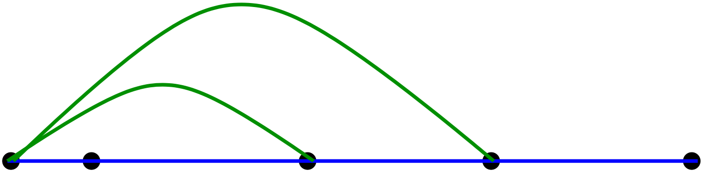
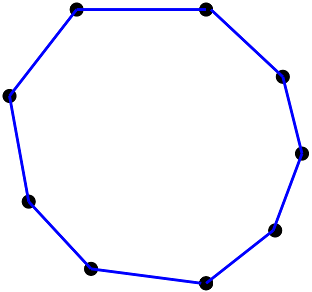


Sharpness Example

Minimum degree 2 is not enough! Simply take any cycle.



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- Other conditions for a chorded cycle.
- Some specified number of disjoint chorded cycles.
- Some specified number of disjoint doubly chorded cycles.
- Cycles with a designated minimum number of chords.

Theorem

If G is a graph on $n \geq 4k$ vertices with minimum degree $\delta(G) \geq 3k$, then G contains at least k vertex disjoint chorded cycles.

Some History (45 years later) - D. Finkel, 2008

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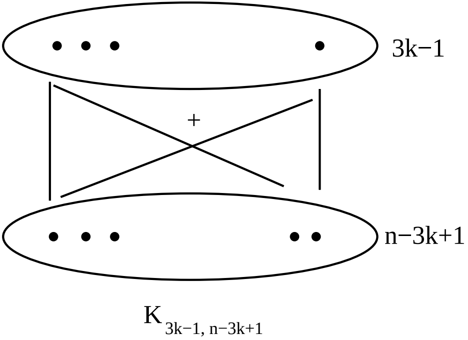
$\delta(G) \geq 2$ **implies cycle.**

Sharpness

Clearly, $n \geq 4k$ is needed as the cycles need at least 4 vertices each.

For $n \geq 6k$, the graph $K_{3k-1,n-3k+1}$ has $\delta = 3k - 1$ and no collection of k vertex disjoint chorded cycles, as chorded cycles here require 3 vertices from each partite set.



Bialostocki, Finkel and Gyarfas, 2008

Conjecture

Let r, s be nonnegative integers and G a graph with order at least $3r + 4s$ and minimum degree $\delta(G) \geq 2r + 3s$.
Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.

They proved this conjecture for $r = 0$, $s = 2$ and for $s = 1$ and every r.

Bialostocki, Finkel and Gyarfas, 2008

Theorem

Let G be a graph with order at least 8 and $\delta(G) \geq 6$, then G contains two vertex disjoint chorded cycles.

They also settled the extremal problem of the minimum number of edges in a graph on n vertices ensuring two vertex disjoint chorded cycles.

Chiba, Fujita and Gao, 2010

Settled the conjecture completely, actually proving more.

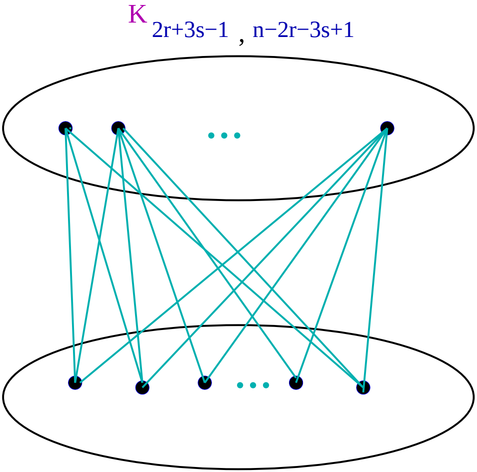
Theorem

Let r and s be integers with $r + s \geq 1$. Let G be a graph of order at least $3r + 4s$. If

$$\sigma_2(G) \geq 4r + 6s - 1,$$

then G contains a collection of $r + s$ vertex disjoint cycles, such that s of them are chorded.

Sharpness example



Y. Gao and L. Guojun -2012

Theorem

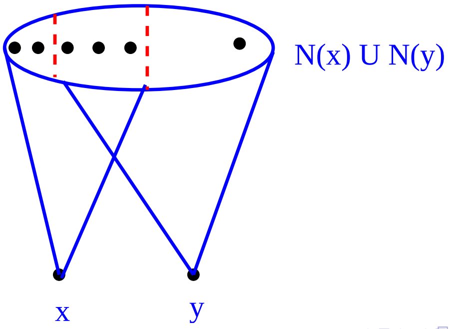
Let k be a positive integer and G a graph of order $n \geq 4k$ with $\sigma_2(G) \geq 6k-1$. Then G contains k vertex disjoint chorded cycles.

Theorem

If G is a graph on $n \geq 4k$ vertices such that for any pair of non-adjacent vertices x, y,

$$|N(x,y)| \geq 4k + 1,$$

then H contains at least k vertex disjoint chorded cycles.



Theorem

If G is a graph with at least $4k$ vertices and minimum degree at least $\lceil \frac{7k}{2} \rceil$, then G contains k vertex disjoint cycles, each with at least 2 chords.

Call such cycles *doubly chorded cycles (or DCC's).*

Theorem

If G is a graph on $n \geq 6k$ vertices with

$$\sigma_2(G) \geq 6k - 1,$$

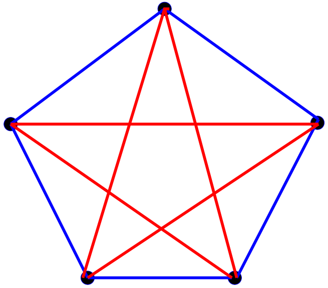
then G contains k vertex disjoint doubly chorded cycles.

Question

How many chords should we expect or hope to find?

Cycles with many chords?

A special case: Cliques



K_5: 4-regular but with 5 chords

In general:

Given a K_{k+1}: It is k-regular with

$$f(k) = \frac{(k-2)(k+1)}{2}$$

chords. We think of $f(k)$ chorded cycles as "loose" K_{k+1} cliques.

Note: There are no single chorded cliques.

Theorem

Ali, Staton - 1999

If $\delta(G) = k$, then G contains a

$$\left\lceil \frac{k(k-2)}{2} \right\rceil - \textit{ chorded cycle.}$$

Note: There are no single chorded cliques.

Theorem

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Corollary

If $\delta(G) \geq 3$, then G contains a doubly chorded cycle - that is, a loose K_4.

Theorem

If $\delta(G) = k$, then G contains an

$$f(k) = \frac{(k+1)(k-2)}{2} - \text{ chorded cycle.}$$

RG, P. Horn, C. Magnant: Our first step -

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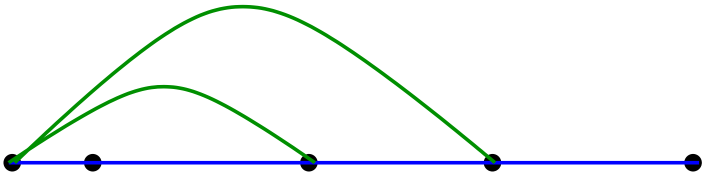

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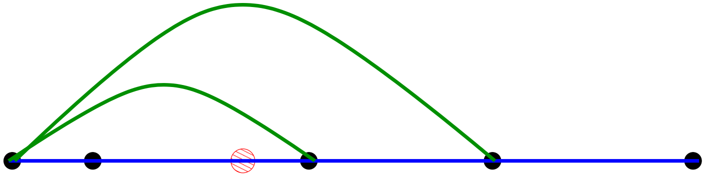

The story so far:

Lower End: $\delta(G) \geq k$ implies an $f(k) = \frac{(k-1)(k+2)}{2}-$chorded cycle.

Upper End:

Theorem

Hajnal and Szemerédi

If $\delta(G) \geq kt$, $|G| = (k+1)t$, then G can be covered by t vertex disjoint K_{k+1}'s.

RG, P. Horn and C. Magnant

Conjecture

If $\delta(G) \geq kt$, and $|G| \geq (k+1)t$ then G contains t

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Tight End: Hajnal-Szemerédi Theorem.
If $t = 1$, this is our first Theorem.
We show it is true for some classes of graphs, and for graphs with some extra "room".

Theorem

There exist k_0, t_0 such that if $\delta(G) \geq kt$, where $k \geq k_0$, $t \geq t_0$ and $n \geq n_0(k,t)$, then G contains t disjoint cycles with at least

$$f(k) = \frac{(k+1)(k-2)}{2}$$

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RG, P. Horn, C. Magnant 2012

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- Bounds for k_0 and t_0 show a tradeoff.
- Bounds for n_0 quite large.

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- minimum degree $\leq k-1$,
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- Problem: Not very useful!

Possibly More Useful:

Theorem

Let $d = \left\lceil \sqrt{\frac{k(k-1)}{2}} \right\rceil$.

- If G has average degree at least $2d$, then G contains a $f(k) = \frac{(k+1)(k-2)}{2}$-chorded cycle.
- There exist graphs with average degree $2d - o(1)$ with no $f(k)$-chorded cycle.

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- Sharpness: Bipartite graph $K_{d,n}$.
- $k = 2, 3, 4$: Trivial induction removing vertex of lowest degree if $< \delta$.
- $k \geq 5$ much tougher induction.

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- Can we control the order of the chorded cycles?
- Can we expand our chorded cycle system to span $V(G)$?

Question

Can we make many short chorded cycles?

with Chen, Hirohata, Ota and Song

Theorem

Let k be a natural number. Then there exists a positive integer n_k such that if G is a graph with $\delta(G) \geq 3k + 8$ and order at least n_k, then G contains k vertex disjoint chorded cycles of the same length.

Theorem

Let G be a multigraph of order n and minimum degree at least 5. Then G contains a chorded cycle of length at most $c_0 \log_2 n$, where $30 \leq c_0 \leq 300$ is a constant.

Specifing certain edges to be chords

Question

Can we make an independent set of k edges the chords of k vertex disjoint cycles?

work with M. Cream, R. Faudree, K. Hirohata

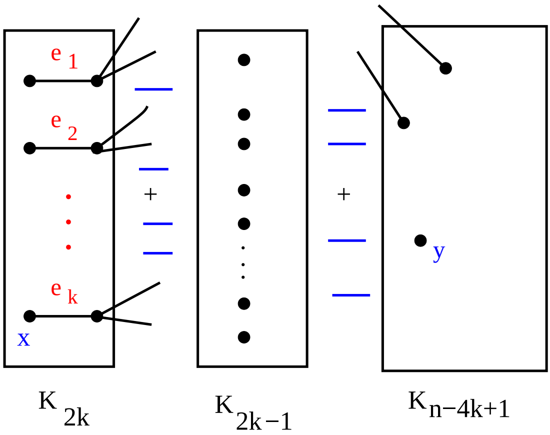
Question

Can we make an independent set of k edges chords of k vertex disjoint cycles?

Theorem

Let $k \geq 1$ be an integer and G be a graph of order $n \geq 14k$. If $\sigma_2(G) \geq n + 3k - 2$, then for any k independent edges $e_1, e_2, \ldots, e_k$ of G, the graph G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that e_i is a chord of C_i for all $1 \leq i \leq k$. Furthermore, $4 \leq |V(C_i)| \leq 5$ for each i.

Sharpness Example



Question- Placing vertices on chorded cycles

Question

When can we distribute k vertices on k disjoint chorded cycles?

Placing vertices on chorded cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 16k - 12$. If $\delta(G) \geq n/2$ then for any set of k vertices $\{v_1, v_2, \ldots, v_k\}$ there exists a collection of k vertex disjoint chorded cycles $\{C_1, \ldots, C_k\}$ such that $v_i \in V(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Placing Edges on Chorded Cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let G be a graph of order $n \geq 18k - 2$ and let $e_1, e_2, \ldots, e_k$ be a set of k independent edges in G. If

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k chorded cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

[with M. Cream, R. Faudree and K. Hirohata]

As a Corollary to the proof we obtain the fact the edges

$$e_1, e_2, \ldots, e_k$$

can be a mix of either chords or edges of the cycles (again one edge per cycle).

Further, we can show that the cycle system can also be extended to span $V(G)$.

Doubly Chorded Cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let G be a graph of order $n \geq 22k - 2$ and let $e_1, \ldots, e_k$ be k independent edges in G. Then if

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k vertex disjoint doubly chorded cycles $C_i, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Corollary

The above system can be extended to span $V(G)$.

Containing Linear Forests

Fact

Given independent path $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ with each $r_i \geq 2$ let $r = \sum r_i$. Then the number of interior vertices in this path system is $r - 2k$.

Theorem

Let $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ be a linear forest in a graph G of order $16k + r - 2$ with

$$\delta(G) \geq n/2 + r - 1 - k.$$

Then there exists a system of k chorded cycles $C_1, \ldots C_k$ such that the path P_{r_i} lies on the cycle C_i and $|V(C_i)| \leq r_i + 4$.