

When Rolling Two Dice

4	5	6
(1, 4)	(1, 5)	(1, 6)
(2, 4)	(2, 5)	(2, 6)
(3, 4)	(3, 5)	(3, 6)
(4, 4)	(4, 5)	(4, 6)
(5, 4)	(5, 5)	(5, 6)
(6, 4)	(6, 5)	(6, 6)

TABLE 1.2: The Sample Space of Sums When Rolling a Pair of Dice

die 1 → die 2 ↓	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

on Principle in action. This is
In fact, the rule is not limited

Principle: For k independent
the number of ways to perform

é's dice problems. In the one-
there are a total of $6^4 = 1296$
ie with six outcomes per roll.
It is easier to count those that
are only five outcomes per roll
. Thus, the outcomes with at
portion (remember Cardano!)

one 6 than there are ways to
ould mean you win more often

pair of dice rolled 24 times?
on Principle tells us that the
nes (see Table 1.2). Of these
the 36 outcomes of our task

are bad for us. Considering this fraction then, the proportion of outcomes with
no double 6s is

$$\left(\frac{35}{36}\right)^{24} = .5086 > \frac{1}{2}.$$

That is, more than half of the outcomes have no double 6 in 24 rolls of a pair
of dice. Thus, these are losing outcomes in game 2 and we should expect to
lose this game more often than win.

The Chevalier de Méré had his sad news. His reasoning had been flawed
and the truth could be seen by making a careful examination of the sample
spaces for the two games and then determining the proportion of favorable or
non-favorable outcomes in each space.

The reader should be aware that the form of the sample space is all up
to you. It should be based on the experiment at hand and the information
needed. We have seen one sample space for rolling a pair of dice, but the same
general experiment could have other sample spaces. For example, if we roll a
pair of dice and want to know the total sum rolled, then our sample space
could be as shown in Table 1.2.

Exercises:

1.2.1 Build a table for the sample space of sums when rolling a pair of 4-sided
dice.

1.2.2 Build the choice tree for the problem of flipping a fair coin three times.

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- 1.2.3 Build the choice tree for the problem of rolling one dice and then flipping a fair coin.
- 1.2.4 Use the multiplication principle to determine the order of the set of outcomes obtained by selecting one card from a standard deck of 52 cards, with 13 in each of 4 suits, and then flipping a coin.
- 1.2.5 Use the multiplication principle to determine the order of the set of possible outcomes obtained by selecting one of the five teams in the National League Eastern Division and then selecting one day of the week to watch that team play.
- 1.2.6 How big is the set of outcomes obtained by selecting one player from a 12-man basketball team and one player from a 53-man NFL team?
- 1.2.7 How large is the set of outcomes if you select one card from a standard deck and then select a second card? How does the count change if you return the first card to the deck before you select the second card?
- 1.2.8 In your own words, explain the multiplication principle.
- 1.2.9 How should The Chevalier de Méré expect to do if his original one-die game had only three rolls in order to roll a 6? What if he had five rolls in order to roll a 6?
- 1.2.10 What is the size of the sample space for a game where the outcomes are: you roll a die and then flip a coin? Create a choice tree to verify your answer.
- 1.2.11 How might you represent the outcomes for the game in the previous problem?
- 1.2.12 What is the sample space for the experiment of drawing one card from a standard deck of 52 cards? (Note: a standard deck will always have 52 cards composed of 4 suits, each with 13 cards, Ace, two, . . . , ten, Jack, Queen, and King.)
- 1.2.13 What is the sample space for drawing one card if our deck consists only of all the hearts and all the Aces from a standard deck?
- 1.2.14 How large is the sample space for the experiment of rolling one die, then drawing one card from a standard deck?
- 1.2.15 Create a choice tree to show the possible outcomes for flipping a coin three times. Verify the number of outcomes by using the Multiplication Principle.
- 1.2.16 Create a choice tree to show the possible outcomes from the experiment of flipping a coin, then rolling one 4-sided die, then flipping a second coin. Use the Multiplication Principle to verify the number of outcomes.

Multiplication Principle we know this set has $6^3 = 216$ outcomes. Thus, we certainly do not wish to write them all out as we did for two dice in Table 1.2. But how can we think about this sample space? Probably the easiest way is to think of the three dice as being of different colors, say red, white, and blue. Then we can record the outcomes of rolling the three dice based on their colors. If when we roll, the red die is 6, the white die is 4, and the blue die is 3, then we could write this outcome as $(6, 4, 3)$. That is, this *ordered triple* records that particular outcome perfectly. The set of all 216 of these ordered triples

(red die, white die, blue die)

comprises our sample space. We have now rediscovered what Galileo had noticed 400 years ago!

So ordered pairs were used to record the outcomes from flipping a coin twice. Now ordered triples help us record the outcomes when we roll three dice. In general, an ordered k -tuple (x_1, x_2, \dots, x_k) can be used to show the results of $k \geq 2$ experiments.

Thus, we have seen the use of ordered k -tuples to be a practical way of recording outcomes of multi-events. Flipping a coin two times can be recorded as ordered pairs

(first flip, second flip)

and, in general, rolling k dice can be recorded as ordered k -tuples. This form of modeling our sample space will continue to be useful throughout the text.

Exercises:

- 1.3.1 What are the possible triples representing outcomes if you flip a fair coin three times?
- 1.3.2 For the experiment in the previous problem, what is the probability of obtaining (H, H, T) ? What is the probability of obtaining two heads and one tail?
- 1.3.3 Describe the set of possible outcomes for the teams playing in the superbowl.
- 1.3.4 What is the set of possible outcomes for your grade in this class?
- 1.3.5 A bowl contains 2 white balls, 3 silver balls, 1 yellow ball, and 2 green balls.
 1. What is the set of possible outcomes for drawing one ball at random from this bowl?
 2. What is the set of possible outcomes for drawing two balls at random from this bowl (without replacement)?

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(placement)?

1.3.6 An experiment involves rolling a pair fair dice. Describe the set of all possible outcomes for the following observations:

1. you observe a sum of 6.
2. you observe an even sum.
3. you observe a sum less than 4.
4. you observe a sum that is a perfect square.

1.3.7 You roll two fair dice. Describe the set of all possible outcomes of the following observations:

1. you observe a sum less than 4.
2. you observe two odd numbers.
3. you observe one value at least 5 and the other at most 2.
4. you observe a sum strictly between 4 and 7.

1.3.8 Create a table for the probabilities of each possible sum when rolling a pair of dice.

1.3.9 What is the probability of drawing a spade from a standard deck? What is the probability of drawing an Ace from a standard deck?

1.3.10 What is the probability of drawing a King or a Queen from a standard deck?

1.3.11 Flip a coin and draw a card from a standard deck. Now determine the size of the sample space for this experiment. What is the probability of any single element in this sample space?

1.3.12 What is the probability of drawing a heart that is not an Ace from a standard deck?

1.3.13 What is the probability of drawing a diamond that is not a Jack, Queen, or King from a standard deck?

1.3.14 Suppose you play the following game: you roll one die and you flip one coin. Winning payments are as described in the table below. Determine the probabilities for each of the winning payments.

	1	2,3,4	5,6
H	3	0	2
T	2	1	1

1.3.15 What is the probability of first rolling a 6 with one die, then drawing a club from a standard deck?

1.3.16 You first flip a coin, then roll a die, then draw a card from a standard deck.

1. What is the probability of obtaining a head? Of rolling a 3? of drawing a club?
2. What is the probability of obtaining a head, a 3, and a club on one try of this three part game?
3. What is the probability of obtaining a tail, an even number, and an Ace?
4. What is the probability of obtaining a tail, a number at least as large as 3, and a Jack, Queen, or King?

1.4 The Laws That Govern Us

In this section we will develop some of the fundamental laws of probability. The reader well versed in probability theory might skip this section. Those just learning probability will wish to go carefully through this section.

In the last section we took for granted several important rules. We state them here for completeness.

Rule 1.4.1. *The sum of the probabilities of all events in the sample space equals one.*

This is fundamental. A probability is a proportion of the time that an outcome should occur. We cannot then exceed one in summing all these proportions. If we fail to reach one, then some outcome has been overlooked or some probability of an outcome(s) is incorrectly given.

Rule 1.4.2. *If event E has probability p , the probability event E does not happen is $1 - p$.*

This is a very useful rule and a consequence of the first rule. Sometimes it will be easier for us to compute the probability of the failure of the event E , which we denote as \overline{E} (the standard set complement of the set E), rather than computing the probability of success for event E . Thus, restating this rule we have

$$P(E) = 1 - P(\overline{E}),$$

which makes finding $P(E)$ easy when finding $P(\overline{E})$ is also easy.

Exercises:

1.4.1 You roll one fair die.

1. What is the probability you roll an even number?
2. What is the probability you roll a 4, 5, or 6?
3. What is the probability you roll a 4 or a 5?

1.4.2 You roll a pair of fair dice.

1. What is the probability the sum is 5, 6, or 7?
2. What is the probability the sum is at most 4?
3. What is the probability the sum is odd?
4. What is the probability the sum is odd if the first die shows an even number?
5. What is the probability the sum is even if the first die shows an even number?
6. What is the probability the first die has a value less than 5 and the second die has a value less than 4?

1.4.3 Given a standard deck of 52 cards:

1. find the probability you are dealt a 5 on the first card.
2. find the probability your first two cards are a Jack and a 10.
3. find the probability you are dealt a heart or a Jack, Queen, or King on the first card.
4. find the probability you are dealt a red Ace or red King.

1.4.4 What is the probability of drawing any 7 or the Ace of Clubs when drawing one card from a standard deck?

1.4.5 What is the probability of drawing an 8 or a heart in drawing one card from a standard deck?

1.4.6 You flip a coin five times. How large is the sample space of outcomes of the five flips? What is the probability of any one of these equally likely outcomes?

1.4.7 What is the probability of rolling a total of 4 when rolling a pair of dice?

1.4.8 What is the probability of rolling doubles when rolling a pair of dice?

1.4.9 What is the probability of rolling a total of 8 or doubles when rolling a pair of dice?

1.4.10 What is the probability of rolling three 6s when rolling three dice? What about exactly two 6s when rolling three dice? What about exactly one 6 when rolling three dice? What about no 6s when rolling three dice?

1.4.11 Former football players have sustained a serious knee injury with probability .3. If two former football players are selected at random, what is the probability they both had suffered a serious knee injury?

1.4.12 Former NFL players are known to have sustained a concussion while playing with probability .15. If two former players are selected at random, what is the probability at least one of them suffered a concussion?

1.4.13 Suppose the probability any MLB player stays with the same team for 10 years or more is $1/500$. If two players, A and B, start with the same team on the same day:

1. what is the probability A will play for this team less than 10 years?
2. what is the probability both A and B will play with this team less than 10 years?
3. what is the probability A and B will both play with this team at least 10 years?
4. what is the probability one of A or B will play with this team for at least 10 years?

1.4.14 Five cards are drawn from a standard deck of 52 cards. Given the two events: E_1 : all cards are hearts, and E_2 : the five cards are an Ace, King, Queen, Jack, and 10.

1. Define the event $E_1 \cap E_2$.
2. Define the event $E_1 \cup E_2$.
3. Find the probability of E_1 .
4. Find the probability of E_2 .
5. Find the probability of $E_1 \cap E_2$.
6. Find the probability of $E_1 \cup E_2$.

1.4.15 Two fair dice are tossed. Let A be the event the first die shows an odd number and let B be the event that the second die shows a number greater than 3.

1. Find the probability of event A .
2. Find the probability of event B .
3. Find the probability of event $A \cap B$.
4. Find the probability of event $A \cup B$.

1.4.16 A basketball player makes 60 percent of his free throw attempts. What is the probability he makes exactly two of his next three shots?

1.4.2

1.4.17 Flip a coin and toss one die. Suppose the event A is the occurrence of a head and an even number and event B is the occurrence of a head and a number greater than 2.

1. Find $P(A)$ and $P(B)$.
2. Find $P(A \cap B)$ and $P(A \cup B)$.
3. Define the conditional events $A|B$ and $B|A$.
4. Find $P(A|B)$ and $P(B|A)$.

1.4.18 A bowl contains 100 coins. One has a head on each side, while the other 99 have a head on one side and a tail on the other side. One coin is picked at random and flipped two times.

1. What is the probability we get two heads?
2. What is the probability we get two tails?

1.5

1.4.19 Upon rolling two dice, what is the probability the sum is 8 given that both die show an even number? What is the probability the sum is 12 given both die show an even number? What is the probability the sum is 10 given both die show an odd number?

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1.4.20 On a game show you are offered several bowls from which to choose tickets. Bowl I contains three tickets to an NFL game and seven tickets to an MLB game. Bowl II contains six NFL tickets and four MLB tickets. You will select one ticket from one bowl.

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1. Compute the probability you selected an NFL ticket.
2. Given that you selected an NFL ticket, what is the conditional probability that it was drawn from bowl II?

1.4.21 Police records show that arrests at a certain soccer stadium during a game occur with probability .35. The probability that an arrest and conviction will occur is .14. What is the probability that the person arrested will be convicted?

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1.4.22* In the NFL, it is known that the probability is .82 that a first round draft choice who attends all of training camp will have a productive first season and that the corresponding probability for those that do not attend all of camp is .53. If 60% of the first round picks attend all of training camp, what is the probability that a first round pick who had a productive first season will have attended all of camp?

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1.4.23* A hockey team noticed that 37% of its goals are scored by centers, 30% by left wings, 20% by right wings, and 13% by defensemen. If 20% of the goals scored by centers are power play goals, while 15% scored by wings and 10% scored by defensemen are power play goals:

1. what is the probability that a power play goal was scored by a defenseman?
2. what is the probability it was scored by a left winger?
3. what is the probability it was scored by a right winger?
4. what is the probability it was scored by a center?

1.5 Poker Hands versus Batting Orders

It should be apparent that to compute probabilities for large events or in large sample spaces, we need to be able to count the number of outcomes that fit our event without having to list them all. This is not always an easy task. In fact, there is an old joke that says, "There are three kinds of mathematicians, those that can count and those that can't."

In order to begin counting in more complicated or larger situations, it will be helpful to recognize the difference between a poker hand and a batting order! In a batting order we have a sequence of items (the hitters) whose position is fixed; that is, the order of their appearance (once and only once in the batting order) is what matters here. But with a poker hand, it does not matter in what order you receive the cards, all that really matters is the *set of cards* (your hand) you have at the end of dealing. This difference between order mattering and not mattering plays a fundamental role in counting. Let us begin with an example where order matters.

Example 1.5.1. Suppose your baseball team (of exactly nine players) has a game and you must decide the batting order. How many possible batting orders are there?

Solution: This is an application of the multiplication principle to the "tasks" of choosing hitters. There are clearly nine choices for the first hitter, then only eight choices for the second hitter, seven choices for the third hitter, and so forth. Thus, it should be clear that there are

$$9 \times 8 \times 7 \times \dots \times 3 \times 2 \times 1 = 362,880$$

possible batting orders. \square

Each of the different batting orders is called a *permutation*. That is, formally a permutation is an ordered arrangement of the objects from some set.