Graphs and Combinatorics (2020) 36:1927–1945
https://doi.org/10.1007/s00373-020-02227-z

ORIGINAL PAPER

On Degree Sum Conditions and Vertex-Disjoint Chorded Cycles

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Received: 20 December 2019 / Revised: 15 July 2020 / Published online: 21 September 2020

Abstract
In this paper, we consider a general degree sum condition sufficient to imply the existence of k vertex-disjoint chorded cycles in a graph G. Let $\sigma_t(G)$ be the minimum degree sum of t independent vertices of G. We prove that if G is a graph of sufficiently large order and $\sigma_t(G) \geq 3kt - t + 1$ with $k \geq 1$, then G contains k vertex-disjoint chorded cycles. We also show that the degree sum condition on $\sigma_t(G)$ is sharp. To do this, we also investigate graphs without chorded cycles.

Keywords Vertex-disjoint chorded cycles · Minimum degree sum · Degree sequence · Biconnected components · Blocks

1 Introduction

The study of cycles in graphs is a rich and an important area. One question of particular interest is to find conditions that guarantee the existence of k vertex-disjoint cycles. Let G be a graph. Corrádi and Hajnal [2] first considered a minimum degree condition to imply a graph must contain k vertex-disjoint cycles, proving that if $|G| \geq 3k$ and the minimum degree $\delta(G) \geq 2k$, then G contains k vertex-disjoint cycles. For an integer $t \geq 1$, let

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$$\sigma_t(G) = \min \left\{ \sum_{v \in X} \deg_G(v) \ : \ \begin{array}{c} X \text{ is an independent vertex} \\ \text{set of } G \text{ with } |X| = t. \end{array} \right\},$$

and $\sigma_t(G) = \infty$ when the independence number is $t-1$ or less. Enomoto [3] and Wang [11] independently extended the Corrádi and Hajnal result, requiring a weaker condition on the minimum degree sum of any two non-adjacent vertices. They proved that if $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$, then G contains k vertex-disjoint cycles. In 2006, Fujita et al. [5] proved that if $|G| \geq 3k+2$ and $\sigma_3(G) \geq 6k-2$, then G contains k vertex-disjoint cycles, and in [7], this result was extended to $\sigma_4(G) \geq 8k-3$. Recently, Ma and Yan [10] proved a conjecture from [7] by showing that if G has sufficiently large order and $\sigma_t(G) \geq 2kt - t + 1$, then G contains k vertex-disjoint cycles.

A *chord* of a cycle is an edge between two non-consecutive vertices of the cycle. An extension of the study of vertex-disjoint cycles is that of vertex-disjoint chorded cycles. We say a cycle is *chorded* if it contains at least one chord. In 2008, Finkel [4] proved the following result on the existence of k vertex-disjoint chorded cycles which can be viewed as an extension of the Corrádi and Hajnal result.

Theorem 1 (Finkel [4]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint chorded cycles.*

In 2010, Chiba et al. [1] extended the above result by using the $\sigma_2(G)$ condition.

Theorem 2 (Chiba, Fujita, Gao, Li [1]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\sigma_2(G) \geq 6k-1$, then G contains k vertex-disjoint chorded cycles.*

Recently, Theorem 2 was extended as follows.

Theorem 3 [8] *Let $k \geq 1$ be an integer. If G is a graph of order at least $8k+5$ and $\sigma_3(G) \geq 9k-2$, then G contains k vertex-disjoint chorded cycles.*

The last result was further extended to $\sigma_4(G)$ in a submitted paper by Gould, Hirohata, and Keller.

Theorem 4 *Let $k \geq 1$ be an integer. If G is a graph of order at least $11k+7$ and $\sigma_4(G) \geq 12k-3$, then G contains k vertex-disjoint chorded cycles.*

In this paper, we prove the following result in Sect. 4.

Theorem 5 *For $k \geq 1$ and $t \geq 1$, if G is a graph of order $n \geq (10t-1)(k-1) + 12t + 13$ and $\sigma_t(G) \geq 3kt - t + 1$, then G contains k vertex-disjoint chorded cycles. Further, this degree condition is sharp.*

Remark To see the sharpness of the degree condition of Theorem 5, for n sufficiently large order, consider the complete bipartite graph $B = K_{3k-1, n-3k+1}$. Then $\sigma_t(B) = t(3k-1)$. Further, it is not possible to construct k vertex-disjoint chorded cycles in B, as any chorded cycle must use three vertices from the partite set of order $3k-1$.

All graphs considered here are simple, undirected, and finite. For terminology and notation not defined here, see [6]. Let G be a graph, H a subgraph of G, and $S \subseteq V(G)$. For $u \in V(G)$, we denote the set of neighbors of u in G by $N_G(u)$, $\deg_G(u) = |N_G(u)|$, $N_H(u) = N_G(u) \cap V(H)$, and $\deg_H(u) = |N_H(u)|$. Also we denote $\deg_H(S) = \sum_{u \in S} \deg_H(u)$. If $H = G$, then $\deg_G(S) = \deg_H(S)$. The subgraph of G induced by S is denoted by $\langle S \rangle$. Let $G - S = \langle V(G) - S \rangle$ and $G - H = \langle V(G) - V(H) \rangle$. If $S = \{u\}$, then we write $G - u$ for $G - S$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. If G is one vertex, that is, $V(G) = \{u\}$, then we simply write u instead of G. For two disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2. Let Q be a path or a cycle with a given orientation and $x \in V(Q)$. Then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of Q from x to y (including x and y) in the given direction. The reverse sequence of $Q[x, y]$ is denoted by $Q^-[y, x]$. We also write $Q(x, y] = Q[x^+, y]$, $Q[x, y) = Q[x, y^-]$ and $Q(x, y) = Q[x^+, y^-]$. If Q is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t(, x_1)$, then we assume that an orientation of Q is given from x_1 to x_t. If P is a path connecting x and y, then we denote the path P with an orientation from x to y as $P[x, y]$. The reverse sequence of $P[x, y]$ is denoted by $P^-[y, x]$. For an integer $r \geq 1$ and two vertex-disjoint subgraphs A, B of G, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from A to B such that $\deg_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. In this paper, since it is sufficient to consider the case of equality in the above inequality, when we write $(d_1, d_2, \ldots, d_r)$, we assume $\deg_B(v_i) = d_i$ for each $1 \leq i \leq r$. For two disjoint $X, Y \subseteq V(G)$, $E(X, Y)$ denotes the set of edges of G connecting a vertex in X and a vertex in Y. A cycle of length ℓ is called a *ℓ-cycle*. For a graph G, comp(G) is the number of components of G. Let R be a graph. If G has no induced subgraph isomorphic to R, then G is called *R-free*.

2 Graphs with No Chorded Cycles

In this section, we examine some useful properties of graphs that contain no chorded cycles. Our ultimate goal is to show they contain large independent sets of small degree sum. This will be important in our proof later.

Lemma 1 *Let T be a tree of order $n \geq 2$. Then the following statements hold.*

(i) *T has at least $n/2 + 1$ vertices of degree at most 2.*
(ii) *T contains an independent set I of order at least $n/4$ with each vertex of I having degree at most 2 in T.*

Proof Let $\{v_1, \ldots, v_b\}$ be the set of branch vertices in T. Let ℓ be the number of leaves in T and s be the number of stem vertices. Clearly $\ell + s + b = n$. Since T has $n - 1$ edges, the degree sum of T is

$$2(n-1) = \ell + 2s + \sum_{i=1}^{b} \deg_T(v_i) \geq \ell + 2(n - \ell - b) + 3b,$$

which implies $\ell \geq b + 2$. Consequently,

$$\ell + s \geq (b+2) + s = (b+s) + 2 = (n - \ell) + 2$$
$$2\ell + s \geq n + 2$$
$$\ell + \frac{s}{2} \geq \frac{n}{2} + 1.$$

If L is the set of all leaves and stems in T, then

$$|L| = \ell + s \geq \ell + \frac{s}{2} \geq \frac{n}{2} + 1.$$

Thus (i) holds.

Since T is bipartite, one of the partite sets contains at least half the vertices of L. Thus T contains an independent subset $I \subset L$ with $|I| \geq |L|/2 \geq n/4$, and (ii) holds. □

Definitions A *biconnected graph* is a non-separable graph. Note that any two vertices (two edges) of a biconnected graph lie on a common cycle. A *non-chorded* graph is a graph not containing any chorded cycles. A *leaf* is a vertex of degree 1. A *stem* is a vertex of degree 2. A *branch* is a vertex of degree at least 3.

Lemma 2 *If H is a non-chorded graph of order n, then H contains an independent set I of order at least $n/12$ with each vertex of I having degree at most 2 in H.*

Before proving Lemma 2, we state and prove some helpful propositions.

Proposition 1 *Every non-chorded biconnected graph H of order at least four is triangle-free.*

Proof Suppose H contains a triangle on vertices a, b, c. Since H is connected, without loss of generality, we can say a has some neighbor $d \in V(H) - \{b, c\}$. Since H is biconnected, edges ab and ad must lie on a common cycle in H. Let C be such a cycle. If C contains edge bc, then ac is a chord on the cycle, a contradiction. If C does not contain bc, then $\langle C \cup c \rangle$ contains a cycle with chord ab, a contradiction. □

Proposition 2 *Let $k \geq 1$ be an integer. If H is a non-chorded biconnected graph of order at least four, then E(H) can be decomposed into*

- *a cycle $C = F_0$, and*
- *if C is not a hamiltonian cycle in H, then a sequence of paths $P_1, \ldots, P_k$ (each with at least two edges) where the endpoints of P_i are a_i, b_i $(a_i \neq b_i)$,*

such that there exists a sequence of subgraphs $F_1, \ldots, F_k$ of H, where for all $1 \leq i \leq k$,

(i) $F_i = P_i \cup F_{i-1}$,
(ii) $V(P_i) \cap V(F_{i-1}) = \{a_i, b_i\}$,
(iii) F_i *is a non-chorded biconnected graph, and*
(iv) $F_k = H$ (see Fig. 1).

Proof Let C be a cycle in H. Note that H is triangle-free by Proposition 1, and in particular, C is not a triangle. Let $F_0 = C$ and let $E_1 = E(H) \setminus E(F_0)$. If C is a hamiltonian cycle in H, then since H is non-chorded, $E_1 = \emptyset$. For each $i \geq 1$, if $E_i \neq \emptyset$, do the following: Select any $f \in E(F_{i-1})$ and any $e_i \in E_i$. Since H is biconnected, there exists a cycle C_i in H containing f and e_i. Let P_i be a path in C_i containing e_i so that the endpoints of P_i are in $V(F_{i-1})$. Note that $|E(P_i)| \geq 2$. Call these endpoints a_i, b_i, and assume that $V(P_i) \cap V(F_{i-1}) = \{a_i, b_i\}$. Let $F_i = P_i \cup F_{i-1}$. Since H is non-chorded biconnected, the graph F_i is also non-chorded biconnected. Let $E_{i+1} = E_i \setminus E(P_i)$. Let $k+1$ be the minimum index so that E_{k+1} is empty. Then $F_k = H$. □

Proposition 3 *Let $k \geq 1$ be an integer. Let $C = F_0$ be any cycle of order at least four, let $P_1, \ldots, P_k$ be a sequence of paths (each with at least two edges) such that for each $1 \leq i \leq k$, P_i is a path from a_i to b_i ($a_i \neq b_i$), and let $F_1, \ldots, F_k$ be a sequence of graphs such that for each $1 \leq i \leq k$,*

(i) $F_i = P_i \cup F_{i-1}$,
(ii) $V(P_i) \cap V(F_{i-1}) = \{a_i, b_i\}$, *and*
(iii) F_i *is a non-chorded biconnected graph.*
Then for each $1 \leq i \leq k$, there exists some vertex $v \in P_i(a_i, b_i)$ such that $\deg_{F_k}(v) = 2$. Further, there exist distinct vertices $x, x' \in V(C) \setminus \bigcup_{i=1}^{k} V(P_i)$ such that $\deg_{F_k}(x) = \deg_{F_k}(x') = 2$.

Proof Suppose for a contradiction that for some $1 \leq \ell \leq k$, $\deg_{F_k}(v) \geq 3$ for all $v \in P_\ell(a_\ell, b_\ell)$. Let $P_\ell : v_0 = a_\ell, v_1, \ldots, v_{t-1}, v_t = b_\ell$, and let $F = F_{\ell-1} \setminus \{v_0, v_t\}$. Note

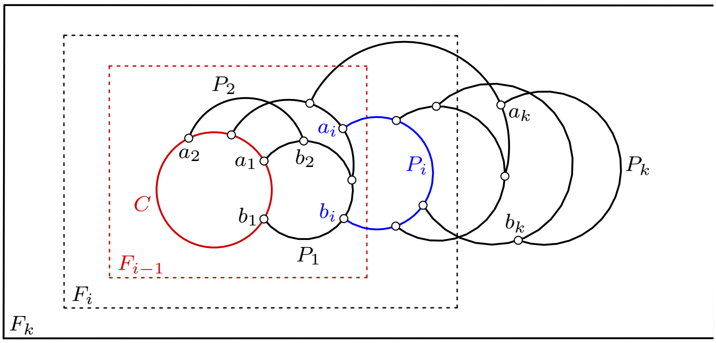


Fig. 1 The graph H

that since F_k is non-chorded biconnected graph of order at least four, F_k is triangle-free by Proposition 1.

Claim 1 *For each* $1 \leq i \leq t-2$, *there exists a path* S_i *in* F_k *from* v_i *to* v_j *for some* $i+2 \leq j \leq t$ *such that* $S_i(v_i, v_j) \cap V(P_\ell) = \emptyset$ *and* $V(S_i) \cap V(F) = \emptyset$.

Proof We prove Claim 1 by induction. Since $\deg_{F_k}(v) \geq 3$ for all $v \in P_\ell(a_\ell, b_\ell)$, there exists a neighbor u_i of v_i with $u_i \notin \{v_{i-1}, v_{i+1}\}$ for each $1 \leq i \leq t-2$. Since F_k is biconnected, there exists a path S_i in F_k starting with $v_i, u_i, \ldots$, terminating at v_j with $i \neq j$ such that $S_i(v_i, v_j) \cap V(P_\ell) = \emptyset$.

First we prove the case where $i = 1$. Suppose $V(S_1) \cap V(F) \neq \emptyset$. Then there exists a vertex $w \in V(S_1) \cap V(F)$ such that $S_1(v_1, w) \cap V(F) = \emptyset$. Since $F_{\ell-1}$ is biconnected, there exists a cycle C_1 in $F_{\ell-1}$ containing v_0 and w. We assume that an orientation of C_1 is given from v_0 to w clockwise. Suppose $v_t \in V(C_1)$, so $v_t \in C_1(v_0, w)$ or $v_t \in C_1(w, v_0)$. Without loss of generality, we may assume that $v_t \in C_1(v_0, w)$. Then

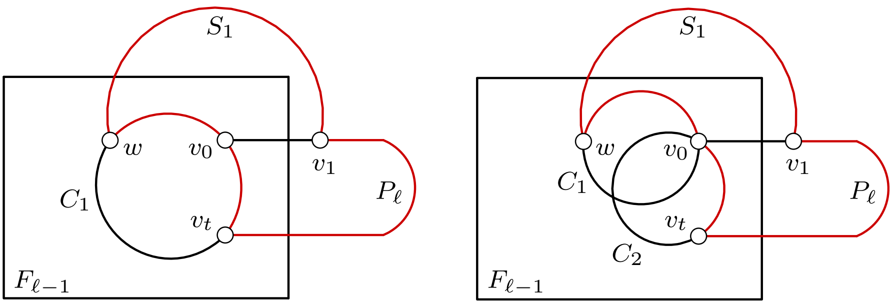
$$P_\ell[v_1, v_t], C_1^-[v_t, v_0], C_1^-[v_0, w], S_1^-[w, v_1]$$

is a cycle with chord v_0v_1, a contradiction (see Fig. 2a). Thus $v_t \notin C_1(v_0, w)$. Similarly $v_t \notin C_1(w, v_0)$, hence $v_t \notin V(C_1)$. Since $F_{\ell-1}$ is biconnected, there exists a cycle C_2 in $F_{\ell-1}$ containing v_0 and v_t. We assume that an orientation of C_2 is given from v_0 to v_t clockwise. Without loss of generality, we may assume that $w \notin C_2^-(v_t, v_0)$. If $C_2^-(v_t, v_0) \cap V(C_1) = \emptyset$, then

$$P_\ell[v_1, v_t], C_2^-[v_t, v_0], C_1^-[v_0, w], S_1^-[w, v_1]$$

is a cycle with chord v_0v_1, a contradiction (see Fig. 2b). Thus we may assume that $C_2^-(v_t, v_0) \cap V(C_1) \neq \emptyset$.

Let z be a vertex such that $z \in C_2^-(v_t, v_0) \cap V(C_1)$ and $C_2^-(v_t, z) \cap V(C_1) = \emptyset$. By assumption, $z \neq w$. If $z \in C_1(v_0, w)$, then



(a) $v_t \in C_1(v_0, w)$ **(b)** $C_2^-(v_t, v_0) \cap V(C_1) = \emptyset$

Fig. 2 The construction of chorded cycles

$$P_\ell[v_1, v_t], C_2^-[v_t, z], C_1^-[z, v_0], C_1^-[v_0, w], S_1^-[w, v_1]$$

is a cycle with chord v_0v_1, a contradiction. Otherwise, $z \in C_1^-(v_0, w)$, and similarly

$$P_\ell[v_1, v_t], C_2^-[v_t, z], C_1[z, v_0], C_1[v_0, w], S_1^-[w, v_1]$$

is a cycle with chord v_0v_1, a contradiction. Thus $V(S_1) \cap V(F) = \emptyset$. Next suppose $j \in \{0, 2\}$, that is, $v_j \in \{v_0, v_2\}$. If $j = 0$, then

$$P_\ell[v_1, v_t], C_2[v_t, v_0], S_1^-[v_0, v_1]$$

is a cycle with chord v_0v_1, a contradiction. If $j = 2$, then similarly, we can find a cycle with chord v_1v_2, a contradiction.

For induction, assume that Claim 1 is true for $i - 1$. Thus there exists a path S_{i-1} in F_k from v_{i-1} to $v_{j'}$ for some $i + 1 \leq j' \leq t$ satisfying the conditions of Claim 1. Suppose that every path S_i starting at vertex v_i, proceeding to u_i, and continuing in that direction passes through some vertex $x \in V(F) \cup S_{i-1}(v_{i-1}, v_{j'})$ before reaching any v_j with $i \neq j$. Then select a vertex x such that $S_i(v_i, x) \cap (V(F) \cup S_{i-1}(v_{i-1}, v_{j'})) = \emptyset$. First suppose $x \in V(F)$. Since $F_{\ell-1}$ is connected, there exists a path Q_1 in $F_{\ell-1}$ from x to v_0. Then

$$P_\ell[v_0, v_{i-1}], S_{i-1}[v_{i-1}, v_{j'}], P_\ell^-[v_{j'}, v_i], S_i[v_i, x], Q_1[x, v_0]$$

is a cycle with chord $v_{i-1}v_i$, a contradiction. Next suppose $x \in S_{i-1}(v_{i-1}, v_{j'})$. Since $F_{\ell-1}$ is connected, there exists a path Q_2 in $F_{\ell-1}$ from v_t to v_0. Then

$$P_\ell[v_0, v_{i-1}], S_{i-1}[v_{i-1}, x], S_i^-[x, v_i], P_\ell[v_i, v_t], Q_2[v_t, v_0]$$

is a cycle with chord $v_{i-1}v_i$, a contradiction. Thus S_i is a path from v_i to v_j not containing any vertex in $V(F) \cup S_{i-1}(v_{i-1}, v_{j'})$. If $j \geq i + 2$, then Claim 1 holds. Thus we may assume that $j \leq i + 1$. Suppose $j \leq i - 2$. Then

$$P_\ell[v_j, v_{i-1}], S_{i-1}[v_{i-1}, v_{j'}], P_\ell^-[v_{j'}, v_i], S_i[v_i, v_j]$$

is a cycle with chord $v_{i-1}v_i$, a contradiction. If $j = i - 1$, then using the above path Q_2,

$$P_\ell[v_0, v_{i-1}], S_i^-[v_{i-1}, v_i], P_\ell[v_i, v_t], Q_2[v_t, v_0]$$

is a cycle with chord $v_{i-1}v_i$, a contradiction. If $j = i + 1$, then similarly, we can find a cycle with chord v_iv_{i+1}, a contradiction. Thus for each $1 \leq i \leq t - 2$, there exists a path S_i in F_k from v_i to v_j for some $i + 2 \leq j \leq t$ satisfying the conditions of Claim 1. □

By Claim 1, there exists a path S_{t-2} from v_{t-2} to v_t such that $S_{t-2}(v_{t-2}, v_t) \cap V(P_\ell) = \emptyset$ and $V(S_{t-2}) \cap V(F) = \emptyset$. Since $\deg_{F_k}(v_{t-1}) \geq 3$ by our assumption, there exists a neighbor u_{t-1} of v_{t-1} with $u_{t-1} \notin \{v_{t-2}, v_t\}$. Since F_k is biconnected, there exists a path S_{t-1} in F_k starting with $v_{t-1}, u_{t-1}, \ldots$, terminating at v_j with $j \neq t - 1$ such that $S_{t-1}(v_{t-1}, v_j) \cap V(P_\ell) = \emptyset$. Since $F_{\ell-1}$ is biconnected, there exists a cycle C_1 containing v_0 and v_t. We assume that an orientation of C_1 is given from v_0 to v_t

clockwise. Suppose that every path S_{t-1} starting at v_{t-1} passes through some vertex $x \in V(F) \cup S_{t-2}(v_{t-2}, v_t)$ before reaching v_j with $j \neq t-1$. Then we take a vertex x such that $S_{t-1}(v_{t-1}, x) \cap (V(F) \cup S_{t-2}(v_{t-2}, v_t)) = \emptyset$. First suppose $x \in V(F)$. Since $F_{\ell-1}$ is biconnected, two vertices v_0 and x must lie on a common cycle C_2 in $F_{\ell-1}$. We assume that an orientation of C_2 is given from v_0 to x clockwise. Then we may assume that $v_t \notin C_2(x, v_0)$. Thus

$$P_\ell[v_0, v_{t-2}], S_{t-2}[v_{t-2}, v_t], P_\ell^-[v_t, v_{t-1}], S_{t-1}[v_{t-1}, x], C_2[x, v_0]$$

is a cycle with chord $v_{t-2}v_{t-1}$, a contradiction. Next suppose $x \in S_{t-2}(v_{t-2}, v_t)$. Then

$$P_\ell[v_0, v_{t-2}], S_{t-2}[v_{t-2}, x], S_{t-1}^-[x, v_{t-1}], P_\ell[v_{t-1}, v_t], C_1[v_t, v_0]$$

is a cycle with chord $v_{t-2}v_{t-1}$, a contradiction. Thus S_{t-1} is a path from v_{t-1} to v_j not containing any vertex in $V(F) \cup S_{t-2}(v_{t-2}, v_t)$. If $j \leq t-2$, then

$$P_\ell[v_j, v_{t-2}], S_{t-2}[v_{t-2}, v_t], P_\ell^-[v_t, v_{t-1}], S_{t-1}[v_{t-1}, v_j]$$

is a cycle with chord $v_{t-2}v_{t-1}$, a contradiction. If $j = t$, then

$$P_\ell[v_0, v_{t-1}], S_{t-1}[v_{t-1}, v_t], C_1[v_t, v_0]$$

is a cycle with chord $v_{t-1}v_t$, a contradiction. Thus, for each $1 \leq i \leq k$, there exists some vertex $v \in P_i(a_i, b_i)$ such that $\deg_{F_k}(v) = 2$.

Next consider $F_1 = P_1 \cup C$. We assume that an orientation of C is given from a_1 to b_1 clockwise. Then $C[a_1, b_1], P_1^-[b_1, a_1]$ is a cycle in F_k. By the above result, there exists some vertex $x \in C(b_1, a_1)$ with $\deg_{F_k}(x) = 2$. Similarly, since $P_1[a_1, b_1], C[b_1, a_1]$ is a cycle in F_k, there exists some vertex $x' \in C(a_1, b_1)$ with $\deg_{F_k}(x') = 2$. This completes the proof of Proposition 3. □

Proposition 4 *Every non-chorded biconnected graph H of order n has at least $(n-2)/3 + 2$ stem vertices.*

Proof Let C and $P_1, \ldots, P_k$ be a cycle and paths satisfying the conclusions of Proposition 2. Then by Proposition 3, there exist at least $k+2$ stem vertices in H. Also, since H is biconnected, every vertex in H is either a stem vertex or a branch vertex. Now consider the endpoints of P_i for each $1 \leq i \leq k$. By Proposition 2, there exist at most $2k$ branch vertices in H. Thus there exist at least $n - 2k$ stem vertices in H. Consequently, the number of stem vertices in H is at least $\max\{k+2, n-2k\}$, which is always at least $(n-2)/3 + 2$. □

Definition A *biconnected component* in a graph is a maximal biconnected subgraph. In this paper, we do not consider a single edge to be a biconnected component, and we handle these edges separately. Every cycle in a graph is contained in exactly one biconnected component. The following intuitive proposition is shown in [9].

Proposition 5 (Harary, Prins [9]) *If B_1, B_2 are distinct biconnected components in a graph, then $E(B_1) \cap E(B_2) = \emptyset$.*

Proposition 6 *Let $k \geq 1$ be an integer, and let H be a non-chorded connected graph containing k biconnected components. Then $E(H)$ can be decomposed into*

- *a sequence of non-chorded biconnected components $B_1, \ldots, B_k$, and*
- *a sequence of edge-disjoint paths $P_2, \ldots, P_\ell$ (some of which might be just a single vertex) with $\ell \geq k$, where the endpoints of P_i are a_i, b_i for each $2 \leq i \leq \ell$,*

so that there exists a sequence of induced subgraphs $F_1, F_2, \ldots, F_\ell$ of H with the following properties:

(i) $F_1 = B_1$,
(ii) *for each* $2 \leq i \leq k$, $F_i = F_{i-1} \cup P_i \cup B_i$, $V(P_i) \cap V(F_{i-1}) = \{a_i\}$, $V(P_i) \cap V(B_i) = \{b_i\}$, *and* $V(F_{i-1}) \cap V(B_i) = \emptyset$ *unless* $a_i = b_i$, *in which case* $V(F_{i-1}) \cap V(B_i) = \{a_i\}$,
(iii) *for each* $k+1 \leq i \leq \ell$, $F_i = F_{i-1} \cup P_i$, $V(P_i) \cap V(F_{i-1}) = \{a_i\}$, $\deg_H(b_i) = 1$, $|P_i| \geq 2$, *and*
(iv) $F_\ell = H$.

Proof Since H is non-chorded, every biconnected component in H must be non-chorded. Choose any biconnected component in H to be $F_1 = B_1$ (satisfying (i)). We claim that $|V(B) \cap V(F_{i-1})| \leq 1$ for any biconnected component B in $H \backslash E(F_{i-1})$ and for each $2 \leq i \leq k$. For some $2 \leq i \leq k$, suppose that there exists a biconnected component B in $H \backslash E(F_{i-1})$ with $|V(B) \cap V(F_{i-1})| \geq 2$. Then for some $u, v \in V(B) \cap V(F_{i-1})$, there exists a path Q_1 from u to v in F_{i-1} and a path Q_2 from u to v in B such that $Q_1 \cup Q_2$ forms a cycle Q. This cycle Q is in H. Thus Q is contained in some biconnected component B'. Since Q_1 is in F_{i-1}, it is edge-disjoint from B, Q is not in B and $B' \neq B$. But B and B' share some edge of Q_2, contradicting Proposition 5. Thus the claim holds.

First suppose that there exists a biconnected component B in $H \backslash E(F_{i-1})$ with $V(B) \cap V(F_{i-1}) = \{v\}$ for some vertex v. In this case, let $B_i = B$, $P_i = v$, and $F_i = F_{i-1} \cup P_i \cup B_i$, with $a_i = b_i = v$. Next suppose that all biconnected components in $H \backslash E(F_{i-1})$ are vertex-disjoint from F_{i-1}. Let B_i be a biconnected component in $H \backslash E(F_{i-1})$ such that a path from B_i to F_{i-1} in H is edge-disjoint from every other biconnected component in $H \backslash E(F_{i-1})$, and let this path be P_i. Since H is connected, such a B_i, P_i exist. Let $F_i = F_{i-1} \cup P_i \cup B_i$, $V(P_i) \cap V(F_{i-1}) = \{a_i\}$, and $V(P_i) \cap V(B_i) = \{b_i\}$. Thus (ii) is satisfied.

Clearly F_k is a connected graph containing all the cycles in H, and $H \backslash E(F_k)$ is a forest. Then there exists no path P in $H \backslash E(F_k)$ with both endpoints in $V(F_k)$, otherwise $F_k \cup P$ would contain a cycle not in F_k. If $E(H) \backslash E(F_{i-1}) \neq \emptyset$, then do the following: Select some edge $e \in E(H) \backslash E(F_{i-1})$ that is incident to a leaf vertex v in H. Let P_i be a path from $v = b_i$ to $V(F_{i-1})$ with $V(P_i) \cap V(F_{i-1}) = \{a_i\}$. Let $F_i = F_{i-1} \cup P_i$. Since P_i contains edge e, $|P_i| \geq 2$, and since $v = b_i$ is a leaf in H, $\deg_H(b_i) = 1$, satisfying (iii).

Since H is finite, there exists some $\ell \geq k$ for which $E(H) \backslash E(F_\ell) = \emptyset$, satisfying (iv). □

Now we finally prove Lemma 2.

Proof of Lemma 2 If H is acyclic, then applying Lemma 1 (ii) to each connected component of H gives the result. Thus we may assume that H has at least one cycle. Hence H contains a biconnected component. Let $B_1, \ldots, B_k$ and $P_2, \ldots, P_\ell$ be a decomposition of $E(H)$ into biconnected components and paths as described by the conclusion of Proposition 6 with the corresponding subgraphs $F_1, \ldots, F_\ell$ in H. For each B_i, $1 \leq i \leq k$, let $L_i = \{v \in V(B_i) : \deg_{B_i}(v) \leq 2\}$. By Proposition 4, each L_i has order at least $(|B_i| - 2)/3 + 2$. Let $S = \{v \in V(H) : \deg_H(v) \leq 2\}$. We will show that $|S| \geq |H|/6$. First, let $S_i = \{v \in V(F_i) : \deg_{F_i}(v) \leq 2\}$ for each $1 \leq i \leq k$, and we claim the following.

Claim 1 *For each* $1 \leq i \leq k$, $|S_i| \geq |F_i|/5 + 2$.

Proof First suppose $i = 1$. Then recall $F_1 = B_1$. If $|B_1| \geq 5$, then $|S_1| = |L_1| \geq (|B_1| - 2)/3 + 2 \geq |F_1|/5 + 2$. If $|B_1| \leq 4$, then B_1 is a 3-cycle or a 4-cycle, since these are the only biconnected components on at most 4 vertices. Then clearly $|S_1| \geq |F_1|/5 + 2$.

Next suppose $2 \leq i \leq k$. Then recall $F_i = F_{i-1} \cup P_i \cup B_i$, and assume by inductive assumption that F_{i-1} contains a set S_{i-1} of vertices of degree at most 2, where $|S_{i-1}| \geq |F_{i-1}|/5 + 2$. We have the following two cases.

Case 1 For some $2 \leq i \leq k$, $|P_i| = 1$.

Then $a_i = b_i$. By Proposition 6 (ii), $V(F_{i-1}) \cap V(B_i) = \{a_i\}$. Thus $|F_i| = |F_{i-1}| + |B_i| - 1$. While a_i may have degree 2 in each of F_{i-1}, B_i separately, it has degree greater than 2 in F_i. Thus

$$\begin{aligned} |S_i| &\geq (|S_{i-1}| - |\{a_i\}|) + (|L_i| - |\{a_i\}|) \\ &\geq \left(\frac{|F_{i-1}|}{5} + 2\right) + \left(\frac{|B_i| - 2}{3} + 2\right) - 2 \\ &= \frac{|F_{i-1}| + |B_i| - 1}{5} + \frac{2|B_i| + 23}{15} \\ &= \frac{|F_i|}{5} + \frac{2|B_i| + 23}{15}. \end{aligned} \tag{1}$$

If $|B_i| \geq 4$, then, by (1), we have $|S_i| \geq |F_i|/5 + 2$. Thus we may assume that $|B_i| \leq 3$. Then B_i is a 3-cycle and $|L_i| = 3$, in which case the inequality is easily shown.

Case 2 For some $2 \leq i \leq k$, $|P_i| \geq 2$.

Then $a_i \neq b_i$. By Proposition 6 (ii), $V(F_{i-1}) \cap V(B_i) = \emptyset$. Thus

$$\begin{aligned} |F_i| &= |F_{i-1}| + |B_i| + |P_i| - |\{a_i, b_i\}| \\ &= |F_{i-1}| + |B_i| + |P_i| - 2. \end{aligned}$$

Note that $\deg_{P_i}(v) \leq 2$ for each $1 \leq i \leq \ell$ and every vertex $v \in V(P_i)$. While a_i, b_i may have degree 2 in each of F_{i-1}, B_i or P_i separately, they have degree greater than 2 in F_i. Thus

$$\begin{aligned}|S_i| &\geq (|S_{i-1}| - |\{a_i\}|) + (|L_i| - |\{b_i\}|) + (|P_i| - |\{a_i, b_i\}|)\\ &\geq \left(\frac{|F_{i-1}|}{5} + 2\right) + \left(\frac{|B_i| - 2}{3} + 2\right) + |P_i| - 4\\ &= \frac{|F_{i-1}| + |B_i| + |P_i| - 2}{5} + \frac{2|B_i| + 12|P_i| - 4}{15}\\ &= \frac{|F_i|}{5} + \frac{2|B_i| + 12|P_i| - 4}{15}.\end{aligned} \tag{2}$$

Note that $|P_i| \geq 2$. If $|B_i| \geq 5$, then, by (2), we have $|S_i| \geq |F_i|/5 + 2$. Thus we may assume that $|B_i| \leq 4$. Then B_i is a 3-cycle or a 4-cycle, and $|L_i| = 3$ or 4. In either case, the inequality is again easily shown. □

In particular, Claim 1 shows

$$|S_k| \geq |F_k|/5 + 2. \tag{3}$$

Let $t = |S_k \cap \bigcup_{i=k+1}^{\ell} a_i|$. Enumerate the components $T_1, T_2, \ldots, T_w$ of $\bigcup_{i=k+1}^{\ell} \langle V(P_i) \rangle$, and note that $w \geq t$. Clearly

$$t \leq |S_k|. \tag{4}$$

Claim 2 *We have* $|S| \geq |H|/6$.

Proof Each component T_i, $1 \leq i \leq w$, is a tree, so by Lemma 1 (i), it has at least $|T_i|/2 + 1$ vertices of degree at most 2. Each component contains exactly one vertex $v \in V(F_k)$, while the rest are in $H - F_k$, and this one vertex v may have degree at least 2 in F_k, so the number of vertices of degree at most 2 in $H - F_k$ is

$$\begin{aligned}|S \cap (H - F_k)| &\geq \sum_{i=1}^{w} \frac{|T_i|}{2} = \sum_{i=1}^{w} \left(\frac{|T_i| - 1}{2} + \frac{1}{2}\right) = \frac{|H| - |F_k|}{2} + \frac{w}{2}\\ &\geq \frac{|H| - |F_k| + t}{2}.\end{aligned}$$

Also $|S \cap F_k| = |S_k| - t$. Then

$$\begin{aligned}|S| &= |S \cap F_k| + |S \cap (H - F_k)|\\ &\geq |S_k| - t + \frac{|H| - |F_k| + t}{2} = \frac{|H| - |F_k| - t}{2} + |S_k|.\end{aligned} \tag{5}$$

Combining (3), (4) and (5) gives

$$|S| \geq \frac{|H| - |F_k| + |S_k|}{2} \geq \frac{|H|}{2} - \frac{2|F_k|}{5} + 1.$$

Since $|S| \geq |S_k|$, by (3),

$$|S| \geq \frac{|F_k|}{5} + 2.$$

Thus $|S| \geq \max\{|H|/2 - 2|F_k|/5 + 1, |F_k|/5 + 2\}$, which is at least $|H|/6$ for all values of $|F_k|$. □

We claim that $\langle S \rangle$ is a forest or H is a cycle. Suppose $\langle S \rangle$ is not a forest. Then $\langle S \rangle$ contains a cycle C. If $H = C$, then the claim holds. Thus $H \neq C$, that is, $V(H) \setminus V(C) \neq \emptyset$. Note that $\deg_H(v) \leq 2$ for each $v \in S$. Since H is connected by the assumption, we get a contradiction. Thus the claim holds. If $\langle S \rangle$ is a forest, then it is bipartite. Since $|S| \geq |H|/6$ by Claim 2, there exists an independent subset $I \subseteq S$ of order at least $(|H|/6)/2 = n/12$. If H is a cycle, then clearly Lemma 2 also is true. This completes the proof of Lemma 2. □

3 Other Lemmas

In this section, we state several known lemmas that will be used in the proof of our main result. Note that a *minimal* set of r vertex-disjoint cycles $C_1, \ldots, C_r$ is a set with $|\bigcup_{i=1}^{r} C_i|$ as small as possible.

Lemma 3 [8] *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. If $|C_i| \geq 7$ for some $1 \leq i \leq r$, then C_i has at most two chords. Furthermore, if C_i has two chords, then these chords must be crossing.*

Lemma 4 [8] *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. Then $\deg_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G) - \bigcup_{i=1}^{r} V(C_i)$. Furthermore, for some $C \in \mathscr{C}$ and some $x \in V(G) - \bigcup_{i=1}^{r} V(C_i)$, if $\deg_C(x) = 4$, then $|C| = 4$, and if $\deg_C(x) = 3$, then $|C| \leq 6$.*

Lemma 5 [8] *Suppose there exist at least five edges connecting two vertex-disjoint paths P_1 and P_2 with $|P_1 \cup P_2| \geq 7$. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$ not containing at least one vertex of $\langle P_1 \cup P_2 \rangle$.*

4 Proof of Theorem 5

Suppose Theorem 5 does not hold. We first consider the case where $k = 1$. Then $n \geq 12t + 13$ and $\sigma_t(G) \geq 2t + 1$. Noting $\lceil n/12 \rceil \geq t + 2$, by Lemma 2, G contains an independent set I of order t with each vertex of I having degree at most 2 in G. Then $\deg_G(I) \leq 2t$, a contradiction. Thus we assume $k \geq 2$. Let G be an edge-maximal counter-example. If G is complete, then G contains k vertex-disjoint chorded cycles. Thus we may assume G is not complete. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$, the graph obtained from G by adding the edge xy. By the edge-maximality of G, G' is not a counter-example. Thus G' contains k vertex-disjoint chorded cycles $C_1, \ldots, C_k$. Without loss of generality, we may assume $xy \notin \bigcup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ vertex-disjoint chorded cycles. Over all sets of $k-1$ vertex-disjoint chorded cycles, choose $C_1, \ldots, C_{k-1}$, where $\mathscr{C} = \bigcup_{i=1}^{k-1} C_i$ and $H = G - \mathscr{C}$, such that:

(A1) $|\mathscr{C}|$ is as small as possible,
(A2) subject to (A1), $\text{comp}(H)$ is as small as possible, and
(A3) subject to (A1) and (A2), the number of K_4's in $\mathscr{C}$ is as large as possible.

We may also assume H does not contain a chorded cycle, otherwise, G contains k vertex-disjoint chorded cycles, a contradiction. Theorem 5 holds by Theorems 1-4 for all $t \leq 4$. Thus we also assume $t \geq 5$.

Claim 1 *H has order at least* $12t + 13$.

Proof Suppose this claim fails to hold, that is, suppose $|H| \leq 12t + 12$. First we prove the following subclaim.

Subclaim 1 For each $1 \leq i \leq k-1$, $|C_i| \leq 10t - 1$.

Proof Suppose Subclaim 1 fails to hold, that is, $|C_i| \geq 10t$ for some $1 \leq i \leq k-1$. Without loss of generality, let $|C_1| \geq |C_2| \geq \cdots \geq |C_{k-1}|$. In fact, let $|C_1| = st + r \geq 10t \geq 50$, with $s \geq 10$ and $0 \leq r \leq t-1$.

Subclaim 1.1. *For $s \geq 10$, the cycle C_1 contains s vertex-disjoint sets $X_1, \ldots, X_s$ each with t independent vertices such that* $\deg_{C_1}(\bigcup_{i=1}^{s} X_i) \leq 2st + 4$.

Proof For any st vertices of C_1, their degree sum in C_1 is at most $2st + 4$, since by Lemma 3, C_1 has at most two chords. Thus, it only remains to show that C_1 contains s vertex-disjoint sets of t independent vertices each. Recall $|C_1| = st + r \geq 10t$. Start anywhere on C_1 and label the first st vertices of C_1 with labels 1 through s in order, starting over again with 1 after using label s. If $r \geq 1$, then label the remaining r vertices of C_1 with the labels $s+1, \ldots, s+r$. The labeling above yields s vertex-disjoint sets of t vertices each, where all the vertices labeled with 1 are one set, all the vertices labeled with 2 are another set, and so on. Given this labeling, any vertex in C_1 has a different label than the vertex that precedes it on C_1 and the vertex that succeeds it on C_1. Let C_0 be the cycle obtained from C_1 by removing all chords. Then the vertices in each of the sets are independent in C_0. Thus, the only way vertices in the same set are not independent in C_1 is if the endpoints of a chord of C_1 were given the same label. Note any vertex labeled i is distance at least $s \geq 10$ in C_0 from any other vertex labeled i. Thus, if a vertex and the neighbor preceding it on C_0 or the neighbor succeeding it on C_0 have their labels exchanged, then the vertices in each of the classes are independent in C_0.

Case 1 No chord of C_1 has endpoints with the same label.

Then there exist s vertex-disjoint sets of t independent vertices each in C_1.

Case 2 Exactly one chord of C_1 has endpoints with the same label.

Recall C_1 contains at most two chords, and if C_1 contains two chords, then these chords must be crossing. Since $|C_1| \geq 50$, even if C_1 contains two chords, each chord has an endpoint such that one of the endpoint's neighbors in C_1 is not an endpoint of the other chord. Choose such an endpoint of the chord whose endpoints were assigned the same label, and exchange the label of this vertex for its non-endpoint neighbor. The vertices in each of the resulting classes are still independent in C_1, and now no chord of C_1 has endpoints with the same label. Thus there exist s vertex-disjoint sets of t independent vertices each in C_1.

Case 3 Two chords of C_1 each have endpoints with the same label.

In this case, note two chords are crossing. Suppose an endpoint of one chord of C_1 is adjacent to an endpoint of the other chord on C_1. Now exchange the labels of these adjacent endpoints. Then the vertices in each of the resulting classes are still independent in C_1, and now no chord of C_1 has endpoints with the same label. Thus there exist s vertex-disjoint sets of t independent vertices each in C_1.

Next suppose no endpoint of one chord of C_1 is adjacent to an endpoint of the other chord on C_1. Let x_1x_2, y_1y_2 be the two distinct chords of C_1. Since the two chords are crossing, without loss of generality, we may assume x_1, y_1, x_2, y_2 are in that order on C_1, and the label of x_1 is 1. Then the label of x_1^+ is 2. Now we exchange the labels of x_1 for x_1^+, that is, the label of x_1 is 2 and the label of x_1^+ is 1. Next we exchange the labels of y_2 for y_2^-. Note $y_2 \neq x_1^-$ by our assumption that no endpoint of one chord of C_1 is adjacent to an endpoint of the other chord on C_1. Thus, the vertices in each of the resulting classes are independent in C_1, and no chord of C_1 has endpoints with the same label. Hence there exist s vertex-disjoint sets of t independent vertices each in C_1, completing the proof of Subclaim 1.1.

□

Recall that, by assumption, $|H| \leq 12t + 12$ and $|C_1| \geq 50$. Let $X_1, X_2, \ldots, X_s$ be as in Subclaim 1.1, and let $\mathcal{X} = \bigcup_{i=1}^{s} X_i$. Further, note that $\deg_{C_1}(v) \leq 2$ for every $v \in V(H)$ or a shorter chorded cycle would exist by Lemma 4, contradicting (A1). Thus

$$|E(H, C_1)| \leq 2(12t + 12). \tag{1}$$

First suppose that $k = 2$. Then C_1 is the only cycle in $\mathscr{C}$. By Subclaim 1.1,

$$\begin{aligned} |E(C_1, H)| &\geq \deg_G(\mathcal{X}) - \deg_{C_1}(\mathcal{X}) \\ &\geq s(3kt - t + 1) - (2st + 4) \\ &= s(6t - t + 1) - (2st + 4) \\ &= 3st + s - 4, \end{aligned}$$

but since $s \geq 10$ and $t \geq 5$, we see that $3st + s - 4 \geq 30t + 6 > 2(12t + 12)$, contradicting (1). Thus we may assume that $k \geq 3$. Then, by Subclaim 1.1 and (1),

$$\begin{aligned} |E(\mathcal{X}, \mathscr{C} - C_1)| &= \deg_G(\mathcal{X}) - \deg_{C_1}(\mathcal{X}) - \deg_H(\mathcal{X}) \\ &\geq s(3kt - t + 1) - (2st + 4) - 2(12t + 12) \\ &= 3kst - 3st + s - 24t - 28. \end{aligned} \tag{2}$$

Since $s \geq 10$, we have $3st \geq 30t = 24t + 6t$. Thus

$$24t \leq 3st - 6t. \tag{3}$$

By (2) and (3), we have

$$\begin{aligned} 3kst - 3st + s - 24t - 28 &\geq 3kst - 3st + s - (3st - 6t) - 28 \\ &= 3st(k-2) + s + 6t - 28 \\ &\geq 3st(k-2) + 12. \end{aligned}$$

Thus $|E(\mathcal{X}, C')| > 3st$ for some C' in $\mathscr{C} - C_1$. Let $h = \max\{\deg_{C'}(v) : v \in \mathcal{X}\}$. Let $v^* \in \mathcal{X}$ with $\deg_{C'}(v^*) = h$. Since $|\mathcal{X}| = st$, if $h \leq 3$, then $|E(\mathcal{X}, C')| \leq 3st$, a contradiction. Thus we may assume that $h \geq 4$. By the maximality of C_1, $|C'| \leq |C_1| = st + r$. It follows that $h = \deg_{C'}(v^*) \leq |C'| \leq st + r$. Recall $s \geq 10$, $t \geq 5$ and $0 \leq r \leq t - 1$. Then

$$\begin{aligned} |E(\mathcal{X} - \{v^*\}, C')| &\geq (3st + 1) - \deg_{C'}(v^*) \geq (3st + 1) - (st + r) \\ &= 2st + 1 - r \geq 2st + 1 - (t - 1) \\ &= 2st - t + 2 \\ &\geq 97. \end{aligned} \tag{4}$$

Since $h = \deg_{C'}(v^*) \geq 4$, let v_1, v_2, v_3, v_4 be neighbors of v^* in that order on C'. These vertices partition C' into four intervals $C'[v_i, v_{i+1})$ for each $1 \leq i \leq 4$, where $v_5 = v_1$. By (4), there exist at least 97 edges from $C_1 - v^*$ to C'. Thus some interval clearly receives at least 25 of these edges. Without loss of generality, say $C'[v_4, v_1)$ is such an interval. Then, by Lemma 5, $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$ contains a chorded cycle not containing at least one vertex of $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$. Also, $v^*, C'[v_1, v_3], v^*$ is a cycle with chord v^*v_2, and it uses no vertices from $C'[v_4, v_1)$. Thus we have two shorter vertex-disjoint chorded cycles in $\langle C_1 \cup C' \rangle$, contradicting (A1). Hence Subclaim 1 holds. □

Now as $n \geq (10t - 1)(k - 1) + 12t + 13$ and $|\mathscr{C}| \leq (10t - 1)(k - 1)$ by Subclaim 1, we have $|H| \geq 12t + 13$, a contradiction. This completes the proof of Claim 1. □

By Claim 1, $|H| \geq 12t + 13$. Noting $\lceil |H|/12 \rceil \geq t + 2$, by Lemma 2, there exists an independent set I^* of order $t + 2$ in H such that the degree in H of each vertex of I^* is at most 2. We now select an independent set I of order t from I^* as follows. If H is connected, we select any subset I of order t. If H is not connected, then each component has a longest path with endpoints of degree at most 2 in H (or else the component contains a chorded cycle). If two of these endpoints are in I^*, we select at least two of them, say s_1 and s_2, from different components of H to be in I. Note that each of s_1 and s_2 is not a cut-vertex for its component. If s_1 and s_2 (one or both) are not in I^*, then they might have adjacencies in I^*. We can remove the at most two adjacencies of say s_1 from I^*, and place s_1 in I^*. We can do the same for s_2 if necessary. Then I^* still contains at least t independent vertices with degree at most 2 in H. We select a subset I of order t in I^* that contains both s_1 and s_2. Note that

$$\begin{aligned}\deg_{\mathscr{C}}(I) &= \deg_G(I) - \deg_H(I)\\ &\geq (3kt - t + 1) - 2t\\ &= 3kt - 3t + 1\\ &= 3t(k-1) + 1.\end{aligned}$$

Therefore, there exists a cycle C in $\mathscr{C}$ such that I sends at least $3t + 1$ edges to C. Thus, by Lemma 4, since no vertex of H sends more than four edges to a cycle of $\mathscr{C}$, we see that the degree sequence D of edges from I to C is of the form $(4,4,4,4,\ldots)$, $(4,4,4,\ldots)$, $(4,4,3,\ldots,3,2)$ or $(4,3,\ldots,3)$. Note that if $D = (4,4,4,\ldots)$, then $D = (4,4,4,3,\ldots)$, that is, D contains at least one 3, or $D = (4,4,4,2,2)$ for $t = 5$. Further, since any of these degree sequences contains at least one 4, by Lemma 4 we see that $|C| = 4$. In fact, C induces a K_4, otherwise, the vertex of degree 4 along with a triangle in C would produce a K_4, contradicting (A3). Let $C = w_1, w_2, w_3, w_4, w_1$.

If D has at least two 4's and at least two 3's, then it is simple to construct two vertex-disjoint chorded 4-cycles from C and these vertices of I, as the two vertices of degree 3 are adjacent to the ends of an edge of C and the two vertices of degree 4 are adjacent to the ends of a different independent edge of C. This produces two vertex-disjoint chorded cycles, implying G contains k vertex-disjoint chorded cycles, a contradiction. Thus we have only to consider the two cases where $D = (4,4,4,2,2)$ and $D = (4,3,\ldots,3)$.

First consider $D = (4,4,4,2,2)$. Let z_1 be a vertex of I with degree 2 to C and z_2, z_3, z_4 be the vertices of I with degree 4. Without loss of generality, we may assume that $w_1, w_2 \in N_C(z_1)$. Then z_1, w_2, z_2, w_1, z_1 is a cycle with chord w_1w_2. Also, z_3, w_3, z_4, w_4, z_3 is a second cycle with chord w_3w_4, implying G contains k vertex-disjoint chorded cycles, a contradiction.

Next consider $D = (4,3,\ldots,3)$. Let $\deg_C(z_0) = 4$ and $\deg_C(z_i) = 3$ for each $1 \leq i \leq 4$. First we prove that

$$H \text{ has no component with one vertex of degree 4 and at least three vertices of degree 3.} \tag{5}$$

Suppose not, that is, H has a component H_0 containing z_i for each $0 \leq i \leq 3$. Since H_0 is connected, there exists a path P from z_0 to z_i for some $1 \leq i \leq 3$. Without loss of generality, we may assume that $i = 1$ and P contains neither z_2 nor z_3. Since $\deg_C(z_i) = 3$ for each $i \in \{2,3\}$, we may assume that $w_1, w_2 \in N_C(z_i)$. Then z_2, w_2, z_3, w_1, z_2 is a cycle with chord w_1w_2. Since $\deg_C(z_1) = 3$, without loss of generality, we may assume that $w_3 \in N_C(z_1)$. Then $P[z_0, z_1], w_3, w_4, z_0$ is a second cycle with chord z_0w_3, a contradiction. Thus (5) holds.

Therefore, we assume that H is not connected, that is, $\text{comp}(H) \geq 2$. Let $H_1, H_2, \ldots, H_{\text{comp}(H)}$ be the components of H. Note that it is sufficient to consider the case where each component of H has at least one vertex contained in the degree sequence $D = (4,3,\ldots,3)$. Without loss of generality, for each $i \in \{1,2\}$, we may assume that $s_i \in V(H_i)$ and $\deg_C(s_1) \geq \deg_C(s_2)$. Recall, for each $i \in \{1,2\}$, s_i is not a cut-vertex for H_i.

Case 1 For each $i \in \{1,2\}$, $\deg_C(s_i) = 3$.

In this case, without loss of generality, we may assume that $s_i = z_i$ for each $i \in \{1,2\}$.

Subcase 1 Suppose $\mathrm{comp}(H) = 2$.

Without loss of generality, we may assume that $z_0 \in V(H_1)$. By (5), we may assume that $z_4 \in V(H_2)$. For each $i \in \{1,2\}$, since $\deg_C(s_i) = 3$, we may assume that $w_1, w_2 \in N_C(s_i)$. Then $C' = s_1, w_2, s_2, w_1, s_1$ is a 4-cycle with chord w_1w_2. Since $\deg_C(z_4) = 3$, without loss of generality, we may assume that $w_3 \in N_C(z_4)$. Since $\deg_C(z_0) = 4$, $w_4 \in N_C(z_0)$. Then there exists a path z_0, w_4, w_3, z_4 connecting H_1 and H_2. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note that $H_i - s_i$ is connected for each $i \in \{1,2\}$. Then $\mathrm{comp}(H') \leq \mathrm{comp}(H) - 1$. This contradicts (A2).

Subcase 2 Suppose $\mathrm{comp}(H) \geq 3$.

Subcase 2.1 For some $i \in \{1,2\}$, $z_0 \in V(H_i)$.

Without loss of generality, we may assume that $z_0 \in V(H_1)$, and $z_4 \in V(H_3)$ by our assumption that each component of H has at least one vertex contained in the degree sequence $D = (4, 3, \ldots, 3)$. By the same arguments as Subcase 1, we can reduce the number of components of H, a contradiction.

Subcase 2.2 For some $i \in \{1, 2, \ldots, \mathrm{comp}(H)\} - \{1,2\}$, $z_0 \in V(H_i)$.

Without loss of generality, we may assume that $z_0 \in V(H_3)$. Now consider the cycle C' as in Subcase 1. If $z_3 \in V(H_i)$ for some $i \in \{1, 2, \ldots, \mathrm{comp}(H)\} - \{3\}$, then we apply the same arguments as Subcase 1. Thus we may assume that $z_3 \in V(H_3)$. Since $\deg_C(z_3) = 3$, without loss of generality, we may assume that $w_3 \in N_C(z_3)$. Since H_3 is connected, there exists a path P from z_0 to z_3. Then $P[z_0, z_3], w_3, w_4, z_0$ is a second cycle with chord z_0w_3, a contradiction.

Case 2 Suppose $\deg_C(s_1) = 4$ and $\deg_C(s_2) = 3$.

In this case, note that $s_1 = z_0$. Without loss of generality, we may assume that $s_2 = z_1$.

Subcase 1 Suppose $\mathrm{comp}(H) = 2$.

Subcase 1.1 For some $2 \leq i \leq 4$, $z_i \in V(H_1)$.

Without loss of generality, we may assume that $z_2 \in V(H_1)$. Since $\deg_C(z_2) = 3$ and $\deg_C(s_2) = 3$, $N_C(z_2) \cap N_C(s_2) \neq \emptyset$. Without loss of generality, we may assume that $w_1 \in N_C(z_2) \cap N_C(s_2)$. Since $\deg_C(s_1) = 4$, $C' = s_1, w_2, w_3, w_4, s_1$ is a 4-cycle with chord s_1w_3. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note that $H_1 - s_1$ is connected. Then $\mathrm{comp}(H') \leq \mathrm{comp}(H) - 1$. This contradicts (A2).

Subcase 1.2 For each $2 \leq i \leq 4$, $z_i \in V(H_2)$.

Since $\deg_C(s_2) = 3$, without loss of generality, we may assume that $w_i \in N_C(s_2)$ for each $1 \leq i \leq 3$. If $w_4 \in N_C(z_i)$ for some $2 \leq i \leq 4$, then we apply the same arguments as Subcase 1.1. Thus we may assume that $N_C(s_2) = N_C(z_i)$ for each $2 \leq i \leq 4$. Then $C' = s_2, w_1, w_4, w_2, s_2$ is a 4-cycle with chord w_1w_2. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note that $H_2 - s_2$ is connected. Since $w_3 \in N_C(s_1) \cap N_C(z_2)$, $\mathrm{comp}(H') \leq \mathrm{comp}(H) - 1$. This contradicts (A2).

Subcase 2 Suppose $\mathrm{comp}(H) \geq 3$.

Without loss of generality, we may assume that $z_2 \in V(H_3)$ by our assumption that each component of H has at least one vertex contained in the degree sequence

D. By the same arguments as Subcase 1.1, we can reduce the number of components of H, a contradiction.

This completes the proof of Theorem 5. □

5 Conclusion

We believe that Lemma 2 may be improved to guarantee a larger independent set of low-degree vertices in every non-chorded connected graph. In particular, we conjecture the following.

Conjecture 1 *If H is a non-chorded connected graph of order n, then H contains an independent set I of order at least $n/6$ with each vertex of I having degree at most 2 in H.*

This 1/6 proportion of vertices would be best possible, as we demonstrate with two examples G_1 and G_2.

First, define the graph H with 6 vertices to be the graph containing a 5-cycle $x_1, x_2, x_3, x_4, x_5, x_1$ and where the sixth vertex x_6 is adjacent to x_2 and x_5. To form G_1, take k copies of H called $H^1, H^2, \ldots, H^k$. Let $x_i^j \in V(H^j)$ with $1 \leq i \leq 6$ and $1 \leq j \leq k$, and let $x_6^j x_1^{j+1} \in E(G_1)$ for each $1 \leq j \leq k-1$. Aside from H^1 and H^k, each copy of H has exactly two vertices of degree 2, and only one of these can be included in the independent set I. Each of H^1 and H^k have two independent vertices of degree 2, so $|I| = n/6 + 2$.

Second, construct G_2 by starting with a triangle, and for each of its vertices, connect it by an edge to a new triangle. Then for each vertex of degree 2 in this graph, connect it by an edge to a new triangle. Repeat this process k times. In G_2, every vertex of degree 2 is adjacent to another vertex of degree 2, so only one of each pair can be in I. By adding a triangle adjacent to each vertex of degree 2 in the pair, we can increase the size of I by 1, and we have added 6 vertices. That means the limit

$$\lim_{k \to \infty} \frac{|I|}{n} = \frac{1}{6},$$

so no larger proportion than 1/6 of the vertices of G_2 can be in I.

We also note the following easy-to-prove facts about graphs with no chorded cycles. We did not use these facts in our proof of Theorem 5 but they may be of interest to the reader.

Fact 1 If G is a graph of order n with no chorded cycles, then there exists an ordering of the vertices of G such that each vertex has at most two neighbors preceding it in this ordering. Further G is a tripartite graph.

Fact 2 If G is a graph of order n containing no chorded cycles, then $|E(G)| \leq 2n - 4$.

Acknowledgements The authors would like to thank the referees for their suggestions and comments, and thank Ariel Keller Rorabaugh for the helpful insights she provided. The second author is supported by the

Heilbrun Distinguished Emeritus Fellowship from Emory University Emeritus College. The third author is supported by Japan Society for the Promotion of Science KAKENHI Grant Number JP19K03610.

References

1. Chiba, S., Fujita, S., Gao, Y., Li, G.: On a sharp degree sum condition for disjoint chorded cycles in graphs. Graphs Combin. **26**(2), 173–186 (2010)
2. Corrádi, K., Hajnal, A.: On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hungar. **14**(3–4), 423–439 (1963)
3. Enomoto, H.: On the existence of disjoint cycles in a graph. Combinatorica **18**(4), 487–492 (1998)
4. Finkel, D.: On the number of independent chorded cycles in a graph. Discrete Math. **308**(22), 5265–5268 (2008)
5. Fujita, S., Matsumura, H., Tsugaki, M., Yamashita, T.: Degree sum conditions and vertex-disjoint cycles in a graph. Australas. J. Combin. **35**, 237–251 (2006)
6. Gould, R.J.: Graph Theory. Dover Pub. Inc., Mineola, N.Y. (2012)
7. Gould, R.J., Hirohata, K., Keller, A.: On vertex-disjoint cycles and degree sum conditions. Discrete Math. **341**(1), 203–212 (2018)
8. Gould, R.J., Hirohata, K., Keller, A.: On independent triples and vertex-disjoint chorded cycles in graphs. Australas. J. Combin. **77**(Part 3), 355–372 (2020)
9. Harary, F., Prins, G.: The block-cutpoint-tree of a graph. Publ. Math. Debr. **13**, 103–107 (1966)
10. Ma, F., Yan, J.: The confirmation of a conjecture on disjoint cycles in a graph. Discrete Math. **341**(10), 2903–2911 (2018)
11. Wang, H.: On the maximum number of independent cycles in a graph. Discrete Math. **205**(1–3), 183–190 (1999)